RICE'S (NORMAL VIEW)

$\mathcal{P}$ = SOME PROPERTY OF RE SETS
OR SOME PROPERTY OF REC FUNCTIONS
OR SOME PROPERTY OF COMPUTABLE FUNCTIONS

$\mathcal{P}$ DIVIDES RE SETS INTO TWO SUBSETS

$S_\mathcal{P} = \{ a \mid W_a$ HAS PROPERTY $\mathcal{P} \}$

$\overline{S_\mathcal{P}} = \{ a \mid W_a$ DOES NOT HAVE PROP. $\mathcal{P} \}$

OBVIOUSLY, CAN ALSO VIEW AS

$S_\mathcal{P} = \{ f \mid dom(f)$ HAS PROPERTY $\mathcal{P} \}$

$\overline{S_\mathcal{P}} = \{ f \mid dom(f)$ DOES NOT HAVE PROP. $\mathcal{P} \}$

THIS IS CAST AS BEING ABOUT DOMAINS OR SEMI-DECISION PROCEDURES FOR THE RE SETS

WE ARE NOT ALLOWED TO DROP f INTO $S_\mathcal{P}$ IF $Dom(f) = Dom(g)$ AND $g \in \overline{S_\mathcal{P}}$ OR VICE VERSA. IN OTHER WORDS PROPERTY IS BASED PURELY ON DOMAIN, NOT ON HOW FUNCTION IS IMPLEMENTED

RICE'S (ALTERNATE VIEWS)

$\mathcal{P}$ DIVIDES RE SETS INTO TWO SUBSETS

$S_{\mathcal{P}} = \{ f \mid \text{RANGE}(f) \text{ HAS PROPERTY } \mathcal{P} \}$

$\overline{S_{\mathcal{P}}} = \{ f \mid \text{RANGE}(f) \text{ DOES NOT HAVE PROP. } \mathcal{P} \}$

THIS IS BASED ON PROCEDURES THAT LIST RE SETS AND SO IS ABOUT RANGES

IF $\text{RANGE}(f) = \text{RANGE}(g)$ THEN $f \in S_{\mathcal{P}} \Leftrightarrow g \in S_{\mathcal{P}}$

$\mathcal{P}$ DIVIDES PROCEDURES INTO TWO SUBSETS

$S_{\mathcal{P}} = \{ f \mid \text{THE MAPPING DEFINED BY } f \text{ HAS PROPERTY } \mathcal{P} \}$

$\overline{S_{\mathcal{P}}} = \{ f \mid \text{THE MAPPING DEFINED BY } f \text{ DOES NOT HAVE PROP. } \mathcal{P} \}$

THIS VERSION REQUIRES A STRONG PROPERTY THAT, IF $\forall x\, f(x) = g(x)$

THEN $f \in S_{\mathcal{P}} \Leftrightarrow g \in S_{\mathcal{P}}$

AGAIN, THIS IGNORES IMPLEMENTATION PROPERTIES

RICE'S THEOREMS AND NON-TRIVIAL

RICE'S THEOREM ONLY APPLIES TO
PROPERTIES $\mathcal{P}$ WHERE

$$S_{\mathcal{P}} \neq \emptyset \quad \text{AND} \quad \overline{S_P} \neq \emptyset$$

SO SOME SETS/FUNCTIONS MUST HAVE PROPERTY AND SOME MUST NOT HAVE PROPERTY

AS SUCH, WE CAN ASSUME $\exists a, a \in S_{\mathcal{P}}$
AND $\exists b, b \in \overline{S_P}$

WE WILL, WLOG, ASSUME ALL INDICES, f, OF RE SETS (REC. FUNCTIONS) THAT HAVE THE PROPERTY THAT

$$\text{Dom}(f) = \text{RANGE}(f) = \emptyset$$

THAT IS, $\forall x\ f(x)\uparrow$
ARE IN $\overline{S_P}$ (NOTE ALL ~~ARE~~ IN SAME SET)
IF NOT, WE WILL PROVE $\overline{S_P}$ IS UNDEC.

