COT 3100 8/29/24 10^{30} am

Finishing up section on Boolean Logic + Proof Techniques. [Next T → Sets]

① ~~3~~4 $\forall, \exists$ questions

② Look at 4 "proof archetypes"

③ Random Problems

P1

$$\forall y \in \mathbb{Z}^+ \exists x \in \mathbb{Q}^+ \left[\frac{x}{y} = x^2 - x\right]$$

$$\frac{x}{y} = x^2 - x \longrightarrow \frac{x}{x^2-x} = y$$

So long as $x^2 - x \neq 0$.

$$\frac{x}{x(x-1)} = y$$

$$\frac{1}{x-1} = y$$

$\cancel{y = 1}$

$$\frac{1}{y} = x - 1$$

$$x = 1 + \frac{1}{y} = \boxed{\frac{y+1}{y}}$$

The assertion is true. For an arbitrarily chosen positive integer y, let $x = \frac{y+1}{y}$. Then the stmt is true (This is a value of x that makes it true.)

$$LHS = \frac{x}{y} = \frac{\frac{y+1}{y}}{y} = \frac{y+1}{y^2} ✓$$

$$RHS = x^2 - x = x(x-1)$$

$$= \frac{(y+1)}{y}\left(\frac{y+1}{y} - \frac{y}{y}\right) = \left(\frac{y+1}{y}\right) \times \frac{1}{y} = \frac{y+1}{y^2} ✓$$

LHS = RHS

Done

P2

Prove or disprove for all real numbers

$$\exists x \in R\ \forall y \in R\ [y^2 - 6y + x \geq 0]$$

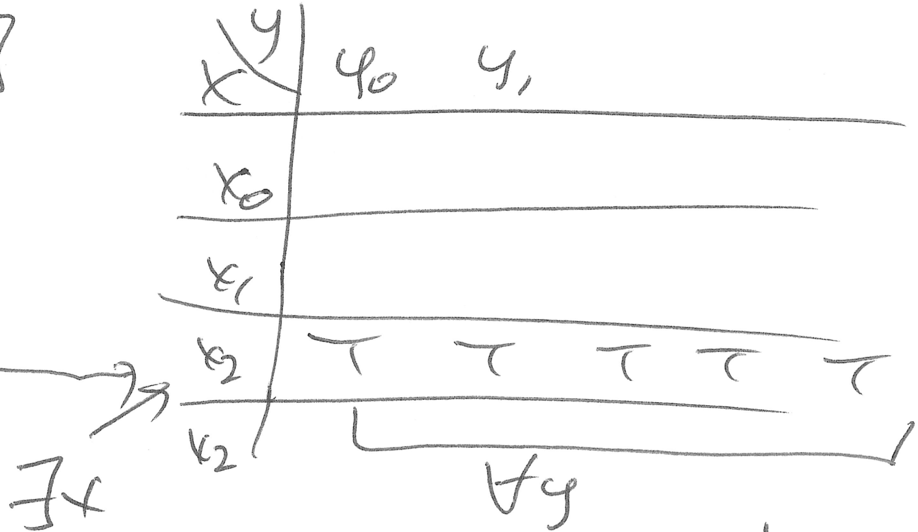
This is true. Consider $x = 9$.

$$y^2 - 6y + x = y^2 - 6y + 9$$
$$= (y-3)^2$$

≥ 0, because $y - 3 \in R$ and all reals squared are non-neg.

P3

$\exists x\ \forall y\ [P(x,y)]$



one now w/all trues!

Is it the case necessarily that

$\forall y\ [\exists x\ (P(x,y)]$

every row has at least 1 T

$\exists x \forall y [P(x,y)] \longrightarrow \forall y [\exists x (P(x,y))]$

BUT the implication isn't true in the other direction.

True: $\forall y \exists x [x+y=0]$ BUT

y \ x	0	-1	1	2	-2
0	T	F	F	F	F
-1	F	F	T	F	F
1	F	T	F	F	F
2	F	F	F	F	T
-2	F	F	F	T	F

$\exists x \forall y [x+y=0]$ is false

$\exists x \forall y [xy=0] \longrightarrow \forall y \exists x [xy=0]$

true
$x=0$ is the one that exists

for arbitrarily chosen y, let $x=0$ and
$xy = 0 \cdot y = 0$ ✓

P4

Prove or Disprove $\exists x \in R \forall y \in R [xy=0]$

TRUE

P5

Prove or disprove over all Reals

$\forall x \in R \exists y \in R [xy=1]$

FALSE
for $x=0$, there is no value of y that works.

