Propositional Logic

Propositions

- A *proposition* is a declarative sentence that is either true or false.
- Constructing Propositions
  - Propositional Variables: *p*, *q*, *r*, *s*, …
  - The proposition that is always true is denoted by **T** and the proposition that is always false is denoted by **F**.
  - Compound Propositions; constructed from logical connectives and other propositions
    - Negation $\neg$
    - Conjunction $\wedge$
    - Disjunction $\vee$
    - Implication $\rightarrow$
    - Biconditional $\leftrightarrow$

Truth Table

- These operators are defined by their *truth tables*, which specify the truth value when propositions are combined by these operators.

- The *negation* of a proposition p is denoted by $\neg p$ and has this truth table:

p	$\neg p$
T	F
F	T

Truth Tables

Consider propositions *p* and *q*. The truth table for p∧q, p∨q, p→q, p↔q

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Truth Table

- Construct a truth table for $p \vee q \rightarrow \neg r$

p	q	r	$\neg r$	$p \vee q$	$p \vee q \rightarrow \neg r$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Equivalent Propositions

- Two propositions are **e***quivalent* if they always have the same truth value.
- **Example**: Show using a truth table that the conditional is equivalent to the contrapositive.

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Tautologies, Contradictions, and Contingencies

- A tautology is a proposition which is always true.
  - Example: $p \vee \neg p$
- A *contradiction* is a proposition which is always false.
  - Example: $p \wedge \neg p$
- A *contingency* is a proposition which is neither a tautology nor a contradiction, such as *p*

P	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Logically Equivalent

- Two compound propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.
- We write this as $p \Leftrightarrow q$ or as $p \equiv q$ where p and q are compound propositions.
- Two compound propositions p and q are equivalent if and only if the columns in a truth table giving their truth values agree.
- This truth table show $\neg p \vee q$ is equivalent to $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Logical Equivalences

1) $\neg(p \wedge q) \equiv \neg p \vee \neg q, \ \ \neg(p \vee q) \equiv \neg p \wedge \neg q$ (De Morgan's Laws)
2) $p \wedge T \equiv p, \quad p \vee F \equiv p$ (Identity Laws)
3) $p \vee T \equiv T, \quad p \wedge F \equiv F$ (Domination Laws)
4) $p \vee p \equiv p, \ p \wedge p \equiv p$ (Idempotent laws)
5) $\neg(\neg p) \equiv p$ (Double Negation Law)
6) $p \vee \neg p \equiv T, \ p \wedge \neg p \equiv F$ (Negation Laws)
7) $p \vee q \equiv q \vee p, \quad p \wedge q \equiv q \wedge p$ (Commutative Laws)
8) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ (Associative Laws)
$(p \vee q) \vee r \equiv p \vee (q \vee r)$
9) $(p \vee (q \wedge r) \equiv (p \vee q)) \wedge (p \vee r)$ (Distributive Laws)
$(p \wedge (q \vee r)) \equiv (p \wedge q) \vee (p \wedge r)$
10) $p \vee (p \wedge q) \equiv p, \ p \wedge (p \vee q) \equiv p$ (Absorption Laws)

Logical Equivalences

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \lor q$
$p \to q \equiv \neg q \to \neg p$
$p \lor q \equiv \neg p \to q$
$p \land q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \land \neg q$
$(p \to q) \land (p \to r) \equiv p \to (q \land r)$
$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$
$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$
$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Equivalence Proofs

Example: Show that $\neg(p \vee (\neg p \wedge q))$

is logically equivalent to $\neg p \wedge \neg q$

Solution:

$\neg(p \vee (\neg p \wedge q))$	$\equiv$	$\neg p \wedge \neg(\neg p \wedge q)$	by the second De Morgan law
	$\equiv$	$\neg p \wedge [\neg(\neg p) \vee \neg q]$	by the first De Morgan law
	$\equiv$	$\neg p \wedge (p \vee \neg q)$	by the double negation law
	$\equiv$	$(\neg p \wedge p) \vee (\neg p \wedge \neg q)$	by the second distributive law
	$\equiv$	$F \vee (\neg p \wedge \neg q)$	because $\neg p \wedge p \equiv F$
	$\equiv$	$(\neg p \wedge \neg q) \vee F$	by the commutative law for disjunction
	$\equiv$	$(\neg p \wedge \neg q)$	by the identity law for **F**

Equivalence Proofs

Example: Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution:

$$
\begin{aligned}
(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{by truth table for } \rightarrow \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{by the first De Morgan law} \\
&\equiv (\neg p \vee p) \vee (\neg p \vee \neg q) && \text{by associative and} \\
& && \text{commutative laws} \\
& && \text{laws for disjunction} \\
&\equiv T \vee T && \text{by truth tables} \\
&\equiv T && \text{by the domination law}
\end{aligned}
$$

Predicate Logic

- If we have:

  “All men are mortal.”

  “Socrates is a man.”

  Does it follow that “Socrates is mortal?”

- To draw inferences: Need a language that talks about objects, their properties, and their relations.