RICE'S THEOREM (S)

THERE ARE THREE (3) CONSTRAINTS

1. IT IS ABOUT PROPERTIES OF COMPUTABLE FUNCTIONS (PROGRAMS, PROCEDURES)
   IT SPLITS THE INDICES OF FUNCTIONS INTO TWO CLASSES. FOR SOME PROPERTY, P, WE HAVE SETS S_P AND $\overline{S}_P$
   $S_P = \{f \mid \varphi_f \text{ HAS PROPERTY } P\}$
2. IT CAN BE APPLIED ONLY TO NON-TRIVIAL PROPERTIES. P IS NON-TRIVIAL IF
   $S_P \neq \emptyset$ <u>AND</u> $\overline{S}_P \neq \emptyset$
3. IT MUST BE A PROPERTY ABOUT FUNCTIONAL BEHAVIOR, NOT IMPLEMENTATION OR EFFICIENCY.
   THAT IS, IF f AND g HAVE SAME BEHAVIOR THEN BOTH ARE IN S_P OR BOTH ARE IN $\overline{S}_P$

RICE'S VARIANTS BASED ON BEHAVIOR TYPES

I. WEAK BEHAVIOR TYPE BASED ON DOMAIN

f AND g HAVE SAME BEHAVIOR IF

$Dom(f) = Dom(g)$ – STANDARD RICE

II. WEAK BEHAVIOR TYPE BASED ON RANGE

f AND g HAVE SAME BEHAVIOR IF

$Rng(f) = Rng(g)$

III. STRONG BEHAVIOR TYPE BASED ON ACTUAL MAPPINGS FROM INPUT TO OUTPUT

f AN g HAVE SAME BEHAVIOR IF

$\forall x\; f(x) = g(x)$

THIS MEANS SAME VALUE IF CONVERGE AND BOTH DIVERGE OTHERWISE

III ⇒ I AND II

BUT I ⇏ III, II ⇏ III AND

I & II ⇏ III

WANT TO SHOW THAT

IF $\mathcal{P}$ IS A NON-TRIVIAL PROPERTY OF RE SETS (REC. FUNCTIONS)
AND $\mathcal{P}$ IS IMMUNE TO IMPLEMENTATION

THEN $\mathcal{P}$ IS UNDECIDABLE

WE WILL SHOW

$$\text{HALT} \leq_1 S\mathcal{P}$$

$$\text{HALT} = \{\langle x, y\rangle \mid \varphi_x(y)\downarrow\}$$

RICE'S THEOREM STATEMENT

IF P IS A NON-TRIVIAL PROPERTY OF FUNCTION INDICES THAT IS IMMUNE TO IMPLEMENTATION (AN I/O PROPERTY) THEN P IS UNDECIDABLE (NON-RECURSIVE)

KEYS TO PROOF ARE:

1. ALL FUNCTIONS WITH EMPTY DOMAINS/RANGES WILL EITHER HAVE PROPERTY P OR NOT HAVE PROPERTY P.
2. ASSUME ALL FUNCTIONS WITH EMPTY DOMAINS/RANGES ARE IN S_P. IF NOT WE PROVE THE COMPLEMENT OF P TO BE UNSOLVABLE
3. AS P IS NON-TRIVIAL, THERE IS SOME FUNCTION INDEX IN S_P. CALL THIS INDEX r

PROOF OF RICE'S THEOREM (S)

LET $\mathcal{P}$ BE SOME NON-TRIVIAL, IMPLEMENTATION INDEPENDENT PROPERTY OF REC. FUNCTIONS (RE SETS) WHERE

$$S_P = \{ z \mid \varphi_z \text{ HAS PROPERTY } \mathcal{P} \}$$

AS $\mathcal{P}$ IS NON-TRIVIAL, $\exists R \ni \varphi_R$ HAS PROPERTY $\mathcal{P}$ AND SO $R \in S_P$