Proof Archetypes

1. Direct Proof
2. Proof by Cases
3. Proof by Contradiction
4. Proof of Contrapositive

$p \to q$

1. Assume p is true.
2. Logical Deductions

real math

K. $\boxed{q}$

Direct Proof

$p \to q$

$p \to (r \vee s)$

$r \to q \wedge$

$s \to q$

1. Break up p into cases that cover its entirety
2. For each case, r, Prove $r \to q$ via Direct Proof

$p \to q$

1. Assume $\overline{q}$
2. Logical Steps
3. Use p (or assume p)

CONTRADICTION

Pf Contra

Proof of contrapositive $(p \to q)$

Contrapositive $\overline{q} \to \overline{p}$ (equivalent) prove this insted.

Proof by Cases

If $n \in \mathbb{Z}$, then $n^2+n \in$ Even

Case 1: $n \in$ Even

Assume $n \in$ Even.

$\exists x \in \mathbb{Z} \mid n = 2x$

$$n^2+n = (2x)^2 + 2x$$
$$= 4x^2+2x$$
$$= 2(2x^2+x)$$

Since $x \in \mathbb{Z}$, $2x^2+x \in \mathbb{Z}$

It follows that n^2+n is even in this case.

Case 2: $n \in$ Odd

Assume $n \in$ Odd

$\exists x \in \mathbb{Z} \mid n = 2x+1$

$$n^2+n = (2x+1)^2+(2x+1)$$
$$= 4x^2+4x+1+2x+1$$
$$= 4x^2+6x+2$$
$$= 2(2x^2+3x+1)$$

Since $x \in \mathbb{Z}$, $2x^2+3x+1 \in \mathbb{Z}$,

It follows that n^2+n is even in this case.

Let x, y be odd prove xy is odd

Assume $x, y \in$ Odd

$\exists x' \in \mathbb{Z} \mid x = 2x'+1$

$\exists y' \in \mathbb{Z} \mid y = 2y'+1$

$$xy = (2x'+1)(2y'+1)$$
$$= 4x'y'+2x'+2y'+1$$
$$= 2(2x'y'+x'+y')+1$$

Since $x', y' \in \mathbb{Z}$, $(2x'y'+x'+y') \in \mathbb{Z}$

Thus we've proven $xy \in$ Odd.

Prime # is an integer >1 which only has 2 divisors : 1 and itself.

Composite int, n, is int >1 that can be expressed as $n=ab$, , $a,b \in \mathbb{Z}^+ \wedge a>1 \wedge b>1$.

If n is a composite integer, it has at least 1 divisor, a, such that $1 < a \leq \sqrt{n}$.

Prove via contradiction.

Assume the opposite, that n is composite but has no divisors $\leq \sqrt{n}$

non-trivial

$$n = ab, \qquad a>\sqrt{n} \text{ and } b>\sqrt{n}$$

$$> \sqrt{n}\sqrt{n}$$

$$= n$$

$\Rightarrow n > n$ is a contradiction.

It follows that ~~lee~~ our initial assumption was incorrect. Thus n must have at least 1 divisor $\leq \sqrt{n}$.

non-trivial

Problem

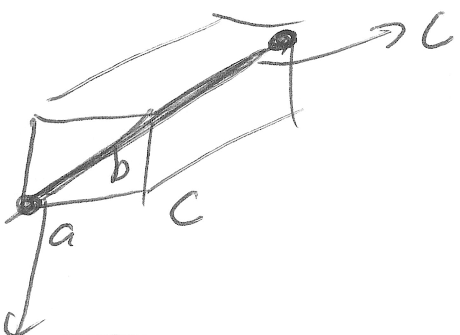
rect box P, distinct edge lengths a, b and c.

Sum of lengths of 12 edges is ~~12~~ 13.

Sum of areas of 6 faces is $\frac{11}{2}$.

Volume of P is $\frac{1}{2}$.

What is the length of the longest interior diagonal?



