WEEK #12

USEFUL FUNCTIONS (STP, VALUE)

WHEN DEVELOPING UNIVERSAL FUNCTION
WE CREATED A PREDICATE STP
TO SEE IF FUNCTION HALTS IN SOME
FIXED TIME FOR SOME FIXED INPUT

$STP(f, x, t) = $ TRUE

IFF $\Phi_f(x)$ CONVERGES IN t
OR FEWER STEPS

WE ALSO NEEDED A WAY TO EXTRACT
OUTPUT IF $\Phi_f(x)$ HALTS

$VALUE(f, x, t) = f(x)$

PROVIDED $STP(f, x, t)$
ELSE $VALUE(f, x, t)$ GIVES
A MEANINGLESS VALUE (WE USE 0)

QUANTIFICATION

S IS DECIDABLE IFF $\exists$ AN ALG. χ_S (CALLED S's CHARACTERISTIC FUNCTION) SUCH THAT

$x \in S \Leftrightarrow \chi_S(x)$ $\quad \chi_S$ IS A PREDICATE

S IS RE IFF $\exists$ AN ALG. $A_S \ni$

$x \in S \Leftrightarrow \exists t\, A_S(x,t)$

CAN SHOW WITH $A_S(x,t) = STP_{g_S}(x,t) = STP(g_S, x, t)$

WHERE $S = Dom(g_S)$

$\bar{S}$ IS CO-RE IFF $\exists$ AN ALG. $A_S \ni$

$x \in \bar{S} \Leftrightarrow \forall t\, A_S(x,t)$

JUST COMPLEMENT ABOVE FOR RE

A NON-RE, NON-CO-RE PROBLEM

$f \in TOTAL \Leftrightarrow \forall x \exists t\, [STP(f,x,t)]$

CANNOT REDUCE # OF ALTERNATING QUANTIFIERS

QUANTIFICATION IS FIRST ESTIMATE OF COMPUTATIONAL BOUND (UPPER)

USES OF STP AND VALUE

$\langle f, x \rangle \in HALT \Leftrightarrow \exists t\, [STP(f,x,t)]$

$f \in TOTAL \Leftrightarrow \forall x \exists t\, [STP(f,x,t)]$

$f \in HasZero \Leftrightarrow \exists \langle x,t \rangle [STP(f,x,t)\ \&\&$
$VALUE(f,x,t) == 0]$

$f \in MonoIncr \Leftrightarrow \forall x \exists t\, [STP(f,x,t)\ \&\&$
$STP(f,x+1,t)\ \&\&$
$VALUE(f,x,t) <$
$VALUE(f,x+1,t)]$

$f \in EMPTY \Leftrightarrow \forall \langle x,t \rangle [\, \overline{STP(f,x,t)}\,]$

CATEGORIES (P ALGORITHMIC PREDICATE)

$x \in S \Leftrightarrow P(x)$ RECURSIVE

$x \in S \Leftrightarrow \exists y\, P(x,y)$ RE

$x \in S \Leftrightarrow \forall y\, P(x,y)$ CO-RE

$x \in S \Leftrightarrow \forall y \exists z\, P(x,y,z)$ NON-RE, NON-CORE

$x \in S \Leftrightarrow \exists y \forall z\, P(x,y,z)$ NON-RE NON-CORE

More Quantification & Upper Bounds

$\{f \mid f(x)\downarrow$ implies it does so in $> 2x+2$ steps$\}$

$\forall x \ \overline{STP(f,x,2*x+2)}$ Co-RE

$\{\langle f,x\rangle \mid f(x)\downarrow$ implies it does so in $> 2x+2$ steps$\}$

$\overline{STP(f,x,2*x+2)}$ REC

$\{f \mid \forall x\ f(x)=0\}$

$\forall x\, \exists t\, [STP(f,x,t)\ \&\ VALUE(f,x,t)=0]$

NRNC

$\{f \mid$ for some x $f(x)=0\}$

$\exists\langle x,t\rangle\, [STP(f,x,t)\ \&\ VALUE(f,x,t)=0]$

RE

Latter two are amenable to Rice's theorem; first two are not

Sample Question#5

5. Let **S** be an re (recursively enumerable), non-recursive set, and **T** be an re, possibly recursive non-empty set. Let

   E = { z | z = x + y, where x ∈ S and y ∈ T }.

   Answer with proofs, algorithms or counterexamples, as appropriate, each of the following questions:

   $f_E(\langle x,y \rangle) = f_S(x) + f_T(y)$

   (a) Can **E** be non re?

   (b) Can **E** be re non-recursive?

   (c) Can **E** be recursive?