WLOG, ASSUME $e \notin S_P$ WHENEVER $\text{Dom}(\varphi_e) = \text{RANGE}(\varphi_e) = \emptyset$. THAT IS, $\forall x\, \varphi_e(x)\uparrow$

LET $\text{HALT} = \{ \langle x, y \rangle \mid \varphi_x(y)\downarrow \}$

GIVE ARB. x, y AND EXISTENCE OF R

DEFINE

$$f_{x,y,R}(z) = \varphi_x(y) - \varphi_x(y) + \varphi_R(z)$$

IF $\varphi_x(y)\downarrow$ THEN $f_{x,y,R}(z) = \varphi_R(z) \ \forall z$
AND $f_{x,y,R} \in S_P$

IF $\varphi_x(y)\uparrow$ THEN $f_{x,y,R}(z)\uparrow \ \forall z$
AND $f_{x,y,R} \in \overline{S_P}$

THUS, $\text{HALT} \leq S_P$

COMMENT ABOUT R

AS $R \in S_P$ AND $e \in \overline{S_P}$ WHEN

$$\text{DOM}(\varphi_e) = \text{RANGE}(\varphi_e) \; \& \; \forall x \, \varphi_e(x)\uparrow$$

WE KNOW THAT

$$\text{DOM}(\varphi_R) \neq \emptyset$$

$$\text{RANGE}(\varphi_R) \neq \emptyset$$

AND $\exists z \, \varphi_R(z)\downarrow$

SUMMARY

IF $\varphi_x(y)\downarrow$

$\forall z \; f_{x,y,R}(z) = \varphi_R(z)$ STRONG VERSION

$\mathrm{Dom.}(f_{x,y,R}) = \mathrm{DOM}(\varphi_R)$ *NORMAL VERSION

$\mathrm{RANGE}(f_{x,y,R}) = \mathrm{RANGE}(\varphi_R)$ * WEAK 2

IF $\varphi_x(y)\uparrow$

$\mathrm{RANGE}(f_{x,y,R}) = \emptyset \neq \mathrm{RANGE}(\varphi_R)$

$\mathrm{Dom}(f_{x,y,R}) = \emptyset \neq \mathrm{Dom}(\varphi_R)$

$\exists z \; f_{x,y,R}(z) \neq \varphi_R(z)$

AS $\exists z \; \varphi_R(z)\downarrow$

WEAK 1 USAGE

EMPTY = $\{ f \mid \forall x \; f(x)\uparrow \}$

EMPTY IS NON-TRIVIAL AS $\uparrow \in$ EMPTY & $C_0 \notin$ EMPTY

LET f, g BE ARB. INDICES $\ni$ dom(f) = dom(g)

$f \in$ EMPTY IFF dom$(f) = \emptyset$ IFF dom$(g) = \emptyset$ IFF $g \in$ EMPTY

THUS, EMPTY IS AN UNDEC. PROPERTY

FINITE = $\{ f \mid f(x)\downarrow$ ON ONLY A FINITE NUMBER OF INPUTS $x \}$

FINITE IS NON-TRIVIAL AS $\uparrow \in$ FINITE & $C_0 \notin$ FINITE

LET f, g BE ARB. INDICES $\ni$ dom(f) = dom(g)

$f \in$ FINITE IFF $|$dom$(f)|$ IS FINITE
IFF $|$dom$(g)|$ IS FINITE IFF $g \in$ FINITE

THUS, FINITE IS AN UNDEC. PROPERTY

TOTAL = $\{ f \mid \forall x \; f(x)\downarrow \}$

TOTAL IS NON-TRIVIAL AS $\uparrow \notin$ TOTAL & $C_0 \in$ TOTAL

LET f, g BE ARB. INDICES $\ni$ dom(f) = dom(g)

$f \in$ TOTAL IFF $|$dom$(f)| = \mathbb{N}$ IFF dom$(g) = \mathbb{N}$
IFF $g \in$ TOTAL

