

WEEK #12

USEFUL FUNCTIONS (STP, VALUE)

WHEN DEVELOPING UNIVERSAL FUNCTION

WE CREATED A PREDICATE STP

TO SEE IF FUNCTION HALTS IN SOME
FIXED TIME FOR SOME FIXED INPUT

$$\text{STP}(f, x, t) = \text{TRUE}$$

IFF $Q_f(x)$ CONVERGES IN t
OR FEWER STEPS

WE ALSO NEEDED A WAY TO EXTRACT
OUTPUT IF $Q_f(x)$ HALTS

$$\text{VALUE}(f, x, t) = f(x)$$

PROVIDED $\text{STP}(f, x, t)$

ELSE $\text{VALUE}(f, x, t)$ GIVES

A MEANINGLESS VALUE (WE USE 0)

QUANTIFICATION

S IS DECIDABLE IFF \exists AN ALG. χ_S (CALLED S 'S CHARACTERISTIC FUNCTION) SUCH THAT
 $x \in S \Leftrightarrow \chi_S(x)$ χ_S IS A PREDICATE

S IS RE IFF \exists AN ALG. $A_S \Rightarrow$
 $x \in S \Leftrightarrow \exists t A_S(x, t)$

CAN SHOW WITH $A_S(x, t) = \text{STP}_{g_S}(x, t) = \text{STP}(g_S, x, t)$
WHERE $S = \text{Dom}(g_S)$

\bar{S} IS CO-RE IFF \exists AN ALG. $A_S \Rightarrow$
 $x \in \bar{S} \Leftrightarrow \forall t A_S(x, t)$

JUST COMPLEMENT ABOVE FOR RE

A NON-RE, NON-CO-RE PROBLEM

$f \in \text{TOTAL} \Leftrightarrow \forall x \exists t [\text{STP}(f, x, t)]$

CANNOT REDUCE # OF ALTERNATING
QUANTIFIERS

QUANTIFICATION IS FIRST ESTIMATE
OF COMPUTATIONAL BOUND (UPPER)

USES OF STOP AND VALUE

$$\langle f, x \rangle \in \text{HALT} \Leftrightarrow \exists t [\text{STOP}(f, x, t)]$$

$$f \in \text{TOTAL} \Leftrightarrow \forall x \exists t [\text{STOP}(f, x, t)]$$

$$f \in \text{HASZERO} \Leftrightarrow \exists \langle x, t \rangle [\text{STOP}(f, x, t) \ \&\& \ \text{VALUE}(f, x, t) = 0]$$

$$f \in \text{MONOTONIC} \Leftrightarrow \forall x \exists t [\text{STOP}(f, x, t) \ \&\& \ \text{STOP}(f, x+1, t) \ \&\& \ \text{VALUE}(f, x, t) < \text{VALUE}(f, x+1, t)]$$

$$f \in \text{EMPTY} \Leftrightarrow \forall \langle x, t \rangle [\overline{\text{STOP}(f, x, t)}]$$

CATEGORIES (P ALGORITHMIC PREDICATE)

$$x \in S \Leftrightarrow P(x) \quad \text{RECURSIVE}$$

$$x \in S \Leftrightarrow \exists y P(x, y) \quad \text{RE}$$

$$x \in S \Leftrightarrow \forall y P(x, y) \quad \text{CO-RE}$$

$$x \in S \Leftrightarrow \forall y \exists z P(x, y, z) \quad \text{NON-RE, NON-CORE}$$

$$x \in S \Leftrightarrow \exists y \forall z P(x, y, z) \quad \text{NON-RE, NON-CORE}$$

MORE QUANTIFICATION & UPPER BOUNDS

$\{S \mid f(x) \downarrow \text{IMPLIES IT DOES SO IN } > 2x+2 \text{ STEPS}\}$

$\forall x \overline{\text{STP}(S, x, 2*x+2)}$ CO-RE

$\{S, x \mid f(x) \downarrow \text{IMPLIES IT DOES SO IN } > 2x+2 \text{ STEPS}\}$

$\overline{\text{STP}(S, x, 2*x+2)}$ REC

$\{S \mid \forall x f(x)=0\}$

$\forall x \exists t [\text{STP}(S, x, t) \wedge \text{VALUE}(S, x, t)=0]$
NRNC

$\{S \mid \text{FOR SOME } x f(x)=0\}$

$\exists \langle x, t \rangle [\text{STP}(S, x, t) \wedge \text{VALUE}(S, x, t)=0]$
RE