5(a) NO.

AS S & T ARE NON-EMPTY RE SETS, EACH IS THE RANGE OF SOME ALG., SAY f_S & f_T, RESP.

DEFINE

$$f_E(\langle x,y \rangle) = f_S(x) + f_T(x)$$

f_E IS AN ALG. THAT ENUMERATES THE SUM OF EVERY PAIR IN $S \times T$

AND SO E IS RE

(b) LET T BE $\{0\}$ THEN

$E = S$ WHICH BY DEFN IS RE, NON-REC.

(c) LET S BE $\mathbb{N}$ THEN

$E = \{z \mid z \geq \min(S)\}$

AS S IS NON-EMPTY IT HAS A MIN VALUE

AS S IS NON-EMPTY $z \geq \min(S)$ MUST EXIST AND

$\chi_E(z) = z \geq \min(S)$

SO E IS REC.

SEMI-THUE SYSTEMS

$S = (\Sigma, R)$

R: FINITE SET OF RULES

$\alpha \to \beta \quad \alpha, \beta \in \Sigma^*$

LIKE GRAMMARS BUT NO VARIABLES

CAN SIMULATE TMs (MAYBE DO LATER)

CAN SIMULATE STs BY POST CORRESPONDENCE PROBLEM (PCP)

$\vec{X} = (x_1, \ldots, x_n), \vec{Y} = (y_1, \ldots, y_n), n > 0$

$x_i, y_i \in \Sigma^*$

$\exists\, i_1, \ldots, i_k, k > 0, 1 \le i_j \le n \ni$

$x_{i_1} \cdots x_{i_k} = y_{i_1} \cdots y_{i_k}$

$n = 3, \Sigma = \{a, b\}, \vec{X} = (aba, bb, a), \vec{Y} = (bab, b, baa)$

SOLUTION 2, 3, 1, 2 $\quad$ bbaababb

bbaababb

IF ONE SOLN, $\exists$ AN INF. # OF SOLUTIONS

PROBLEM IS, IN GENERAL, UNDEC.

AMBIGUITY OF CFG

$S \to A \mid B$

$$\left.\begin{array}{l} A \to x_i\, A[i] \mid x_i\, [i] \\ B \to y_i\, B[i] \mid y_i\, [i] \end{array}\right\} 1 \le i \le n$$

$$\left.\begin{array}{l} A \overset{*}{\Rightarrow} x_{i_1} \cdots x_{i_k} [i_k] \cdots [i_1] \\ B \overset{*}{\Rightarrow} y_{i_1} \cdots y_{i_k} [i_k] \cdots [i_1] \end{array}\right\} k > 0$$

AMBIGUOUS IFF SOLUTION TO INSTANCE OF PCP

INTERSECTION OF CFL

$L_A \cap L_B \neq \emptyset$ IFF SOLN OF PCP

THUS, EMPTINESS OF CSL UNDEC.

OUR EXAMPLE $\vec{X} = (aba, bb, a)$, $\vec{Y} = (bab, b, baa)$

$S \to A \mid B$

$A \to aba\,A[1] \mid bb\,A[2] \mid a\,A[3] \mid$
$\quad aba\,[1] \mid bb\,[2] \mid a\,[3]$

$B \to bab\,B[1] \mid b\,B[2] \mid baa\,B[3] \mid$
$\quad bab\,[1] \mid b\,[2] \mid baa\,[3]$

PCP OVER ONE LETTER

$\vec{X} = [x_1, \ldots, x_n]$ $\quad x_i > 0 \quad n > 0$

$\vec{Y} = [y_1, \ldots, y_n]$ $\quad y_i > 0$

CASE 1

$\exists i \quad x_i = y_i$ SOLUTION i

CASE 2

$\forall i \quad x_i > y_i$ SOLUTION $\emptyset$ NONE

CASE 3

$\forall i \quad x_i < y_i$ SOLUTION $\emptyset$ NONE

CASE 4

$\exists i,j \quad x_i > y_i \quad x_j < y_j$

SOLUTION

x_i REPEATED $y_j - x_j$ TIMES THEN

y_j REPEATED $x_i - y_i$ TIMES

$$x_i(y_j - x_j) + x_j(x_i - y_i)$$

$$= x_i y_j - \cancel{x_i x_j} + \cancel{x_i x_j} - x_j y_i = \underline{\underline{x_i y_j - x_j y_i}}$$

$$y_i(y_j - x_j) + y_j(x_i - y_i)$$

$$= \cancel{y_i y_j} - x_j y_i + x_i y_j - \cancel{y_i y_j} = \underline{\underline{x_i y_j - x_j y_i}}$$