THUS, TOTAL IS AN UNDEC. PROPERTY

WEAK 2 USAGE

CAN REDO EMPTY BY SUBST. RANGE FOR DOMAIN

PRIMES $= \{ f \mid$ RANGE$(f) =$ PRIMES $\}$

PRIMES IS NON-TRIVIAL AS PRIME$(x) = p_x \in$ PRIMES & $C_0 \notin$ PRIMES

LET f, g BE ARB. INDICES $\ni$ RANGE$(f) =$ RANGE(g)

$f \in$ PRIMES IFF RANGE$(f) =$ PRIMES IFF $g \in$ PRIMES
IFF RANGE$(g) =$ PRIMES

THUS, PRIMES IS AN UNDEC. PROPERTY

NOTE: STRONG FORM REQUIRED FOR

ORDEREDPRIMES $= \{ f \mid \forall x\ f(x) = p_x \}$

ORDEREDPRIMES IS NON-TRIVIAL AS PRIME$(x) = p_x \in$ ORDEREDPRIMES
& $C_0 \notin$ ORDEREDPRIMES

LET f, g BE ARB. INDICES $\ni \forall x\ f(x) = g(x)$

$f \in$ ORDEREDPRIMES $\Leftrightarrow \forall x\ f(x) = p_x$
$\Leftrightarrow \forall x\ g(x) = p_x$ AS $\forall x\ f(x) = g(x)$
$\Leftrightarrow g \in$ ORDERED PRIMES

THUS, ORDEREDPRIMES IS AN UNDEC. PROPERTY

RICE EXAMPLES (WEAK)

HAS ZERO $= \{ f \mid \exists x\ f(x) = 0 \}$

$C_0 \in$ HASZERO ; $C_1 \notin$ HASZERO

SO NON-TRIVIAL

CAN USE VERSION II OF RICE

LET f AND g BE ARB. INDICES SUCH THAT

RNG(f) = RNG(g)

$f \in$ HASZERO IFF $\exists x\ f(x) = 0$

IFF $0 \in$ RNG(f)

IFF $0 \in$ RNG(g)

IFF $\exists x\ g(x) = 0$

IFF $g \in$ HAS ZERO

BY RICE'S HASZERO IS NON-RECURSIVE

TOTAL $= \{ f \mid \forall x\ f(x){\downarrow} \}$

$C_0 \in$ TOTAL $\mu y[y = y+1] \notin$ TOTAL

SO NON-TRIVIAL

CAN USE VERSION I OF RICE

LET f AND g BE ARB. INDICES SUCH THAT

DOM(f) = DOM(g)

$f \in$ TOTAL $\Leftrightarrow \forall x\ f(x){\downarrow} \Leftrightarrow$ DOM$(f) = \mathbb{N}$

$\Leftrightarrow$ DOM$(g) = \mathbb{N} \Leftrightarrow \forall x\ g(x){\downarrow} \Leftrightarrow g \in$ TOTAL

BY RICE'S TOTAL IS NON-RECURSIVE

RICE EXAMPLE (STRONG)

$\text{MONOINCR} = \{f \mid \forall x\ f(x+1) > f(x)\}$

$S(x) = x+1 \in \text{MONOINCR}$

$C_0(x) = 0 \notin \text{MONOINCR}$

SO NON-TRIVIAL

MUST USE VERSION 3 OF RICE

LET f AND g BE ARB. INDICES SUCH THAT

$\forall x\ f(x) = g(x)$

$f \in \text{MONOINCR}$ IFF $\forall x\ f(x+1) > f(x)$

IFF $\forall x\ g(x+1) > g(x)$

AS $\forall x\ g(x) = f(x)$

IFF $g \in \text{MONOINCR}$

COMPLEXITY OF A PROBLEM

FIND UPPER BOUND

FIND LOWER BOUND

IF POSSIBLE, ACHIEVE LOWER BOUND

TOOLS IN COMPUTABILITY

RICE'S: IS IT UNDEC.