Predicate Logic

- Propositional Functions $P(x)$:
  - Propositional functions become propositions (and have truth values) when their variables are each replaced by a value from the *domain* (or *bound* by a quantifier).
  - The statement $P(x)$ is said to be the value of the propositional function P at x.
- Quantifiers:
  - *Universal Quantifier* $\forall$**:** $\forall x\, P(x)$ asserts $P(x)$ is true for <u>every</u> x in the *domain*.
  - *Existential Quantifier* $\exists$**:** $\exists x\, P(x)$ asserts $P(x)$ is true for <u>some</u> x in the *domain*.
- The quantifiers are said to bind the variable x in these expressions.

Properties of Quantifiers

- The truth value of $\exists x\, P(x)$ and $\forall x\, P(x)$ depend on both the propositional function $P(x)$ and on the domain U.
- **Examples:**
  1. If U is the positive integers and $P(x)$ is the statement "$x < 2$", then $\exists x\, P(x)$ is true, but $\forall x\, P(x)$ is false.
  2. If U is the negative integers and $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true.
  3. If U consists of 3, 4, and 5, and $P(x)$ is the statement "$x > 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true. But if $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are false.

Precedence of Quantifiers

- The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.

- For example, $\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$
- $\forall x\, (P(x) \vee Q(x))$ means something different.

- Unfortunately, often people write $\forall x\, P(x) \vee Q(x)$ when they mean $\forall x\, (P(x) \vee Q(x))$.

Translating to Predicate Logic

Example 1: "Every student in this class has taken a course in Java."

Solution 1: Let *U* be all students in this class, J(x) denote "x has taken a course in Java": $\forall x\, J(x)$.

Solution 2: If *U* is all people, S(x) denotes "x is a student in this class": $\forall x\, (S(x) \rightarrow J(x))$. $\forall x\, (S(x) \wedge J(x))$ is not correct.

Example 2: "Some student in this class has taken a course in Java."

Solution 1: If *U* is all students in this class: $\exists x\, J(x)$

Solution 2: But if *U* is all people: $\exists x\, (S(x) \wedge J(x))$. $\exists x\, (S(x) \rightarrow J(x))$ is not correct.

Equivalences in Predicate Logic

- Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value
  - for every predicate substituted into these statements and
  - for every domain of discourse used for the variables in the expressions.
- The notation $S \equiv T$ indicates that S and T are logically equivalent.
- **Example:** $\forall x \neg\neg S(x) \equiv \forall x\, S(x)$

De Morgan's Laws for Quantifiers

- The rules for negating quantifiers are:

TABLE 2 **De Morgan's Laws for Quantifiers.**

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Order of Nested Quantifiers

Examples:

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers. Then $\forall x \forall y P(x,y)$ and $\forall y \forall x P(x,y)$ have the same truth value.

2. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that U is the real numbers. Then $\forall x \exists y P(x,y)$ is true, but $\exists y \forall x P(x,y)$ is false.

Quantifications of Two Variables

Statement	When True?	When False
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	*P*(*x*,*y*) is true for every pair *x*,*y*.	There is a pair *x, y* for which *P*(*x,y*) is false.
$\forall x \exists y P(x,y)$	For every *x* there is a *y* for which *P*(*x,y*) is true.	There is an x such that *P*(*x,y*) is false for every *y*.
$\exists x \forall y P(x,y)$	There is an *x* for which *P*(*x,y*) is true for every *y*.	For every *x* there is a y for which *P*(x,y) is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair *x, y* for which *P*(*x,y*) is true.	*P*(x,y) is false for every pair *x,y*

Translation from English

Choose the obvious predicates and express in predicate logic.

Example 1: "Brothers are siblings."

Solution: $\forall x \, \forall y \, (B(x,y) \rightarrow S(x,y))$

Example 2: "Siblinghood is symmetric."

Solution: $\forall x \, \forall y \, (S(x,y) \rightarrow S(y,x))$

Example 3: "Everybody loves somebody."

Solution: $\forall x \, \exists y \, L(x,y)$

Example 4: "There is someone who is loved by everyone."

Solution: $\exists y \, \forall x \, L(x,y)$

Example 5: "There is someone who loves someone."

Solution: $\exists x \, \exists y \, L(x,y)$

Example 6: "Everyone loves himself"

Solution: $\forall x \, L(x,x)$

Rules of Inference for Propositional Logic

Modus Ponens

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:
$(p \land (p \rightarrow q)) \rightarrow q$

Modus Tollens

$$\begin{array}{l} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Corresponding Tautology:
$(\neg p \land (p \rightarrow q)) \rightarrow \neg q$

Hypothetical Syllogism

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Corresponding Tautology:
$((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)$

Disjunctive Syllogism

$$\begin{array}{l} p \lor q \\ \neg p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:
$(\neg p \land (p \lor q)) \rightarrow q$