SEMI-THUE

$\alpha \to \beta$ AND TURING MACHINES
(SHORT STRING REPLACEMENT IN IDs)

CHANGES TO TM INSTANTANEOUS DESCRIPTION AS TM MAKES MOVES

ID IS SOME FINITE DESCRIPTION. USE ONE WITH LEFT/RIGHT SIDE STRINGS AND STATE

| . . ? | a q b | ? . . . |

$\underbrace{\qquad\qquad\qquad\qquad}_{\text{INCLUDES ALL NON-BLANK SQUARES}}$

q b R p

| . . ? a b p ? . . . |

q b L p

| . . . ? p a b ? . . . |

q b c P

| . . . ? a p c ? - - - |

TYPICALLY USE SOME LEFT/RIGHT END MARKERS

E.G. | # . . . . # |

SO CAN EXPAND AND CONTRACT

FOR EXAMPLE $aqb\# \Rightarrow abp0\#$ IF RIGHT MOVE

OR $\#qb \Rightarrow \#q0b$ (IF LEFT MOVE

MORE DETALS ON
ST SIMULATING TM

$\# \alpha q a \beta \#$

IN STATE q, READING a
SEE PAGE 360 FOR DETAILED SIM

TM HALT PROGLEM CAN BE
RECAST AS ST WORD PROBLEM

$$\#1^x q_1 0\# \overset{*}{\Longrightarrow} ?\#\alpha q_0 \beta\# \overset{*}{\Longrightarrow} \#q_0\#$$

START

HALT STATE
(USE JUST ONE)

CLEANUP

THUS, IF TM, M, ACCEPT X,
THAT IS, M HALTS WHEN STARTED ON X,
THEN ST_M (ST SIMULATING M) REWRITES
$\#1^x q_1 0\#$ $\Big\}$ $\#1^x q_1 0\# \overset{*}{\Longrightarrow} \#q_0\#$
AS $\#q_0\#$

UNSOLVABILITY OF ST WORD PROBLEM

PRIOR SHOWS A SOLUTION TO ST WORD PROBLEM: GIVEN ST, S, AND WORDS w_1, w_2, IS IT TRUE THAT $w_1 \overset{*}{\underset{S}{\Longrightarrow}} w_2$

IS UNDECIDABLE

UNSOLVABILITY OF PCP

CAN MIMIC ST DERIVATION BY SOLUTION TO INSTANCE OF PCP (POST CORRESPONDENCE PROB)

EXAMPLE: $aba \to ab$; $a \to aa$; $b \to a$ (ST, S)

QUESTION: $bbbb \overset{*}{\underset{S}{\Longrightarrow}} aa$

PCP INSTANCE

$\vec{X}$ $[\,bbbb*,\ ab,\ \underline{ab},\ aa,\ \underline{aa},\ a,\ \underline{a},\ ]$

$\vec{Y}$ $[\ ,\ \underline{aba},\ aba,\ \underline{a},\ a,\ \underline{b},\ b,\ \underline{*}aa]$

MUST START

RULES

MUST END

ALSO $*,\ \underline{*},\ a,\ \underline{a},\ b,\ \underline{b}$

$\underline{*},\ *,\ \underline{a},\ a,\ \underline{b},\ b$

USED FOR PART NOT REWRITTEN

ST SOLUTION MAPPED TO PCP

MUST START [bbbb*
[

AS ONLY PAIR THAT STARTS WITH SAME CHARACTER

[bbbb*
[
MIMIC THREESTEPS OF b→a TO 1,2,4 CHARACTERS
[bbbb*abaa*
[bbbb*
MIMIC ONE STEP OF aba→ab
[bbbb*abaa*aba*
[bbbb*abaa*
MIMIC ONE STEP OF aba→ab
[bbbb*abaa*aba*ab*
[bbbb*abaa*aba*
[bbbb*abaa*
MIMIC ONE STEP OF b→a
[bbbb*abaa*aba*ab*aa
[bbbb*abaa*aba*ab
[bbbb*abaa*aba*ab*aa]
[bbbb*abaa*aba*ab*aa]
[bbbb*abaa*
LAST STEP

PCP CONSEQUENCES

P! $\vec{X} = (X_1, \ldots, X_n)$ $\vec{Y} = (Y_1, \ldots, Y_n)$

① AMBIGUITY OF CFG IS UNDECIDABLE

$S \to A \mid B$

$A \to X_i A [i] \mid X_i [i]$ $\quad 1 \le i \le n$

$B \to Y_i B [i] \mid Y_i [i]$ $\quad 1 \le i \le n$

$k > 0$

P HAS SOLN. IFF $\exists i_1 \ldots i_k \ni$
$$\left\{\begin{array}{l} A \overset{*}{\Rightarrow} X_{i_1} \cdots X_{i_k} [i_k] \cdots [i_1] \\ B \overset{*}{\Rightarrow} Y_{i_1} \cdots Y_{i_k} [i_k] \cdots [i_1] \\ X_{i_1} \cdots X_{i_k} \equiv Y_{i_1} \cdots Y_{i_k} \end{array}\right.$$

