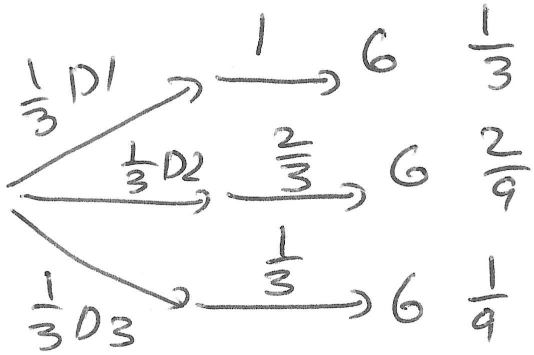
Set 1 COT 3100 11/14/24 9 am

Q1) (a) $E(x) = np =$ $n = 20$ $p = \frac{1}{4}$

$20 \times \frac{1}{4} = \boxed{5}$

(b) $\binom{20}{5}\left(\frac{1}{4}\right)^{5}\left(\frac{3}{4}\right)^{15}$

Q3)



a) $p(6) = \frac{1}{3} + \frac{2}{9} + \frac{1}{9} = \frac{2}{3}$

b) $p(D3 \mid 6) = \frac{p(D3 \text{ and } 6)}{p(6)}$

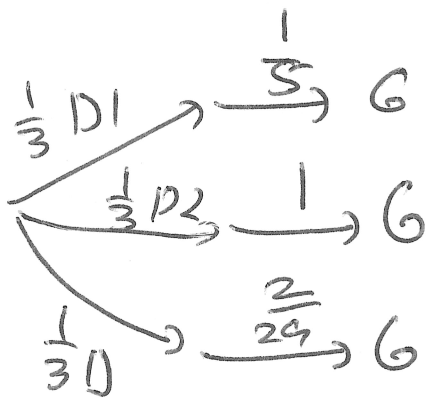
$= \frac{\frac{1}{9}}{\frac{2}{3}} = \frac{1}{6}$

Q3 Alt)

D1 2685

D2 16

D3 26275



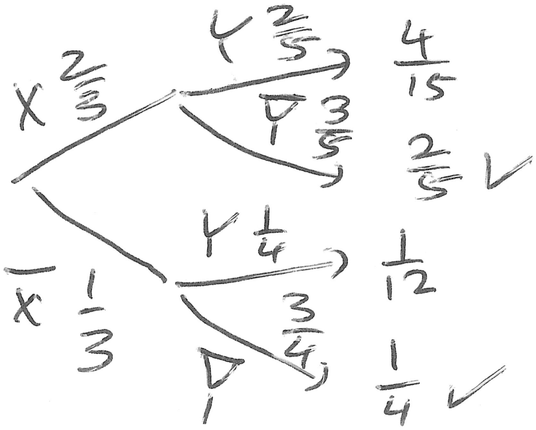
$p(6) = \frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times 1 + \frac{1}{3} \times \frac{2}{29}$

$> \frac{1}{3}$

$\frac{5}{40} = \frac{1}{8}$ of all sold

7) $P(X) = \frac{2}{3}$, $P(Y|X) = \frac{2}{5}$, $P(Y|\bar{X}) = \frac{1}{4}$

Find $P(\bar{Y})$, $P(\bar{X} \cup \bar{Y})$



$$P(\bar{Y}) = \frac{2}{5} + \frac{1}{4} = \frac{\cancel{8+5}}{20} = \frac{13}{20}$$

$$P(\bar{X} \cup \bar{Y}) = P(\bar{X}) + P(\bar{Y}) - P(\bar{X} \cap \bar{Y})$$

$$= \frac{1}{3} + \frac{13}{20} - \frac{1}{4}$$

$$= \frac{1}{12} + \frac{13}{20}$$

$$= \frac{5+39}{60} = \frac{44}{60} = \boxed{\frac{11}{15}}$$

$$P(\bar{X} \cup \bar{Y}) = 1 - P(X \cap Y)$$

$$= 1 - \frac{4}{15} = \boxed{\frac{11}{15}}$$

9) $$\sum_{x=0}^{\infty} k \cdot \left(\frac{2}{3}\right)^x = 1$$

$$\frac{k}{1 - \frac{2}{3}} = 1$$

$$3k = 1$$

$$k = \boxed{\frac{1}{3}}$$

$$P(X = x) = \frac{1}{3}\left(\frac{2}{3}\right)^x, \quad x = 0, 1, 2, \ldots$$

$$E(X) = \sum_{x=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^x \cdot x$$