LATTER TWO ARE AMENABLE TO RICE'S THEOREM; FIRST TWO ARE NOT

Sample Question#5

5. Let **S** be an re (recursively enumerable), non-recursive set, and **T** be an re, possibly recursive non-empty set. Let

$$\mathbf{E} = \{ \mathbf{z} \mid \mathbf{z} = \mathbf{x} + \mathbf{y}, \text{ where } \mathbf{x} \in \mathbf{S} \text{ and } \mathbf{y} \in \mathbf{T} \}.$$

Answer with proofs, algorithms or counterexamples, as appropriate, each of the following questions:

- (a) Can **E** be non re?
(b) Can **E** be re non-recursive?
(c) Can **E** be recursive?

$$f_E(\langle x, y \rangle) = f_S(x) + f_T(y)$$



5(a) NO.

AS S & T ARE NON-EMPTY RE SETS, EACH IS THE RANGE OF SOME ALG., SAY f_S & f_T , RESP.

DEFINE

$$f_E(\langle x, y \rangle) = f_S(x) + f_T(x)$$

f_E IS AN ALG. THAT ENUMERATES ^{THE SUM OF} EVERY PAIR IN $S \times T$ AND SO E IS RE

(b) LET T BE $\{0\}$ THEN $E = S$ WHICH BY DEFIN IS RE, NON-REC,

(c) LET S BE \mathbb{N} THEN

$$E = \{z \mid z \geq \min(S)\}$$

AS S IS NON-EMPTY IT HAS A MIN VALUE

$\chi_E(z) = z \geq \min(S)$ MUST EXIST AND SO E IS REC.

~~NOT ACTUALLY RE~~

11/10/2016

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SEMI-TRUE SYSTEMS

$$S = (\Sigma, R)$$

R: FINITE SET OF RULES

$$\alpha \rightarrow \beta \quad \alpha, \beta \in \Sigma^*$$

LIKE GRAMMARS BUT NO VARIABLES

CAN SIMULATE TMs (MAYBE DO LATER)

CAN SIMULATE STs BY ^{POST} CORRESPONDENCE
PROBLEM. (PCP)

$$\vec{X} = (x_1, \dots, x_n), \vec{Y} = (y_1, \dots, y_n), n > 0$$
$$x_i, y_i \in \Sigma^*$$

$$\exists i_1, \dots, i_k, k > 0, 1 \leq i_j \leq n \Rightarrow$$

$$x_{i_1} \dots x_{i_k} = y_{i_1} \dots y_{i_k}$$

$$n = 3, \Sigma = \{a, b\}, \vec{X} = (aba, bb, a), \vec{Y} = (bab, b, ba)$$

SOLUTION 2, 3, 1, 2

bhaababb
bbaababb

IF ONE SOLN, \exists AN INF. # OF SOLUTIONS

PROBLEM IS, IN GENERAL, UNDEC.

Ambiguity of CFG

$$S \rightarrow A | B$$

$$A \rightarrow x_i A[i] | x_i [i] \quad \left. \vphantom{A \rightarrow} \right\} 1 \leq i \leq n$$

$$B \rightarrow y_i B[i] | y_i [i]$$

$$\left. \begin{aligned} A &\stackrel{*}{\Rightarrow} x_{i_1} \dots x_{i_k} [i_k] \dots [i_1] \\ B &\stackrel{*}{\Rightarrow} y_{i_1} \dots y_{i_k} [i_k] \dots [i_1] \end{aligned} \right\} k > 0$$

Ambiguous iff solution to instance of PCP

Intersection of CFL

$L_A \cap L_B \neq \emptyset$ iff soln of PCP

Thus, Emptiness of CSL undec.

Our Example $\vec{X} = (aba, bba), \vec{Y} = (bab, bba)$

$$\begin{aligned} S &\rightarrow A | B \\ A &\rightarrow aba A[1] | bba A[2] | a A[3] | \\ &\quad aba [1] | bb [2] | a [3] \\ B &\rightarrow bab B[1] | b B[2] | baa B[3] | \\ &\quad bab [1] | b [2] | baa [3] \end{aligned}$$

PCP OVER ONE LETTER

$$\vec{x} = [x_1, \dots, x_n] \quad x_i > 0 \quad n > 0$$
$$\vec{y} = [y_1, \dots, y_n] \quad y_i > 0$$

