Assignment # 7.1a Sample Key

1. For the following languages, either provide a grammar to show it is a CFL or employ the Pumping Lemma to show it is not

   a.) **L = { $a^i b^j$ | j > 2*I }**

   This language is a CFL. A grammar that works is

   $S \rightarrow aSbb \mid Sb \mid b$

Assignment # 7.1b Sample Key

1. b.) **L = { $a^n b^{n!}$ | n>0 }**

   PL: Provides N>0

   We: Choose $a^N b^{N!} \in L$

   PL: Splits $a^N b^{N!}$ into uvwxy, $|vwx| \le N$, $|vx| > 0$, such that $\forall i \ge 0\ uv^iwx^iy \in L$

   We: Choose i=2

   Case 1: vx contains only b's, then we are increasing the number of b's while leaving the number of a's unchanged. In this case uv^2wx^2y is of form $a^N b^{N!+c}$, c>0 and this is not in L.

   Case 2: vx contains some a's and maybe some b's. Under this circumstances uv^2wx^2y has at least N+1 a's and at most N!+N-1 b's. But (N+1)! = N!(N+1) = $N!*N+N \ge N! + N > N!+N-1$ and so is not in L.

   Cases 1 and 2 cover all possible situations, so L is not a CFL

Assignment # 7.2 Sample Key

2. Consider the context-free grammar **G = ({ S } , { a , b } , S , P)**, where **P** is:

   $\mathbf{S} \rightarrow \mathbf{S\,a\,S\,b\,S} \mid \mathbf{S\,b\,S\,a\,S} \mid \mathbf{S\,a\,S\,a\,S} \mid \mathbf{a} \mid \lambda$

   Provide the first part of the proof that

   L (G) = L = { w | w has at least as many **a's** as **b's }**

   That is, show that $\mathbf{L(G)} \subseteq \mathbf{L}$

   To attack this problem we can first introduce the notation that, for a syntactic form α, α_a = the number of **a's** in α, and α_b = the number of **b's** in α. Using this, we show that if $\mathbf{S} \Rightarrow^* \alpha$, then $\alpha_b \leq \alpha_a$ and hence that $\mathbf{L(G)} \subseteq \mathbf{L}$:

   A straightforward approach is to show, inductively on the number of steps, **i**, in a derivation, that, if $\mathbf{S} \Rightarrow^{i} \alpha$, then $\alpha_b \leq \alpha_a$.

Assignment # 7.2 Sample Key

Basis (i=1): Since S $\Rightarrow\alpha$ iff S $\rightarrow$ α and all rhs of S have $\alpha_b \leq \alpha_a$ then the base case holds

IH: Assume if S $\Rightarrow_m\alpha$, then $\alpha_b \leq \alpha_a$, whenever $m \leq n$

IS: Show that if S $\Rightarrow_{n+1}\alpha$, then $\alpha_b \leq \alpha_a$

If S α then S $\Rightarrow_n$ β and $\beta\Rightarrow\alpha$

Since G has only one non-terminal S, the rewriting of β to α involves a single application of one of the S-rules. By the I.H., β has the property that $\beta_b \leq \beta_a$. Since a single application of an S rule either adds no b's or a's, one a, one a and one b, or two b's, we have the three following cases:

Assignment # 7.2 Sample key

Case 0: $\alpha_a = \beta_a$, and $\alpha_b = \beta_b$

In which case, using the IH, we have:

$\beta_b \leq \beta_a \rightarrow \alpha_b \leq \alpha_a$

Case 1: $\alpha_b = \beta_b$, and $\alpha_a = \beta_a + 1$

In which case, using the IH, we have:

$\beta_b \leq \beta_a \rightarrow \alpha_b \leq \alpha_a$

Case 2: $\alpha_b = \beta_b + 1$, and $\alpha_a = \beta_a + 1$

In which case, using the IH, we have:

$\beta_b \leq \beta_a \rightarrow \alpha_b \leq \alpha_a$

Case 3: $\alpha_b = \beta_b$, and $\alpha_a = \beta_a + 2$

In which case, using the IH, we have:

$\beta_b \leq \beta_a \rightarrow \alpha_b \leq \alpha_a$