$$= \frac{1}{3} \sum_{x=0}^{\infty} x \cdot \left(\frac{2}{3}\right)^x$$

$$S = 0 \cdot \frac{2}{3}^0 + 1 \cdot \frac{2}{3}^1 + 2 \cdot \frac{2}{3}^2 + 3 \cdot \frac{2}{3}^3$$

$$- \quad \frac{2}{3}S = \quad 0 \cdot \frac{2}{3}^1 + 1 \cdot \frac{2}{3}^2 + 2 \cdot \frac{2}{3}^3$$

$$\frac{1}{3}S = \quad 1 \cdot \frac{2}{3} + 1 \cdot \frac{2}{3}^2 + 1 \cdot \frac{2}{3}^3 + \ldots$$

$$\frac{1}{3}S = \frac{\frac{2}{3}}{1 - \frac{2}{3}} \rightarrow \frac{1}{3}S = 2$$

$$\boxed{S = 6}$$

$$E(X) = \frac{1}{3}S = \frac{1}{3} \times 6 = \boxed{2}$$

10) a) $\frac{2}{3}$

b) $\underset{\substack{\downarrow \\ \text{Jack} \\ 5,6}}{\frac{1}{3}} \times \underset{\substack{\downarrow \\ \text{Jill} \\ 1,2,3,4}}{\frac{2}{3}} = \frac{2}{9}$

c) Let X = prob Jack wins

$$X = \frac{2}{3} + \underset{\substack{\downarrow \\ 5,6 \\ \text{Jack}}}{\frac{1}{3}} \times \underset{\substack{\downarrow \\ 5,6 \\ \text{Jill}}}{\frac{1}{3}} \times X$$

$$X - \frac{X}{9} = \frac{2}{3}$$

$$\frac{8}{9}X = \frac{2}{3} \rightarrow \boxed{X = \frac{3}{4}}$$

11) a) $\int_0^2 f(x) \cdot x \, \partial x$

$$= \int_0^2 \frac{1}{12}(8x - x^3) \cdot x \, \partial x$$

$$= \frac{1}{12}\int_0^2 (8x^2 - x^4) \, \partial x$$

$$= \frac{1}{12}\left[\frac{8}{3}x^3 - \frac{x^5}{5}\right]\Big|_0^2$$

$$= \frac{1}{12}\left[\frac{64}{3} - \frac{32}{5}\right] = \frac{32}{12}\left[\frac{2}{3} - \frac{1}{5}\right] = \frac{8}{3} \times \frac{(10-3)}{15} = \boxed{\frac{56}{45}}$$

b)

$$\int_0^m \frac{1}{12}(8x - x^3)\,dx = \frac{1}{2}$$

$$\frac{1}{12}\left[4x^2 - \frac{x^4}{4}\right]^m = \frac{1}{2}$$

$$4m^2 - \frac{m^4}{4} = 6$$

$$16m^2 - m^4 = 24$$

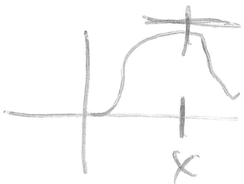
$$m^4 - 16m^2 + 24 = 0$$

256
96
160

$$m^2 = \frac{16 \pm \sqrt{256 - 4(24)}}{2} = 8 \pm \frac{4\sqrt{10}}{2} = \boxed{8 \pm 2\sqrt{10}}$$

$$m^2 = 8 - 2\sqrt{10} \rightarrow m = \sqrt{8 - 2\sqrt{10}}$$

c)



$$f(x) = \frac{1}{12}(8x - x^3)$$

$$f'(x) = \frac{1}{12}(8 - 3x^2) = 0$$

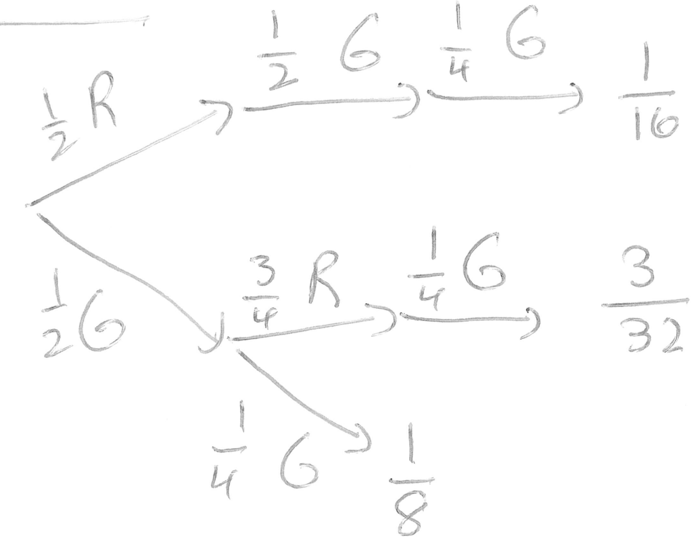
$$3x^2 = 8$$

$$x^2 = \frac{8}{3}$$

$$x = \frac{2\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \boxed{\frac{2\sqrt{6}}{3}}$$

