Assignment # 8.1 Sample Key

1. Use reduction from Halt to show that one cannot decide REPEATS, where
   REPEATS = $\{ f \mid \text{for some } x \text{ and } y,\ x \neq y,\ \varphi_f(x)\downarrow,\ \varphi_f(y)\downarrow \text{ and } \varphi_f(x) == \varphi_f(y) \}$

Let f,x be an arbitrary pair of natural numbers. <f,x> is in Halt iff $\varphi_f(x)\downarrow$

Define g by $\forall y\ \varphi_g(y) = \varphi_f(x)$.

Clearly, $\forall y\ \varphi_g(y) = \varphi_f(x)$, and so, if $\varphi_f(x)\downarrow$ then $\forall y\ \varphi_g(y)\downarrow$ and is the constant $\varphi_f(x)$;

else if $\varphi_f(x)\uparrow$ then $\forall y\ \varphi_g(y)\uparrow$.

Formally,

$\langle f,x\rangle \in$ Halt iff $\forall y\ \varphi_g(y)\downarrow$ and is the constant $\varphi_f(x)$, which implies g $\in$ REPEATS

$\langle f,x\rangle \notin$ Halt iff $\forall y\ \varphi_g(y)\uparrow$, which implies g $\notin$ REPEATS

Halt $\leq_m$ REPEATS as we were to show.

Note: I have not overloaded the index of a function with the function in my proof, but I do not mind if you do such overloading.

Assignment # 8.2 Sample Key

2. Show that REPEATS reduces to Halt. (1 plus 2 show they are equally hard)

Let f be an arbitrary natural number. f is in REPEATS iff for some x and y, $x \neq y$, $\varphi_f(x)\downarrow$, $\varphi_f(y)\downarrow$ and $\varphi_f(x) == \varphi_f(y)$

Define g by $\forall z\ \varphi_g(z) = \exists\langle x,y,t\rangle$ [STP(f,x,t) & STP(f,y,t) & ($x\neq y$) & (VALUE(f,x,t) = (VALUE(f,y,t))].

f $\in$ Repeats iff $\exists x,y$, $x\neq y$, such that $\varphi_f(x)\downarrow$ and $\varphi_f(y)\downarrow$ and $\varphi_f(x) = \varphi_f(y)$ iff $\forall z\ \varphi_g(z) = 1$ which implies g is an algorithm and so <g,0> $\in$ Halt (note: 0 is just chosen randomly)

f $\notin$ Repeats iff $\sim\exists x,y$, $x\neq y$, such that $\varphi_f(x)\downarrow$ and $\varphi_f(y)\downarrow$ and $\varphi_f(x) = \varphi_f(y)$ iff $\forall z\ \varphi_g(z)\uparrow$ which implies <g,0> $\notin$ Halt .

Summarizing, f is in REPEATS iff <g,0> is in Halt and so

REPEATS $\leq_m$ Halt as we were to show.

Assignment # 8.3 Sample Key

3. Use Reduction from Total to show that DOUBLES is not even re, where DOUBLES = { f | for all x, $\varphi_f(x)\downarrow$, $\varphi_f(x+1)\downarrow$ and $\varphi_f(x+1)=2*\varphi_f(x)$ }

Let f be an arbitrary natural number. f is in Total iff $\forall x\ \varphi_f(x)\downarrow$

Define g by $\forall x\ \varphi_g(x) = \varphi_f(x) - \varphi_f(x)$, for all x.

f $\in$ Total iff $\forall x\ \varphi_f(x)\downarrow$ iff $\forall x\ \varphi_g(x) = 0$ which implies $\forall x\ \varphi_g(x+1) = 2*\varphi_g(x) = 0$ which implies g $\in$ DOUBLES.

f $\notin$ Total iff $\exists x\ \varphi_f(x)\downarrow$ iff $\exists x\ \varphi_g(x)\uparrow$ implies g $\notin$ DOUBLES.

Summarizing, f is in Total iff g is in DOUBLES and so

TOTAL $\leq_m$ DOUBLES as we were to show.