DOES NOT GIVE UPPER BOUND ON
HOW BAD IT MIGHT BE IF UNDEC.
CAN BE RE, CO-RE, OR WORSE
BUT IT IS A FORM OF LOWER BOUND

REDUCIBILITY: IF $A \leq_B B$ THEN
B IS AT LEAST AS HARD AS A
AND A IS NO HARDER THAN B

QUANTIFICATION: GIVES UPPER BOUND BUT
CAN GIVE ONE THAT IS NOT
TIGHT

STP PREDICATE / VALUE FUNCTION

$STP(f, \overline{X}, t)$ IS TRUE IFF

$\Phi_f(\overline{X})$ CONVERGES IN AT MOST t STEPS

CAN SHOW PRF BY

$STP(f, \overline{X}, t) = CONFIG(f, \overline{X}, t) == CONFIG(f, \overline{X}, t+1)$

BASED ON FRS TO REC FCN REDUCTION

FROM WHICH WE SHOWED UNIVERSAL MACHINE

$VALUE(f, \overline{X}, t)$ IS $\Phi_f(\overline{X})$ WHENEVER

$STP(f, \overline{X}, t)$ IS TRUE. OTHERWISE

$VALUE(f, \overline{X}, t)$ IS MEANINGLESS, BUT DEFINED.

CAN SHOW PRF BY

$VALUE(f, \overline{X}, t) = EXP(CONFIG(f, \overline{X}, t), 0)$

$= y$ WHERE y IS EXPONENT OF PRIME 2 IN $CONFIG(f, \overline{X}, t)$

THIS IS BASED ALSO ON OUR UNIVERSAL MACHINE AND CONVENTION THAT FRS LEAVES ANSWER AS EXPONENT OF 0-TH PRIME (2).

NotEmpty

$$NE = \{ f \mid \exists x \varphi_f(x)\downarrow \}$$

$HALT \le_m NE$

LET $\langle f,x \rangle$ BE ARB. PAIR

$f(x)\downarrow$

$\langle f,x \rangle \in HALT \Leftrightarrow \varphi_f(x)\downarrow$

DEFINE $\forall y\, G_{f,x}(y) = \varphi_f(x)$

$\langle f,x \rangle \in HALT \Leftrightarrow \varphi_f(x)\downarrow \Leftrightarrow \forall y\, G_{f,x}(y)\downarrow \Rightarrow \exists y\, G_{f,x}(y)\downarrow \Rightarrow G_{f,x} \in NE$

$\langle f,x \rangle \notin HALT \Leftrightarrow \varphi_f(x)\uparrow \Leftrightarrow \forall y\, G_{f,x}(y)\uparrow \Rightarrow \nexists y\, G_{f,x}(y)\downarrow \Rightarrow G_{f,x} \notin NE$ AND SO

THUS, $\langle f,x \rangle \in HALT \Leftrightarrow G_{f,x} \in NE$

$HALT \le_m NE$ (NE UNDEC.)

$NE \le_m HALT$

LET f BE ARB $f \in NE \Leftrightarrow \exists x\, \varphi_f(x)\downarrow$

DEFINE $\forall y\, G_f(y) = \exists \langle x,t \rangle\, STP(f,x,t)$

COULD BE ANY VALUE FOR X

$f \in NE \Leftrightarrow \exists x\, \varphi_f(x)\downarrow \Leftrightarrow \exists \langle x,t \rangle\, STP(f,x,t)$

$\Leftrightarrow \forall y\, G_f(y)\downarrow \Rightarrow \langle G_f, 0 \rangle \in HALT$

$f \notin NE \Leftrightarrow \forall x\, \varphi_f(x)\uparrow \Leftrightarrow \forall \langle x,t \rangle\, \overline{STP(f,x,t)}$

$\Leftrightarrow \forall y\, G_f(y)\uparrow \Rightarrow \langle G_f, 0 \rangle \notin HALT$

THUS, $f \in NE \Leftrightarrow \langle G_f, 0 \rangle \in HALT$ AND SO

$NE \le_m HALT$ (NE IS RE)

COMBINING WE HAVE NE IS RE-COMPLETE

ALSO $NE \equiv_m HALT$

MORE ON NE

USING QUANTIFICATION OF A DECIDABLE PREDICATE

$NE = \{ f \mid \exists \langle x,t \rangle \; STP(f,x,t) \}$

EXISTENTIAL SO RE (SEMI-DEC.)