Set 2

Q1)



$$\frac{1}{16}+\frac{3}{32}+\frac{1}{8}=\frac{2+3+4}{32}$$
$$=\boxed{\frac{9}{32}}$$

Q2)

1, 2, 4, 4, 5, 6 O 2 E 4

1, 2, 3, 3, 5, 6 O 4 E 2

prob ODD sum

$$p(\text{odd sum}) = p_1(E)\times p_2(O) + p_1(O)p_2(E)$$
$$= \frac{2}{3}\times\frac{2}{3}+\frac{1}{3}\times\frac{1}{3}=\boxed{\frac{5}{9}}$$

Q3) 1 ... 11 CHOOSE 6 SUM ODD

$$\text{Sample Space} = \binom{11}{6} = \frac{11\times10\times9\times8\times7}{1\times2\times3\times4\times5} = 42\times11 = 462$$

32

1 ... 11 6 O 5 E

10 5E $\rightarrow \binom{6}{1}\binom{5}{5} = 6$

30 3E $\rightarrow \binom{6}{3}\binom{5}{3} = 200$

50 1E $\rightarrow \binom{6}{5}\binom{5}{1} = 30$

236

$$p=\frac{236}{462}=\boxed{\frac{118}{231}}$$

$3^1 \to 3,\ 3^2 \to 9,\ 3^3 \to 7,\ 3^4 \to 1$ 　 3, 9, 7, 1

$7^1 \to 7,\ 7^2 \to 9,\ 7^3 \to 3,\ 7^4 \to 1$ 　 7, 9, 3, 1

	7	9	3	1	7	9	3	1
3	0	2	6	4				
9	6	8	2	0				
7	4	6	0	8				
1	8	0	4	2				
3								
9								
7								
1								

$$\boxed{p(\text{Unit } 8) = \frac{3}{16}}$$

6) $X = \binom{50}{0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^{50} + \binom{50}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^{48} + \binom{50}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^{46} + \cdots \binom{50}{50}\left(\frac{2}{3}\right)^{50}\left(\frac{1}{3}\right)^0$

$1 = \left(\frac{2}{3}+\frac{1}{3}\right)^{50} = \binom{50}{0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^{50} + \binom{50}{1}\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^{49} + \binom{50}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^{48} + \cdots \quad \binom{50}{50}\left(\frac{2}{3}\right)^{50}\left(\frac{1}{3}\right)^0$

$+\left(\frac{2}{3}-\frac{1}{3}\right)^{50} = \binom{50}{0}\left(\frac{2}{3}\right)^0\left(-\frac{1}{3}\right)^{50} + \binom{50}{1}\left(\frac{2}{3}\right)^1\left(-\frac{1}{3}\right)^{49} + \binom{50}{2}\left(\frac{2}{3}\right)^2\left(-\frac{1}{3}\right)^{48} + \cdots$

$1 + \frac{1}{3}^{50} = 2\left[\binom{50}{0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^{50}\right] + 2\left[\binom{50}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^{48}\right] \cdots 2\left[\binom{50}{50}\left(\frac{2}{3}\right)^{50}\left(\frac{1}{3}\right)^0\right]$

$1 + \left(\frac{1}{3}\right)^{50} = 2X$

$$X = \boxed{\frac{1+\left(\frac{1}{3}\right)^{50}}{2}}$$

$p(\text{Odd H in 51 toss}) = p(\underline{\text{Odd H in 50 toss}}) \times p(\underline{\text{Tail}}) +$

$+ p(\text{Even H 50 toss}) \times p(\text{Heads})$

$$= \frac{1-\left(\frac{1}{3}\right)^{50}}{2} \times \frac{1}{3} +$$

$$\frac{1+\left(\frac{1}{3}\right)^{50}}{2} \times \frac{2}{3}$$

$$= \frac{\frac{1}{3} - \frac{1}{3}\times\left(\frac{1}{3}\right)^{50} + \frac{2}{3} + \frac{2}{3}\times\left(\frac{1}{3}\right)^{50}}{2} = \frac{1+\frac{1}{3}\left(\frac{1}{3}\right)^{50}}{2} = \frac{1+\left(\frac{1}{3}\right)^{51}}{2}$$

7) 3 Shiny 4 Dull

all strings of 3S 4D $\rightarrow \frac{7!}{3!4!} = 35$

What's prob more than 4 pulls to get all Shiny?