$\sqrt{a^2+b^2}$

$\sqrt{a^2+b^2+c^2}$

$$4(a+b+c) = 13$$

$$2(ab+ac+bc) = \frac{11}{2}$$

$$abc = \frac{1}{2}$$

Find $\sqrt{a^2+b^2+c^2}$

$$(a+b+c)^2 = \left(\frac{13}{4}\right)^2$$

$$\boxed{a^2+b^2+c^2} + \underline{2(ab+ac+bc)} = \frac{169}{16}$$

$$a^2+b^2+c^2 + \frac{11}{2} = \frac{169}{16}$$

$$a^2+b^2+c^2 = \frac{169}{16} - \frac{88}{16} = \frac{81}{16}$$

$$\sqrt{a^2+b^2+c^2} = \boxed{\frac{9}{4}}$$

For how many ordered pairs (a,b) of ints does the poly

$$x^3 + ax^2 + bx + 6$$ have 3

distinct integer roots?

$$(x-r_1)(x-r_2)(x-r_3)$$

$$x^3 - (r_1+r_2+r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3$$

$a = -(r_1+r_2+r_3)$

$b = r_1r_2 + r_1r_3 + r_2r_3$

$c = -r_1r_2r_3 = 6$

$r_1r_2r_3 = -6$

$\boxed{5}$

$\pm 1, \pm 2, \pm 3$

$$\boxed{\begin{array}{c} -1, 2, 3 \\ 1, -2, 3 \\ 1, 2, -3 \\ -1, -2, -3 \end{array}}$$

$1, 1, 6$ ×

$-1, -1, -6$ ×

$\boxed{-1, 1, 6}$ ✓

4) (10 pts) With proof, prove or disprove the following statement:

$$\forall y \in Z^{+} \exists x \in Q^{+}[\frac{x}{y} = x^2 - x]$$

Note: The set Z^+ is the set of positive integers, and the set Q^+ is the set of positive rational numbers.

1) (12 pts) Prove or disprove the following statement over the universe of all real numbers for x and y: $\exists x \forall y[y^2 - 6y + x \geq 0]$.

4) (5 pts) For an open statement P(x, y), it is known that $\exists x[\forall y(P(x,y))]$. Is it necessarily true that $\forall y[\exists x(P(x,y))]$? If it is the case, prove it, if the assertion is false, create a specific open statement P(x, y) for which $\exists x[\forall y(P(x,y))]$ is true and $\forall y[\exists x(P(x,y))]$ is false and explain why the first is true and the second is false.

3) (5 pts) Prove or disprove the following assertion over the universe of integers:

$$\exists x \forall y[xy = 0]$$

Please clearly note whether the assertion is true or not, followed a justification of your answer. Most of the points are awarded for the justification.

4) (5 pts) Prove or disprove the following assertion over the universe of real numbers:

$$\forall x \exists y[xy = 1]$$

Problem

Flora the frog starts at 0 on the number line and makes a sequence of jumps to the right. In any one jump, independent of previous jumps, Flora leaps a positive integer distance m with probability $\frac{1}{2^m}$.

What is the probability that Flora will eventually land at 10?

$\textbf{(A)}\ \frac{5}{512} \qquad \textbf{(B)}\ \frac{45}{1024} \qquad \textbf{(C)}\ \frac{127}{1024} \qquad \textbf{(D)}\ \frac{511}{1024} \qquad \textbf{(E)}\ \frac{1}{2}$

A rectangular box P has distinct edge lengths a, b, and c. The sum of the lengths of all 12 edges of P is 13, the sum of the areas of all 6 faces of P is $\frac{11}{2}$, and the volume of P is $\frac{1}{2}$. What is the length of the longest interior diagonal connecting two vertices of P?

$\textbf{(A)}\ 2 \qquad \textbf{(B)}\ \frac{3}{8} \qquad \textbf{(C)}\ \frac{9}{8} \qquad \textbf{(D)}\ \frac{9}{4} \qquad \textbf{(E)}\ \frac{3}{2}$

Problem 14

For how many ordered pairs (a, b) of integers does the polynomial $x^3 + ax^2 + bx + 6$ have 3 distinct integer roots?

$\textbf{(A)}\ 5 \qquad \textbf{(B)}\ 6 \qquad \textbf{(C)}\ 8 \qquad \textbf{(D)}\ 7 \qquad \textbf{(E)}\ 4$

Pf irr irr^{irr} could = rat

Prime div