COT 3100 Fall 2017 Homework 9
Please Consult WebCourses for the due date/time.

1) Let R_1 and R_2 be relations on a set A = {1, 2, 3, 4}.
In particular, let R_1 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3) (4, 4)} and R_2 = {(1, 2), (2, 3), (3, 4), (1, 3), (2, 4)}.

Determine the following:

a) Whether or not R_1 is reflexive, irreflexive, symmetric, anti-symmetric and transitive or not.
b) Whether or not R_2 is reflexive, irreflexive, symmetric, anti-symmetric and transitive or not.
c) The relation $R_1 \circ R_2$.
d) The relation $R_2 \circ R_1$.
e) $R_1 \cup R_2$
f) $R_1 \cap R_2$
g) The reflexive, symmetric and transitive closures of both R_1 and R_2.

2) Let R be a relation on the set Z^+ defined as follows:

R = {(a, b) | $\exists c \in Fibonacci\ such\ a + b = c\ or\ a - b = c$}

Let the set Fibonacci be the set of positive integers that are Fibonacci numbers.

Determine (with proof) whether or not R is reflexive, irreflexive, symmetric, anti-symmetric and transitive or not.

3) Let b(n) equal the number of bits set to 1 in the binary representation of the positive integer n. Prove that the relation, R, defined below over the ***set of positive integers in between 1 and 1024***, inclusive, is an equivalence relation. Into how many equivalence classes does R partition the set described? Explicitly list all of the members of the following equivalence classes: [2] and [520]. Let the set X be the largest of the equivalence classes. What is the smallest integer that belongs to X?

$$R = \{(x, y) \mid b(x) = b(y)\}$$

4) Let R be a relation on the set Z^+ defined as follows:

R = {(a, b) | $|a - b| \leq 3$}

Determine (with proof) whether or not R is reflexive, irreflexive, symmetric, anti-symmetric and transitive or not.

5) How many anti-symmetric relations on the set A = {1, 2, 3, 4, 5, 6, 7} contain the ordered pairs (2, 3), (3, 3) and (6, 6)?

6) Let f(x) = $x^2 + 2x - 35$ with a domain of all real $x \in [-\infty, -1]$. Prove that f is injective. What is the range of f? (You may either use Calculus or complete the square to prove your answers.)

7) Find $f^{-1}(x)$ for the function given in question #6.

8) Let A be a set of 12 elements and B be a set of 20 elements. How many functions can be defined with the domain of A and the co-domain of B?

9) Let f(x) = $5x^2 + 2x - 7$ and g(x) = 3x + 4. Determine f(g(x)) and g(f(x)).

10) Let f(x) = 2x + 3. Let $f^n(x)$ to be the function f composed with itself n times. (For example, $f^3(x)$ = f(f(f(x))).) Using trial and error, conjecture a guess for $f^n(x)$ and use mathematical induction to prove that guess.

11) Give a summary of the life and mathematical contributions of Leonard Euler. Please aim for a length of roughly 200 - 400 words. **Your summary must be typed.** Please state the sources you used in writing your summary.

COT 3100 11/19/24 10^{30} am

A relation is a subset of a Cartesian Product

$A = \{1, 2, 3\}$ $B = \{a, b\}$

$A \times B = \{(1,a), (2,a), (3,a), (1,b), (2,b), (3,b)\}$

I can define a relation over $A \times B$:

$R = \{(1,a), (2,b)\}$

$A = \{$bread, pita, wrap$\}$

$B = \{$peanut butter, ham, chicken$\}$

$R = \{$(bread, peanut butter), (bread, ham), (pita, ham), (pita, chicken), (wrap, chicken)$\}$

\# of possible relations

\# items in $A \times B = |A| \times |B|$

each item can either be in set or not be in set

\# possible relations $= 2^{|A| \times |B|}$

n-ary relation ordered n-tuple over $A \times B \times C \times \ldots$

A = teachers UCF B = classes UCF

$R = \{(a,b) \mid$ a teaches at least 1 section of b$\}$

(Guha, COT 3100), (Guha, CIS 3362), (Guha, COT 4210) focus on Relations over $A \times A$ (same set) $A = \mathbb{Z}$

$R = \{(a,b) \mid a \equiv b \pmod{10}\}$
$= \{(27,17), (27,-3), \ldots\}$

Properties of Binary Relations over $A \times A$

Reflexive: $\forall a \in A \ (a,a) \in R$

Irreflexive: $\forall a \in A \ (a,a) \notin R$

Symmetric: $\forall a,b \in A$ if $(a,b) \in R$, then $(b,a) \in R$.

Anti-symmetric: $\forall a,b \in A$ if $(a,b) \in R \wedge (b,a) \in R$, then $a = b$.