CASE 1

$$\exists i \quad x_i = y_i$$

SOLUTION i

CASE 2

$$\forall i \quad x_i > y_i$$

SOLUTION \emptyset NONE

CASE 3

$$\forall i \quad x_i < y_i$$

SOLUTION \emptyset NONE

CASE 4

$$\exists i, j \quad x_i > y_i \quad x_j < y_j$$

SOLUTION

x_i REPEATED $y_j - x_j$ TIMES THEN

y_j REPEATED $x_i - y_i$ TIMES

$$x_i (y_j - x_j) + x_j (x_i - y_i)$$
$$= x_i y_j - \cancel{x_i x_j} + \cancel{x_i y_i} - x_j y_i = \underline{\underline{x_i y_j - x_j y_i}}$$
$$y_i (y_j - x_j) + y_j (x_i - y_i)$$
$$= \cancel{y_i y_j} - x_j y_i + x_i y_j - \cancel{y_i y_i} = \underline{\underline{x_i y_j - x_j y_i}}$$

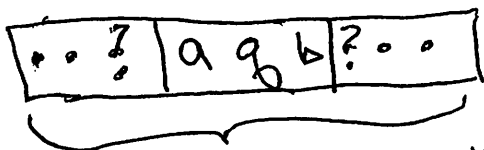
SEMI-TIME

$\alpha \rightarrow \beta$

AND TURING MACHINES
(SHORT STRING REPLACEMENT
IN IDS)

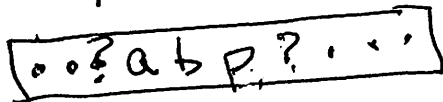
CHANGES TO TM INSTANTANEOUS DESCRIPTION
AS TM MAKES MOVES

ID IS SOME FINITE DESCRIPTION. USE
ONE WITH LEFT/RIGHT SIDE STRINGS AND STATE

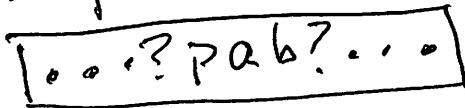


INCLUDES ALL NON-BLANK SQUARES

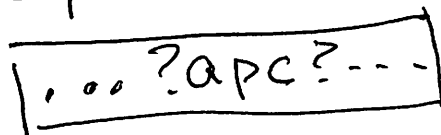
q b R P



q b L P



q b C P



TYPICALLY USE SOME LEFT/RIGHT END MARKERS

E.G. $\# \dots \dots \#$

SO CAN EXPAND AND CONTRACT

FOR EXAMPLE $q a b \# \Rightarrow a b P \#$ IF RIGHT MOVE

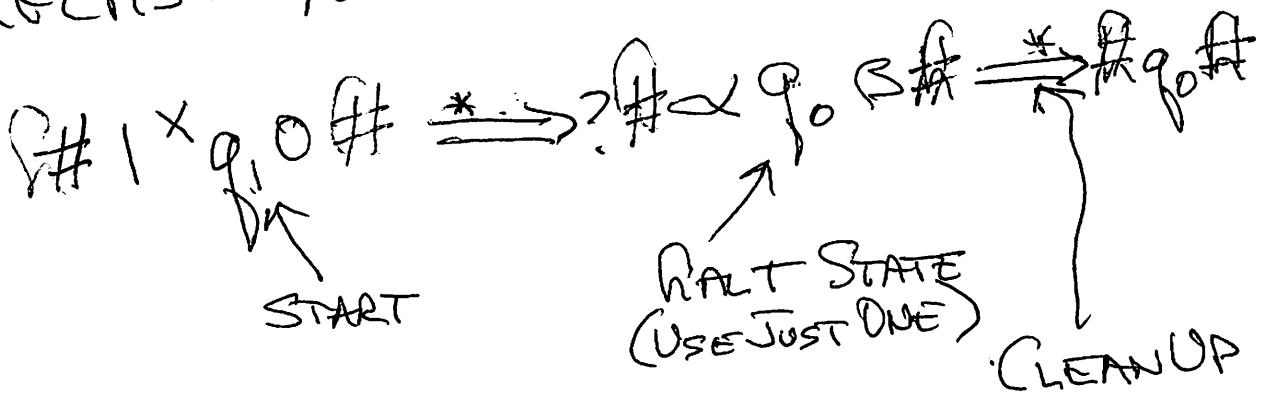
OR $\# q a b \Rightarrow \# q a b$ IF LEFT MOVE