Assignment # 8.3 Alternate Key

3. Use Reduction from Total to show that DOUBLES is not even re, where
   DOUBLES = { f | for all x, $\varphi_f(x)\downarrow$, $\varphi_f(x+1)\downarrow$ and $\varphi_f(x+1)=2*\varphi_f(x)$ }

Let f be an arbitrary natural number. f is in Total iff $\forall x\ \varphi_f(x)\downarrow$

Define g by $\varphi_g(x) = \varphi_f(x) - \varphi_f(x)+2\text{^}x$ for all x.

Clearly, $\varphi_g(x) = 2\text{^}x$, and so $\varphi_g(x+1) = 2*\varphi_g(x) = 2\text{^}(x+1)$ for all x, iff $\forall x\ \varphi_f(x)\downarrow$; otherwise $\varphi_g(x)\uparrow$ for some x.

Summarizing, f is in Total iff g is in DOUBLES and so

TOTAL $\leq_m$ DOUBLES as we were to show.

Assignment # 8.4 Sample Key

4. Show DOUBLES reduces to Total. (3 plus 4 show they are equally hard)

Let f be an arbitrary natural number. f is in DOUBLES iff $\forall x\ \varphi_f(x)\downarrow$, $\varphi_f(x+1)\downarrow$ and $\varphi_f(x+1)=2*\varphi_f(x)$.

Define g by $\forall x\ \varphi_g(x) = \mu y[\varphi_f(x+1) = 2*\varphi_f(x)]$.

$f \in$ DOUBLES iff $\forall x\ \varphi_f(x)\downarrow$, $\varphi_f(x+1)\downarrow$ and $\varphi_f(x+1)=2*\varphi_f(x)$ iff $\forall x\ \varphi_g(x)\downarrow$ iff $g \in$ TOTAL.

Summarizing, f is in DOUBLES iff g is in Total and so

DOUBLES $\leq_m$ TOTAL as we were to show.

Assignment # 8.5 Sample Key

5. Use Rice's Theorem to show that REPEATS is undecidable

First, REPEATS is non-trivial as $C0(x) = 0$ is in REPEATS and $S(x) = x+1$ is not.

Second, REPEATS is an I/O property.

To see this, let f and g are two arbitrary indices such that
$\forall x\ [\varphi_f(x) = \varphi_g(x)]$

$f \in$ REPEATS iff $\exists$ y,z, $y \neq z$, such that $\varphi_f(y)\downarrow$, $\varphi_f(z)\downarrow$ and $\varphi_f(y) = \varphi_f(z)$
iff, since $\forall x\ [\varphi_f(x) = \forall x\ \varphi_g(x)]$, $\exists$ y,z, $y \neq z$, (same y,z as above) such that $\varphi_g(y)\downarrow$, $\varphi_g(z)\downarrow$ and $\varphi_g(y) = \varphi_g(z)$ iff $g \in$ REPEATS.

Thus, $f \in$ REPEATS iff $g \in$ REPEATS.

Assignment # 8.6 Sample Key

6. Use Rice's Theorem to show that DOUBLES is undecidable

First, DOUBLES is non-trivial as C0(x) = 0 (2*0 = 0) is in DOUBLES and S(x) = x+1 is not.

Second, DOUBLES is an I/O property.

To see this, let f and g are two arbitrary indices such that $\forall x\ [\varphi_f(x) = \varphi_g(x)]$.

f ∈ DOUBLES iff for all x, $\varphi_f(x)\downarrow$, $\varphi_f(x+1)\downarrow$ and $\varphi_f(x+1)=2*\varphi_f(x)$ iff, since $\forall x\ [\varphi_f(x) = \varphi_g(x)]$, for all x, $\varphi_g(x)\downarrow$, $\varphi_g(x+1)\downarrow$ and $\varphi_g(x+1)=2*\varphi_g(x)$ iff g ∈ DOUBLES.

Thus, **f ∈ DOUBLES iff g ∈ DOUBLES**.