$R_1 = \{(a,b) \mid a \geq b\}$

Transitive: $\forall a,b,c \in A$ if $(a,b) \in R \wedge (b,c) \in R$, then $(a,c) \in R$

Relation that is reflexive, symmetric + transitive is an equivalence relation

Relation that is reflexive, anti-symmetric + transitive is a partial ordering relation.

$R_n = \{(a,b) \mid a \equiv b \pmod n\}$ over $\mathbb{Z} \times \mathbb{Z}$
$n \in \mathbb{Z}^+$.

reflexive $(a,a) \in R$ because $a \equiv a \pmod n$
by def mod $n \mid (a-a)$, $a - a = 0 = 0 \times n$ ✓

symmetric let $(a,b) \in R$, thus
$a \equiv b \pmod n$ $c \in \mathbb{Z}$
by def mod, $n \mid (a-b) \rightarrow a - b = cn$,
$(-1)(a-b) = (-1)cn$
$b - a = (-c)n$, since $c \in \mathbb{Z}$, $-c \in \mathbb{Z}$, by def mod
$b \equiv a \pmod n \rightarrow (b,a) \in R$.

transitive let $(a,b) \in R \wedge (b,c) \in R$ by def mod
$\underline{a - b = nc_1}$, $c_1 \in \mathbb{Z}$ $\underline{b - c = nc_2}$, $c_2 \in \mathbb{Z}$

$$\begin{array}{rl} & a - b = nc_1 \\ + & b - c = nc_2 \\ \hline & a - c = nc_1 + nc_2 \end{array}$$

$a - c = (c_1 + c_2)n$, Since $c_1, c_2 \in \mathbb{Z}$, $c_1 + c_2 \in \mathbb{Z}$
by def mod $a \equiv c \pmod n$

Draw dot (vertex) for each item in R_5. Draw a directed edge (arrow) btw to vertices a, b if $(a,b) \in R_5$.

CLIQUE

An equivalence relation partitions the set it's over into equivalence classes. We treat each item within an equivalence class as equivalent.

$[0]$ = "the equivalence class of 0"

$= \{0, 5, -5, 10, -10, \ldots\}$

Question 3

how many equivalence classes → 10

$[1]$ → 1 bit on

$520 = 512 + 8$

$1024 = 10000000000_2$ } 11 bits

$[1023]$ → 10 bits 1111111111_2.

$[2] = \{1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024\}$

$[520] = \{3, 5, 6, 9, 10, 12, 17, 18, 20, 24, \ldots\}$

11
101
110
1001
1010
1100
10001

0000000000

let X be the largest of the equivalence classes. What is the smallest member of X?