WHAT ABOUT RICE'S THEOREM?

NE IS NON-TRIVIAL AS $C_0 \in NE$, $\uparrow \notin NE$

LET f, g BE ARB. INDICES SUCH THAT $dom(f) = dom(g)$

$f \in NE \Leftrightarrow \exists x \, \varphi_f(x)\downarrow$ LET x_0 BE SUCH THAT $\varphi_f(x_0)\downarrow$

$\Leftrightarrow \varphi_g(x_0)\downarrow$ AS $\varphi_g(x_0) = \varphi_f(x_0)$

$\Leftrightarrow \exists x \, \varphi_g(x)\downarrow$

$\Leftrightarrow g \in NE$

BY WEAK #1 RICE'S THM., NE IS UNDEC.

RICE'S DOES NOT TELL US IF RE OR NOT

Constant

$$\text{Constant} = \{ f \mid \forall x\, \varphi_f(x)\downarrow \;\&\; \forall \langle x,y \rangle\, \varphi_f(x) = \varphi_f(y) \}$$

TOTAL $\leq_m$ CONSTANT

LET f BE AN ARBITRARY INDEX

$f \in \text{TOTAL} \Leftrightarrow \forall x\, \varphi_f(x)\downarrow$

DEFINE $\forall x\, G_f(x) = \varphi_f(x) - \varphi_f(x)$

$f \in \text{TOTAL} \Leftrightarrow \forall x\, \varphi_f(x)\downarrow \Leftrightarrow \forall x\, G_f(x) = 0 \Rightarrow G_f \in \text{CONSTANT}$

$f \notin \text{TOTAL} \Leftrightarrow \exists x\, \varphi_f(x)\uparrow \Leftrightarrow \exists x\, G_f(x)\uparrow \Rightarrow G_f \notin \text{CONSTANT}$

CONSTANT $\leq_m$ TOTAL

LET f BE AN ARBITRARY INDEX

$f \in \text{Constant} \Leftrightarrow \forall x\, \varphi_f(x)\downarrow \;\&\; \forall \langle x,y \rangle\, \varphi_f(x) = \varphi_f(y)$

DEFINE $\forall x\, G_f(x) = \mu y\, [\varphi_f(x) = \varphi_f(x+1)]$

$f \in \text{Constant} \Leftrightarrow \forall x\, G_f(x) = 0 \Rightarrow G_f \in \text{TOTAL}$

$f \notin \text{Constant} \Leftrightarrow \exists x\, G_f(x)\uparrow \Rightarrow G_f \notin \text{TOTAL}$

COMBINING WE HAVE CONSTANT IS NOT RE

ALSO CONSTANT $\equiv_m$ TOTAL

MORE ON CONSTANT

USING QUANTIFICATION OF A DECIDABLE PREDICATE

$$\text{CONSTANT} = \{ f \mid \forall \langle x,y \rangle \exists t\,[\text{STP}(f,x,t)\ \&\ \text{STP}(f,y,t)\ \&\ \text{VALUE}(f,x,t) == \text{VALUE}(f,y,t)] \}$$

ALTERNATING QUANTIFIERS SO MORE COMPLEX THAN RE (NOTE MANY WAYS TO DESCRIBE)

WHAT ABOUT RICE'S THEOREM?