Rules of Inference for Propositional Logic

Rule	Inference	Corresponding Tautology
Addition	$\dfrac{p}{\therefore p \vee q}$	**Corresponding Tautology:** $p \rightarrow (p \vee q)$
Simplification	$\dfrac{p \wedge q}{\therefore q}$	**Corresponding Tautology:** $(p \wedge q) \rightarrow q$
Conjunction	$\dfrac{\begin{array}{l} p \\ q \end{array}}{\therefore p \wedge q}$	**Corresponding Tautology:** $((p) \wedge (q)) \rightarrow (p \wedge q)$
Resolution	$\dfrac{\begin{array}{l} \neg p \vee r \\ p \vee q \end{array}}{\therefore q \vee r}$	**Corresponding Tautology:** $((\neg p \vee r) \wedge (p \vee q)) \rightarrow (q \vee r)$

Rules of Inference for Quantified Statements

- Universal Instantiation (UI) $$\frac{\forall x P(x)}{\therefore P(c)}$$
- Universal Generalization (UG) $$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$$
- Existential Instantiation (EI) $$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$$
- Existential Generalization (EG) $$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$$

Using Rules of Inference

A *valid argument* is a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference. The last statement is called conclusion.

Example 1: From the single proposition

$$p \land (p \to q)$$

Show that q is a conclusion.

Solution:

Step	**Reason**
1. $p \land (p \to q)$	Premise
2. p	Conjunction using (1)
3. $p \to q$	Conjunction using (1)
4. q	Modus Ponens using (2) and (3)

Proof Methods

- Direct Proofs
- Indirect Proofs
  - Proof of the Contrapositive
  - Proof by Contradiction

Direct Proof

- Proving Conditional Statements: $p \rightarrow q$

Direct Proof: Assume that p is true. Use rules of inference, axioms, and logical equivalences to show that q must also be true.

Example: Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

Solution: Assume that n is odd. Then $n = 2k + 1$ for an integer k. Squaring both sides of the equation, we get:

$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2r + 1,$

where $r = 2k^2 + 2k$, an integer.

We have proved that if n is an odd integer, then n^2 is an odd integer.

Proof by Contraposition

Example: Prove that for an integer n, if n^2 is odd, then n is odd.

Solution: Use proof by contraposition. Assume n is even (i.e., not odd). Therefore, there exists an integer k such that $n = 2k$. Hence,

$$n^2 = 4k^2 = 2(2k^2)$$

and n^2 is even (i.e., not odd).

We have shown that if n is an even integer, then n^2 is even. Therefore by contraposition, for an integer n, if n^2 is odd, then n is odd.

Proof by Contradiction

To prove q, assume $\neg q$ and derive a contradiction such as $q \wedge \neg q$. Since we have shown that $\neg q \rightarrow \mathrm{F}$ is true, it follows that the contrapositive $\mathbf{T} \rightarrow q$ also holds.

Example: Use a proof by contradiction to give a proof that √2 is irrational.

Solution: Suppose √2 is rational. Then there exists integers *a* and *b* with √2 = *a/b*, where $b \neq 0$ and *a* and *b* have no common factors. Then

$$2 = \frac{a^2}{b^2} \qquad 2b^2 = a^2$$

Therefore a^2 must be even. If a^2 is even then *a* must be even (an exercise). Since *a* is even, $a = 2c$ for some integer *c*. Thus,

$$2b^2 = 4c^2 \qquad b^2 = 2c^2$$

Therefore b^2 is even. Again then *b* must be even as well. But then 2 must divide both *a* and *b*. This contradicts our assumption that *a* and *b* have no common factors. We have proved by contradiction that our initial assumption must be false and therefore √2 is irrational .

Disproof by Counter-Example

Example: prove that every prime number *p* of the form 4 *n* + 1 can be written as a sum of two squares. Thus, 5 = 12 + 22 , 13 = 22 + 32 , etc.

Demonstrating the truth of such a statement by a few examples doesn't constitute a proof, because we can not enumerate infinitely many scenarios. However, to *disprove* a statement, a single example that contradicts to the statement suffices. Such an example is called a *counter example*.

- **Conjecture.** If *n* is a positive integer, then *n*2 – *n* + 41 is a prime number.

  Disproof: Let *n* = 41. Then the expression *n*2 – *n* + 41 = 412 – 41 + 41 = 412 , which is not a prime. Thus, the conjecture is disproved by the counter example *n* = 41.

(*Note*: It turns out that the expression *n*2 – *n* + 41 produces prime numbers for $1 \leq n \leq 40$.)

Proof by Cases

Theorem. Let n be an integer. Prove that the expression $n\ (n + 1)$ is always even.

Proof: Since n is an integer, n is either even or odd.

- (Case 1) n is even, that is, $n = 2\ m$ for some integer m. Thus, $n\ (n + 1) = (2\ m)\ (2\ m +1) = 2\ (m\ (2\ m +1))$, by associative law. Thus, it is even by definition.

- (Case 2) n is odd, that is, $n = 2\ m + 1$ for some integer m. Thus, $n\ (n + 1) = (2\ m + 1)\ (2\ m +2) = 2\ (m + 1)\ (2\ m + 1)$, by distributive and commutative laws. Thus, it is even by definition.