X has 5 bits on $\binom{10}{5}$ is max of $\binom{10}{k}$

$0000011111_2 = \boxed{31}$

$R = \{(a,b) \mid \exists c \in Z^+ \mid a = bc\}$ R is over $Z^+ \times Z^+$

reflexive: for all a, $a = 1 \times a$ so $(a,a) \in R$ ✓

anti-symm: let $(a,b) \in R \wedge (b,a) \in R$

$a = bc_1 \wedge b = ac_2$ $c_1, c_2 \in Z^+$

$a = (ac_2c_1)$

$1 = c_2 \times c_1 \} \rightarrow c_2 = 1 \wedge c_1 = 1$

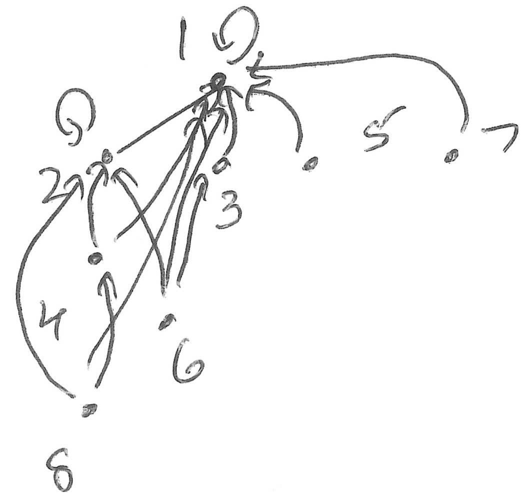
$a = b(1) \rightarrow a = b$ ✓

transitive   let $(a,b) \in R \wedge (b,c) \in R$

$a = bc_1 \qquad b = cc_2$

$a = c(c_2c_1)$

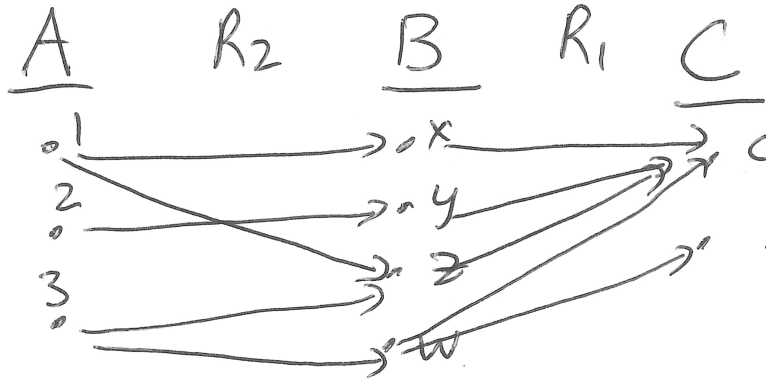
Since $c_1, c_2 \in Z^+$, $c_2 \times c_1 \in Z^+$ thus $(a,c) \in R$.



$R_1 \circ R_2$ = "R_1 composed with R_2"

$R_2 \subseteq A \times B, \quad R_1 \subseteq B \times C$

$R_1 \circ R_2 = \{ (a,c) \mid \exists b \in B \mid (a,b) \in R_2 \wedge (b,c) \in R_1 \}$



$R_1 \circ R_2 = \{ (1,a), (2,a), (3,a), (3,b) \}$

Question 1, c, d

$R_1 = \{ (1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4) \}$

$R_2 = \{ (1,2), (2,3), (3,4), (1,3), (2,4) \}$

$R_1 \circ R_2 = \{ (1,1), (1,2), (2,3), (2,4), (3,3), (3,4)$
$(1,3), (1,4) \}$

$R_2 \circ R_1 = \{ (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,4)$
$(4,4) \}$

Counting # relations satisfy criteria (Question 5)

$A = \{1,2,3,4,5,6,7\}$ how many anti-symmetric relations on $A \times A$ that contain $(2,3)$, $(3,3)$ and $(6,6)$

$(3,3)$, $(6,6) \rightarrow 1$ way

$(1,1),(2,2),(4,4),(5,5),(7,7) \rightarrow 2^5$ ways

$(2,3) \in R \wedge (3,2) \notin R$ (anti-sym) $\rightarrow 1$ way

$\underset{(a,a)}{7^2 - 7} - 2 = 40$ ordered pairs left (a,b) $a \neq b$ not $(2,3)$ or $(3,2)$

Create 20 pairs of ordered pairs

$(1,3)$ and $(3,1) \rightarrow \emptyset, \{(1,3)\}, \{(3,1)\}$

3 choices valid to maintain anti-sym

3^{20} ~~ways to place subsets~~ of these 20 pairs

$\boxed{\text{final ans} = 2^5 \times 3^{20}}$

Question 2

Fib = set POSITIVE fibonacci #s

$R = \{(a,b) \mid \exists c \in Fib \text{ s.t. } a+b=c \underline{\underline{\text{ or }}} a-b=c\}$

reflexive: no, $(2,2) \notin R$ because $2+2=4,\ 4 \notin Fib$
$2-2=0,\ 0 \notin Fib$

irreflexive: no, $(1,1) \in R$ $1+1=2,\ 2 \in Fib$

Symmetric: no, $(7,2) \in R$ because $7-2=5,\ 5 \in Fib$
but $(2,7) \notin R$ because $2+7=9,\ 9 \notin Fib$
$2-7=-5,\ -5 \notin Fib$

anti-symm: no $(3,5) \in R \land (5,3) \in R$ but $3 \neq 5$.
$3+5=8,\ 8 \in Fib$ $5+3=8,\ 8 \in Fib$

transitive: no $(3,5) \in R \land (5,8) \in R$ but $(3,8) \notin R$
$3+5=8$ $5+8=13$ $3+8=11,\ 11 \notin Fib$
$8 \in Fib$ $13 \in Fib$ $3-8=-5\ -5 \notin Fib$.

Question 4

$R = \{ (a,b) \mid |a-b| \leq 3 \}$ over $\mathbb{Z}^+$

reflexive: yes $(a,a) \in R$ $|a-a| = 0, 0 \leq 3$ ✓

Irreflexive: no $(1,1) \in R$ $|1-1| = 0$

symm: yes if $(a,b) \in R$

$|a-b| \leq 3$

$|b-a| = |-(a-b)| = \cancel{x} |a-b| \leq 3$ ✓

anti-symm: no $(2,3) \in R \wedge (3,2) \in R$ $2 \neq 3$

$|2-3| = 1 \leq 3$ $|3-2| = 1 \leq 3$

transitive no $(1,3) \in R \wedge (3,5) \in R$ but $(1,5) \notin R$

$|1-3| = 2 \leq 3$ $|3-5| = 2 \leq 3$ $|1-5| = 4$

$4 > 3$

✓