MORE DETAILS ON ST SIMULATING TM

q_0 B

IN STATE q , READING Q
SEE PAGE 360 FOR DETAILED SIM

TM HALT PROBLEM CAN BE
RECAST AS ST WORD PROBLEM



THUS, IF TM, M , ACCEPT x ,
THAT IS, M HALTS WHEN STARTED ON x ,
THEN ST_M (ST SIMULATING M) REWRITES

$\#1xq_0B\#$ AS $\#q_0\#$ } $\#1xq_0B\# \xrightarrow{*} \#q_0\#$

UNSOLVABILITY OF ST WORD PROBLEM

PRIOR SHOWS A SOLUTION TO ST WORD PROBLEM:
 GIVEN ST, S , AND WORDS w_1, w_2 ,
 IS IT TRUE THAT $w_1 \xrightarrow[S]{*} w_2$

IS UNDECIDABLE

UNSOLVABILITY OF PCP

CAN MIMIC ST DERIVATION BY SOLUTION TO INSTANCE OF PCP (POSTCORRESPONDENCE PROB)

EXAMPLE: $aba \rightarrow ab; a \rightarrow aa; b \rightarrow a$ (ST, S)

QUESTION: $bbbb \xrightarrow[S]{*} aa$

PCP INSTANCE

$\begin{matrix} \rightarrow \\ X \\ \downarrow \\ Y \end{matrix}$ $[$ $bbb*$, ab , \underline{ab} , aa , \underline{aa} , a , \underline{a} , \square
 $[$ \underline{aba} , aba , \underline{a} , a , \underline{b} , b , $\underline{*aa}$ \uparrow MUST END
 MUST START \rightarrow RULES

ALSO $*$, $\underline{*}$, a , \underline{a} , b , \underline{b}
 $\underline{*}$, $*$, \underline{a} , a , \underline{b} , b

USED FOR PART NOT REWRITTEN

ST. SOLUTION MAPPED TO PCP

MUST START $[b b b b^*$

AS ONLY PAIR THAT STARTS WITH SAME CHARACTER

$[b b b b^*$

MIMIC THREESTEPS OF $b \rightarrow a$ TO 1, 2, 4 CHARACTERS

$[b b b b^* \underline{a} \underline{b} \underline{a} \underline{a} \underline{x}]$

$[b b b b^*$

MIMIC ONE STEP OF $aba \rightarrow ab$

$[b b b b^* \underline{a} \underline{b} \underline{a} \underline{a} \underline{*} \underline{a} \underline{b} \underline{a} \underline{*}]$

$[b b b b^* \underline{a} \underline{b} \underline{a} \underline{a} \underline{*}]$

MIMIC ONE STEP OF $aba \rightarrow ab$

$[b b b b^* \underline{a} \underline{b} \underline{a} \underline{a} \underline{x} \underline{a} \underline{b} \underline{a} \underline{*} \underline{a} \underline{b} \underline{x}]$

$[b b b b^* \underline{a} \underline{b} \underline{a} \underline{a} \underline{*} \underline{a} \underline{b} \underline{a} \underline{*}]$

MIMIC ONE STEP OF $b \rightarrow a$

$[b b b b^* \underline{a} \underline{b} \underline{a} \underline{a} \underline{*} \underline{a} \underline{b} \underline{a} \underline{*} \underline{a} \underline{b} \underline{*} \underline{a} \underline{a}]$

$[b b b b^* \underline{a} \underline{b} \underline{a} \underline{a} \underline{*} \underline{a} \underline{b} \underline{a} \underline{*} \underline{a} \underline{b}]$

$[b b b b^* \underline{a} \underline{b} \underline{a} \underline{a} \underline{*} \underline{a} \underline{b} \underline{a} \underline{*} \underline{a} \underline{b} \underline{*} \underline{a} \underline{a}]$

$[b b b b^* \underline{a} \underline{b} \underline{a} \underline{a} \underline{*} \underline{a} \underline{b} \underline{a} \underline{*} \underline{a} \underline{b} \underline{x} \underline{a} \underline{a}]$

LAST STEP

PCP CONSEQUENCES

$P: \vec{x} = (x_1, \dots, x_n) \quad \vec{y} = (y_1, \dots, y_n)$

① AMBIGUITY OF CFG IS UNDECIDABLE

$S \rightarrow A | B$
 $A \rightarrow x_i A [L] | x_i [L] \quad 1 \leq i \leq n$
 $B \rightarrow y_i B [L] | y_i [L] \quad 1 \leq i \leq n$

P HAS SOLN. IFF $\exists i_1, \dots, i_r \Rightarrow$

$A \xrightarrow{*} x_{i_1} \dots x_{i_r} [L_1] \dots [L_r] \quad r > 0$
 $B \xrightarrow{*} y_{i_1} \dots y_{i_r} [L_1] \dots [L_r]$

$x_{i_1} \dots x_{i_r} \equiv y_{i_1} \dots y_{i_r}$

IFF ABOVE GRAMMAR IS AMBIGUOUS

② CONSIDER A-RULES AND B-RULES ABOVE
 $L_A \cap L_B \neq \emptyset$ IFF P HAS SOLN

SO NON-EMPTYNESS (EMPTYNESS) OF INTERSECTION OF CFLS IS UNDEC.