CONSTANT IS NON-TRIVIAL AS $C_0 \in$ CONSTANT, $S \notin$ CONSTANT

LET f, g BE ARB. INDICES SUCH THAT

$\forall x\ \varphi_f(x) = \varphi_g(x)$

$f \in \text{CONSTANT} \Leftrightarrow \forall x\ \varphi_f(x){\downarrow}\ \&\ \forall \langle x,y \rangle\ \varphi_f(x) = \varphi_f(y)$

$\Leftrightarrow \forall x\ \varphi_g(x){\downarrow}\ \&\ \forall \langle x,y \rangle\ \varphi_g(x) = \varphi_g(y)$

AS $\forall x\ \varphi_g(x) = \varphi_f(x)$

$\Leftrightarrow g \in \text{CONSTANT}$

BY STRONG RICE'S THEOREM, CONSTANT IS UNDEC.

RICE'S DOES NOT TELL US HOW BAD

ALTERNATE DESCR. OF CONSTANT

$$\{f \mid \forall x \exists t [STP(f,x,t) \,\&\, STP(f,x+1,t) \,\&\, VALUE(f,x,t) == VALUE(f,x+1,t)]$$

$$\{f \mid \exists C \forall x \exists t [STP(f,x,t) \,\&\, VALUE(f,x,t) = C]$$

FIRST IS MINIMAL

SECOND IS NOT (3 ALTERNATING QUANTIFIERS)

OTHER CONSTANT

HERE WE COULD SAY IF $\varphi_f(x)\downarrow$ AND $\varphi_f(y)\downarrow$ THEN $\varphi_f(x) = \varphi_f(y)$

DESCRIBE AS

$$\{f \mid \forall\langle x,y,t\rangle\ [\,STP(f,x,t)\ \&\ STP(f,y,t) \Rightarrow VALUE(f,x,t) == VALUE(f,y,t)\,]\}$$

ONE UNIVERSAL QUANTIFIER

COMPLEMENT OF RE

CALLED CO-RE

THIS SET IS EQUIVALENT TO $\overline{\text{HALT}}$

$\overline{\text{HALT}} = \{\langle f,x\rangle \mid \varphi_f(x)\uparrow\}$

CONSTANT TIME

$$\{f \mid \exists C \forall x \ STP(f, x, C)\}$$

LOOKS TO HAVE 2 QUANTIFIERS BUT SECOND IS ACTUALLY BOUNDED ABOVE

$$\{f \mid \exists C \forall x_{|x| \leq C} \ STP(f, x, C)\}$$

AS CANNOT TRAVERSE X IN LESS THAN $|x|$ TIME, WHERE $|\ |$ COULD BE LOG OR LINEAR (TM WITH UNARY ALPHABET)

LOG IS TO SOME BASE DETERMINED BY REPRESENTATION

RE $\equiv$ SEMI DEC. (PART 1)

LET S BE RE THEN EITHER

$S = \emptyset$ OR $S = \mathrm{Rng}(f_S)$

WHERE f_S IS AN ALGORITHM

WE CAN BUILD g_S AS

$$g_S(x) = \mu y\,[y = y+1] \text{ IF } S = \emptyset$$

OR AS

$$g_S(x) = \mu y\,[f_S(y) = x]$$

OR

$$g_S(x) = \exists y\,[f_S(y) = x]$$

OR

$$g_S(x) = (\exists y\; f_S(y) = x) * x$$

IN ALL THREE CASES

$$\mathrm{Dom}(g_S) = S$$

IN THIRD CASE

$$\mathrm{Dom}(g_S) = \mathrm{Rng}(g_S) = S$$

THUS,

$$\text{RE} \Rightarrow \text{SEMI DEC.}$$

RE $\equiv$ SEMI-DEC. (PART 2)

LET S BE SEMI-DECIDABLE THEN

$S = \text{Dom}(g_S)$

WHERE g_S IS A PROCEDURE

WE CAN BUILD f_S IF WE ASSUME

$S \neq \emptyset$

WE CAN DO SO SINCE IF $S = \emptyset$ THEN IT IS RE BY DEFINITION

LET $a \in S$ BE AN ARB. ELEMENT OF S

$$f_S(\langle x, t \rangle) = \text{STP}(g_S, x, t) * x + (1 - \text{STP}(g_S, x, t)) * a$$

f_S IS AN ALGORITHM

$\text{RNG}(f_S) = \text{Dom}(g_S)$

NOTE: f_S ENUMERATES EACH ELEMENT OF S INFINITELY OFTEN

RESULT IS

SEMI-DEC $\Rightarrow$ RE

NOTE: f_S IS <u>PRIMITIVE</u> REC.