IFF ABOVE GRAMMAR IS AMBIGUOUS

② CONSIDER A-RULES AND B-RULES ABOVE

$L_A \cap L_B \neq \phi$ IFF P HAS SOLN

SO

NON-EMPTINESS (EMPTINESS) OF INTERSECTION OF CFLs IS UNDEC.

NOTE! INTERSECTION OF CFLs IS A CSL SO EMPTINESS OF CSL IS UNDEC.

③ CAN DO CSL EMPTINESS AS DIRECT PROOF

$S \to X_i S Y_i^R \mid X_i T Y_i^R \quad 1 \le i \le n$

$a T a \to * T *$

$* a \to a *$

$a * \to * a$

$T \to *$

NON-EMPTINESS OF CSL

$S \to x_i S y_i^R \mid x_i T y_i^R \quad 1 \le i \le n$

$aTa \to * T *$

$* a \to a *$

$a * \to * a$

$T \to *$

$\Sigma = \{*\}$

$V = \{S, T\} \cup$ PCF ALPH.

GET STRINGS $*^{2j+1}$ for SOME j's

IFF SOLN TO PCP

ABOVE ALSO SHOWS FINITENESS (INFINITENESS) OF CSL IS UNDEC.

TRACES

A COMPUTATIONAL TRACE IS A WORD OF FORM

$\#X_0\#X_1\# \ldots \#X_k\#$

WHERE $X_i \Rightarrow X_{i+1}$ $0 \le i < k$, EACH X_i AN ID,

AND X_k IS A TERMINAL ID

WE CAN SHOW CFGS THAT ALMOST GET TRACES (OFTEN ALTERNATING BETWEEN IDs AND THE REVERSAL OF IDs), WHAT THEY GET IS EVERY OTHER PAIR RIGHT

$\#X_0\#X_1\# \ldots \#X_k\#$

WHERE $X_{2i} \Rightarrow X_{2i+1}$

BUT $X_{2i+1} \Rightarrow ? X_{2i+2}$

SO EVEN/ODD RIGHT, BUT ODD/EVEN ARE NOT NEC. RIGHT

CAN GET SEPARATE GRAMMAR FOR ODD/EVEN

CAN USE $\cap$ OR QUOTIENT TO GET ALIGNMENT

WITHOUT PROOF (SEE 376), FOR CFL L_1, L_2

L_1/L_2 CAN BE ANY RE SET

COMPLEMENT OF TRACE

WHILE TRACE REQUIRES ALL PAIRS TO BE RIGHT, COMPLEMENT REQUIRES JUST ONE ERROR.

CAN SHOW COMPLEMENTS OF TRACES ARE CFLs. MOREOVER, IF MACHINE FOR WHICH WE HAVE TRACES ACCEPTS $\emptyset$ THEN THERE ARE NO TERMINATING TRACES AND COMPLEMENT IS Σ^*.

THIS IS WHY WE CANNOT DECIDE IF A CFG, G, IS SUCH THAT $\mathcal{L}(G) = \Sigma^*$

WE CAN DECIDE IF $\mathcal{L}(G) = \emptyset$, $|\mathcal{L}(G)| = $ INF.

$$\mathcal{L}(G) = \mathcal{L}(G)^2$$

LET G BE AN ARBITRARY CFG (OR CSG)

LET $L = \mathcal{L}(G)$, WANT TO SHOW $L = L^2$ IS UNDEC.

NOTE: $L^2 = \{xy \mid x, y \in L\}$

FIRST, RECALL THAT $L = \Sigma^*$ IS UNDEC. FOR CFG (AND CSG)

CLAIM: $L = \Sigma^*$ IFF

(1) $\Sigma \cup \{\lambda\} \subseteq L$; AND (NOTE: DECIDABLE)

(2) $L \cdot L = L$

CLEARLY IF $L = \Sigma^*$ THEN (1) AND (2) HOLD

CONVERSELY, $\Sigma^* \subseteq L^* = \bigcup_{n \geq 0} L^n \subseteq L$

FOLLOWS FROM (1)

FOLLOWS FROM (2) SINCE IF $L = L^2$ THEN $L = L^3$, ETC.