What's prob get all Shiny in the 1st 4 pulls

$\underline{S}\ \underline{S}\ \underline{S}\ \underline{D}$ DDD $\quad \binom{4}{3}$ ways to choose 3 slots out of 4 for S

S S $\underline{D}$ S DDD

S D S S DDD $\quad \frac{4}{35}$

D S S S DDD

$$p(\text{more 4 pulls}) = 1 - \frac{4}{35} = \boxed{\frac{31}{35}}$$

8)

a	b	c	a+b+c
O	O	O	even
E	E	E	
E	O	E	
O	E	E	

$\{1,2,3,4,5\}$ 3O 2E

$p(O,O,O) = \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{27}{125}$

$p(-,-,E) = \frac{2}{5}\left(1 - \frac{3}{5} \times \frac{3}{5}\right)$

EE
EO
OE

p(c even)   p a odd   prob b odd

p(c even)(1 − p(ab odd))

$= \frac{2}{5} \times \frac{16}{25} = \frac{32}{125}$

ans $= \frac{27+32}{125} = \boxed{\frac{59}{125}}$

9) Sample Space divisors of $10^{99} = 2^{99} \times 5^{99}$

divisors $= (99+1)(99+1) = 100 \times 100$

mult 10^{88}. $\quad 2^x 5^y, \quad 88 \le x \le 99 \quad 12$

$88 \le y \le 99 \quad 12$

mult req   divisor req

mult 10^{88} also divisors 10^{99} is

12×12

prob $= \frac{\cancel{12}^{3} \times \cancel{12}^{3}}{\cancel{100}_{25} \times \cancel{100}_{25}} = \boxed{\frac{9}{625}}$

10)

let

p = prob H

$$\binom{5}{1} p(1-p)^4 = \binom{5}{2} p^2(1-p)^3$$

$$5p(1-p)^4 = 10p^2(1-p)^3$$

$$1-p = 2p$$

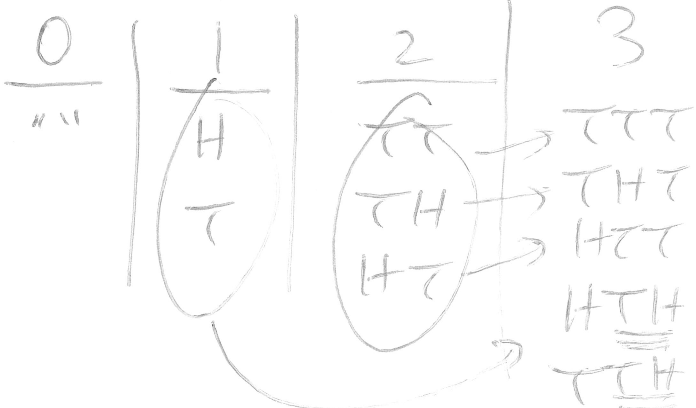
$$3p = 1$$

$$p = \frac{1}{3}$$

$$p(3H) = \binom{5}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 = 10 \cdot \frac{1}{27} \cdot \frac{4}{9} = \boxed{\frac{40}{243}}$$

11) 10 coin tosses prob no 2 H in a row

sample space $= 2^{10} = 1024$



let $t(n)$ = # valid seq of length n

$$t(n) = t(n-1) + t(n-2)$$

$$t(2) = 3 = F_4$$

$$t(3) = 5 = F_5$$

$$t(n) = F_{n+2}$$

$$t(10) = F_{12} = 144$$

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144

$$\text{prob} = \frac{144}{1024} = \boxed{\frac{9}{64}}$$

_ _ _ _ _ _ _ _ _ _] 10 slots no 2 H

$\binom{11}{0}$ 1 ✓ 0 H

$\binom{10}{1}$ ✓ 1 H

$\binom{9}{2}$ _ T _ T _ T _ T _ T _ T _ T T _ 2 H

$\binom{8}{3}$ _ T _ T _ T _ T _ T _ T _ T _ 3 H

$\binom{7}{4}$ _ T _ T _ T _ T _ T _ T _ 4 H

$\binom{6}{5}$ _ T _ T _ T _ T _ T _ 5 H

$$1 + 10 + \frac{9 \times 8}{2} + \frac{8 \times 7 \times 6}{6} + 35 + 6$$

$$\underbrace{1 + 10 + 36}_{47} + \underbrace{56}_{56} + \underbrace{35 + 6}_{41}$$

$$47 + 56 + 41$$

$$103 + 41$$

$$144$$

$$\sum_{i=0}^{\frac{n}{2}} \binom{n-i}{i} = ?$$ Fibonacci Number?