REC & RE

LET S BE A RECURSIVE (DECIDABLE) SET

S HAS AN ASSOCIATED ALGORITHMIC PREDICATE

χ_S, CALLED ITS CHARACTERISTIC FUNCTION SUCH THAT

$$x \in S \iff \chi_S(x)$$

THE COMPLEMENT OF S, $\bar{S}$, IS ALSO RECURSIVE AND

HAS ITS OWN CHARACTERISTIC FUNCTION, $\chi_{\bar{S}}$

$$x \in \bar{S} \iff \chi_{\bar{S}}(x) \iff \chi_S(x) = 0$$

S REC $\Rightarrow$ S IS RE && $\bar{S}$ IS RE

CAN DEFINE g_S & $g_{\bar{S}}$ WHERE $S = Dom(g_S)$, $\bar{S} = Dom(g_{\bar{S}})$

BY

$g_S(x) = \mu y\, \chi_S(x)$ - DIVERGE IFF $\overline{\chi_S(x)}$

$g_{\bar{S}}(x) = \mu y\, \chi_{\bar{S}}(x)$ - DIVERGE IFF $\overline{\chi_{\bar{S}}(x)}$ IFF $\chi_S(x)$

S REC $\Leftrightarrow$ S & $\bar{S}$ ARE BOTH RE

ON PREVIOUS PAGE WE SHOWED

S REC $\Rightarrow$ S + $\bar{S}$ ARE BOTH RE

NEED

S RE & $\bar{S}$ RE $\Rightarrow$ S REC

S RE $\Leftrightarrow$ (i) $S = \emptyset$ OR (ii) $S = \mathrm{RNG}(f_S)$, f_S ALG.

ALSO S RE $\Leftrightarrow$ $S = \mathrm{Dom}(g_S)$, g_S A PROCEDURE

WE WILL USE $S = \mathrm{Dom}(g_S)$, g_S A PROC.

AS $\bar{S}$ IS RE $\bar{S} = \mathrm{Dom}(g_{\bar{S}})$, $g_{\bar{S}}$ A PROC.

$$\chi_S(x) = \mathrm{STP}(g_S, x, \mu t[\mathrm{STP}(g_S, x, t) \,\|\, \mathrm{STP}(g_{\bar{S}}, x, t)])$$

COULD DO OTHER WAY

IF $S = \emptyset$ THEN $\forall x\, \chi_S(x) = 0$ } REC

IF $S = \mathbb{N}$ THEN $\forall x\, \chi_S(x) = 1$

ASSUME $S \neq \emptyset$ AND $S \neq \mathbb{N}$ ($\bar{S} \neq \emptyset$)

AND $S = \mathrm{RNG}(f_S)$ $\bar{S} = \mathrm{RNG}(f_{\bar{S}})$

THEN

$$\chi_S(x) = f_S(\mu y[f_S(y) == x \,\|\, f_{\bar{S}}(y) == x]) == x$$

RE CHARACTERIZATION

$S \neq \emptyset$ IS RE IFF ANY OF BELOW

(a) $S = \mathrm{RNG}(f)$ f PRF (PRIM. REC.)
(b) $S = \mathrm{RNG}(f)$ f TOTAL RECURSIVE (ALG.)
(c) $S = \mathrm{RNG}(f)$ f PARTIAL REC (PROCEDURE)
(d) $S = \mathrm{DOM}(f)$ f PARTIAL REC.