Assignment # 8.1 Key

1. Use reduction from Halt to show that one cannot decide HasOdd, where
   HasOdd = { f | range(f) contains an odd number }

Let f,x be an arbitrary pair of natural numbers. <f,x> is in Halt iff $\varphi_f(x)\downarrow$

Define g by $\varphi_g(y) = \varphi_f(x) - \varphi_f(x) + 1$, for all y.

Clearly, $\varphi_g(y) = 1$, for all y, iff $\varphi_f(x)\downarrow$, and $\varphi_g(y)\uparrow$, for all y, otherwise.

Formally,

<f,x> $\in$ Halt iff $\forall y\ \varphi_g(y) = 1$, which implies g $\in$ HasOdd

<f,x> $\notin$ Halt iff $\forall y\ \varphi_g(y)\uparrow$, which implies g $\notin$ HasOdd

Halt $\leq_m$ HasOdd as we were to show.

Note: I have not overloaded the index of a function with the function in my proof, but I do not mind if you do such overloading.

Assignment # 8.2 Key

2. Show that HasOdd reduces to Halt. (1 plus 2 show they are equally hard)

Let f be an arbitrary natural number. f is in HasOdd iff for some x, $\varphi_f(x)\downarrow$ and $\varphi_f(x)$ is an odd number.

Define g by $\varphi_g(z) = \exists$<x,y,t> [STP(f,x,t) & (VALUE(f,x,t)=2y+1)], for all z.

Clearly, $\varphi_g(z) = 1$, for all z, iff for some x, $\varphi_f(x)\downarrow$ and $\varphi_f(x)$ is an odd number (form 2y+1), and $\varphi_g(z)\uparrow$, for all z, otherwise.

Formally,

f $\in$ HadOdd iff $\exists$x such that $\varphi_f(x)\downarrow$ and is an odd number iff $\forall$z $\varphi_g(z) = 1$ which implies g is an algorithm and so <g,0> $\in$ Halt (note: 0 is just chosen randomly)

f $\notin$ HasOdd iff $\forall$x [$\varphi_f(x)\downarrow$ implies $\varphi_f(x)$ is not an odd number] iff $\forall$z $\varphi_g(z)\uparrow$ which implies <g,0> $\notin$ Halt .

Summarizing, f is in HasOdd iff <g,0> is in Halt and so

HasOdd $\leq_m$ Halt as we were to show.

Assignment # 8.3 Key

3. Use Reduction from Total to show that IsAllOdds is not even re, where
IsAllOdds = { f | range(f) = Set of all odd natural numbers }

Let f be an arbitrary natural number. f is in Total iff $\forall x\ \varphi_f(x)\downarrow$

Define g by $\varphi_g(x) = \varphi_f(x) - \varphi_f(x) + 2x+1$, for all x.

Clearly, $\varphi_g(x) = 2x+1$, for all x, iff $\forall x\ \varphi_f(x)\downarrow$ and so range(g) is all odd numbers, i.e., $g \in$ IsAllOdds; otherwise $\varphi_g(x)\uparrow$ for some x and IsAllOdd and so, for that x, 2x+1 is not in range(g) and thus $g \notin$ IsAllOdds.

Formally,

f $\in$ Total iff $\forall x\ \varphi_f(x)\downarrow$ iff $\forall x\ \varphi_g(x) = 2x+1$ which implies g $\in$ isAllOdds.

f $\notin$ Total iff $\exists x\ \varphi_f(x)\uparrow$ iff $\exists x\ \varphi_g(x)\uparrow$ implies g $\notin$ IsAllOdds.

Summarizing, f is in Total iff g is in IsAllOdds and so

TOTAL $\leq_m$ IsAllOdds as we were to show.

Assignment # 8.4 Key

4. Show IsAllOdds reduces to Total. (3 plus 4 show they are equally hard)

Let f be an arbitrary natural number. f is in IsAllOdds iff $\forall x$ there is a y such that $\varphi_f(y)\downarrow$, $\varphi_f(y)=2x+1$.

Define g by $\varphi_g(x) = \exists\langle y,t\rangle[STP(f,y,t)\ \&\ VALUE(f,y,t)= 2x+1]$, for all x.

Clearly, $\varphi_g(x)\downarrow$ iff $\exists y\ [\varphi_f(y)\downarrow\ \&\ \varphi_f(y)=2x+1]$ and, thus, $\forall x \varphi_g(x)\downarrow$ iff $\forall x$ 2x+1 is in range(f) iff $f \in$ IsAllOdds; otherwise $\varphi_g(x)\uparrow$ for some x and so $g \notin$ Total.

Summarizing, f is in IsAllOdds iff g is in Total and so

IsAllOdds $\leq_m$ TOTAL as we were to show.

Assignment # 8.5 Key

5. Use Rice's Theorem to show that HasOdd is undecidable

First, HasOdd is non-trivial as C1(x) = 1 is in HasOdd and C0(x) = 0 is not.

Second, HasOdd is a property of the range of effective procedures.

**To see this, let f and g are two arbitrary indices such that
∀x [Range(f) = Range(g)]**

f ∈ HasOdd iff 2y+1 ∈ Range(f), for some natural number y, iff, since Range(f) = Range(g), 2y+1 ∈ Range(g), for some natural number y, iff g ∈ HasOdd.

Thus, f ∈ HasOdd iff g ∈ HasOdd.

Thus, by Rice's Weak#2 Theorem, HasOdd is undecidable.

Assignment # 8.6 Key

6. Use Rice's Theorem to show that IsAllOdds is undecidable

First, IsAllOdds is non-trivial as C1(x) = 2x+1 is in IsAllOdds and C0(x) = 0 is not.

Second, IsAllOdds is a property of the range of effective procedures.

To see this, let f and g are two arbitrary indices such that
∀x [Range(f) = Range(g)]

f ∈ IsAllOdds iff 2y+1 ∈ Range(f), for all natural numbers y and Range(f) contains no even numbers, iff, since Range(f) = Range(g), 2y+1 ∈ Range(g), for all natural numbers y and Range(g) contains no even number, iff g ∈ IsAllOdds.

Thus, f ∈ IsAllOdds iff g ∈ IsAllOdds.

Thus, by Rice's Weak#2 Theorem, IsAllOdds is undecidable.