NOTE! INTERSECTION OF CFLS IS A CSL SO EMPTYNESS OF CSL IS UNDEC.

③ CAN DO CSL EMPTYNESS. AS DIRECT PROOF

$S \rightarrow x_i S y_i^R | x_i^T y_i^R \quad 1 \leq i \leq n$
 $a T a \rightarrow * T *$
 $* a \rightarrow a *$
 $a * \rightarrow * a$
 $T \rightarrow *$

NON-EMPTINESS OF CSL

$$S \rightarrow x_i s y_i^R \mid x_i t y_i^R \quad 1 \leq i \leq n$$

$$a t a \rightarrow * T *$$

$$* a \rightarrow a *$$

$$a * \rightarrow * a$$

$$T \rightarrow *$$

$$\Sigma = \{*\}$$

$$V = \{S, T\} \cup \text{PCF ALPH.}$$

GET STRINGS $*^{2j+1}$ for some j 's
IFF SOLN TO PCP

ABOVE ALSO SHOWS FINITENESS
(INFINITENESS) OF CSL IS UNDEC.

TRACES

A COMPUTATIONAL TRACE IS A WORD OF FORM

$\#X_0\#X_1\#\dots\#X_k\#$

WHERE $X_i \Rightarrow X_{i+1}$ $0 \leq i < k$, EACH X_i AN ID,
AND X_k IS A TERMINAL ID

WE CAN SHOW CFG S THAT ALMOST
GET TRACES (OFTEN ALTERNATING BETWEEN
IDS AND THE REVERSAL OF IDS), WHAT
THEY GET IS EVERY OTHER PAIR RIGHT

$\#X_0\#X_1\#\dots\#X_k\#$

WHERE $X_{2i} \Rightarrow X_{2i+1}$

BUT $X_{2i+1} \Rightarrow? X_{2i+2}$

SO EVEN/ODD RIGHT, BUT ODD/EVEN
ARE NOT NEC. RIGHT

CAN GET SEPARATE GRAMMAR FOR ODD/EVEN

CAN USE \cap OR QUOTIENT TO GET
ALIGNMENT

WITHOUT PROOF (SEE 376), FOR CFL L_1, L_2

L_1/L_2 CAN BE ANY RE SET.

COMPLEMENT OF TRACE

WHILE TRACE REQUIRES ALL PAIRS TO BE RIGHT, COMPLEMENT REQUIRES JUST ONE ERROR.

CAN SHOW COMPLEMENTS OF TRACES ARE CFLs. MOREOVER, IF MACHINE FOR WHICH WE HAVE TRACES ACCEPTS \emptyset THEN THERE ARE NO TERMINATING TRACES AND COMPLEMENT IS Σ^* .

THIS IS WHY WE CANNOT DECIDE IF A CFG, G , IS SUCH THAT $L(G) = \Sigma^*$. WE CAN DECIDE IF $L(G) = \emptyset$, $|L(G)| = \text{INF}$.

$$L(G) = L(G)^2$$

LET G BE AN ARBITRARY CFG (OR CSG)

LET $L = L(G)$, WANT TO SHOW $L = L^2$ IS UNDEC.

NOTE: $L^2 = \{xy \mid x, y \in L\}$

FIRST, RECALL THAT $L = \Sigma^*$ IS UNDEC.
FOR CFG (AND CSG)

CLAIM: $L = \Sigma^*$ IFF

(1) $\Sigma \cup \{\lambda\} \subseteq L$; AND (NOTE: DECIDABLE)

(2) $L \cdot L = L$

CLEARLY IF $L = \Sigma^*$ THEN (1) AND (2) HOLD

CONVERSELY, $\Sigma^* \subseteq L^* = \bigcup_{n \geq 0} L^n \subseteq L$

FOLLOWS
FROM (1)

FOLLOWS
FROM (2)
SINCE IF
 $L = L^2$ THEN
 $L = L^3$, ETC.