COT 3100 8/29/24 9am

Finishing up logic section + general outline of proof techniques

✓ (1) 3 $\forall, \exists$ questions

✓ (2) Look at 3 proof archetypes w/examples

(3) Random Problems

P1

$$\forall y \in \mathbb{Z}^+ \, \exists x \in \mathbb{Q}^+ \left[\frac{x}{y} = x^2 - x \right]$$

$$\frac{x}{y} = x^2 - x \longrightarrow \frac{x}{x^2 - x} = y \quad \left(\begin{array}{c}\text{ok} \\ \text{provided} \\ x^2 - x \neq 0\end{array}\right)$$

$$x = yx^2 - yx \qquad y = \frac{x}{x(x-1)} = \frac{1}{x-1}$$

$$0 = yx^2 - yx - x$$

$$x - 1 = \frac{1}{y}$$

$$x = 1 + \frac{1}{y} = \frac{y+1}{y}$$

The assertion is true. For an arbitrarily chosen positive integer y, let $x = \frac{y+1}{y}$, which is a positive rational # then we have

$$LHS = \frac{x}{y} = \frac{\frac{y+1}{y}}{y} = \frac{y+1}{y^2} \checkmark$$

$$RHS = x^2 - x = \left(\frac{y+1}{y}\right)^2 - \frac{y+1}{y}$$

$$= \frac{y^2+2y+1}{y^2} - \frac{y(y+1)}{y^2}$$

$$= \frac{y^2+2y+1-(y^2+y)}{y^2} = \frac{y+1}{y^2} \checkmark$$

$$\exists x^{\in R} \forall y^{\in R} \left[y^2 - 6y + x \geq 0 \right]$$

This is true. Consider $x = 9$

$y^2 - 6y + 9$

$= (y-3)^2$

≥ 0 for all real y since $y-3 \in R$ and all reals squared are non-neg.

$\exists x \forall y \, P(x,y)$

$x \backslash y$	y_0	y_1	...		
x_0					
x_1					
x_2	T	T	T	T	T
.					
.					
.					

for all y

at least 1 row w/all Ts

$\forall x \exists y$

$x \backslash y$	y_0	y_1	y_2
x_0	T	F	F
x_1	F	F	T
x_2		T	

each row have ≥ 1 T.

Proof Archetypes

1. Direct Proof
2. Proof by Cases
3. Proof Contradiction
4. Proof Contrapositive

$p \to q$

1. Assume p
2. Make Logical
3. Deductions
4.

$\vdots$

$\boxed{q}$

Direct Pf

$p \to (r \vee s)$

$r \to q \wedge$

$s \to q$

Split it up into separate cases. Run a direct pf on each case

Proof by cases

if $n \in \mathbb{Z}$

then $n^2 + n \in$ Even

Pf by contradiction

$p \to q$

1. Assume opposite $\overline{q}$
2. Logical steps

$\vdots$

$\downarrow$ p (assume)

$\downarrow$ CONTRADICTION

$\sqrt{2}$ irrational pf.

$p \to q \iff \underbrace{\overline{q} \to \overline{p}}_{\text{Contrapositive}}$

Can prove the contrapositive via direct proof.

Prime # is a pos. int > 1 which is divisible by 1 and itself and no other ints.

Composite # is pos int > 1 with at least 1 divisor > 1 and less then itself.
n can be expressed as $n = ab$, a and b are divisors of $n > 1$.

If n is composite, it has at least 1 divisor x, $1 < x \leq \sqrt{n}$.

n has no non-trivial divisors $\leq \sqrt{n}$

Since n is composite $\exists\, a, b \in \mathbb{Z}$
s.t $a > 1 \wedge b > 1 \wedge n = ab$.
$a < n \wedge b < n$

$$
\begin{aligned}
n &= ab \\
&> \sqrt{n} \times \sqrt{n} \\
&= n
\end{aligned}
$$

$n > n$ is a contradiction

It follows that $a > \sqrt{n} \wedge b > \sqrt{n}$ is false, at least one of these is $\leq \sqrt{n}$.

Prove that it's possible to have $p^q = r$ where p and q are irrational but r is rational.

$\sqrt{2}$ is irrational (previously proven)

let $x = \sqrt{2}^{\sqrt{2}}$. If x is rational, we're done.

Case 1

x is irrational

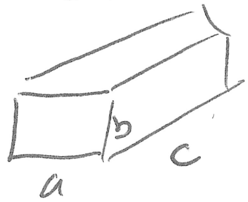
Case 2 Let $y = x^{\sqrt{2}} = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$

$= \sqrt{2}^{2}$

$= 2$ rational

✓

P is a rectangular box edge lengths a, b, c (distinct). Sum of all 12 edges of P is 13, sum of areas of 6 faces of P is $\frac{11}{2}$. Volume of P is $\frac{1}{2}$. What is length of the longest interior diagonal?



$4(a+b+c) = 13$

$2(ab+ac+bc) = \frac{11}{2}$

$ab+ac+bc = \frac{11}{4}$ $\quad abc = \frac{1}{2}$

Find $\sqrt{a^2+b^2+c^2}$

$$(a+b+c)^2 = \left(\frac{13}{4}\right)^2$$

$$\boxed{a^2+b^2+c^2} + \boxed{2ab+2ac+2bc} = \frac{169}{16}$$

$$a^2+b^2+c^2 + \boxed{2(ab+ac+bc)} = \frac{169}{16}$$

$$a^2+b^2+c^2 + 2\left(\frac{11}{4}\right) = \frac{169}{16}$$

$$a^2+b^2+c^2 = \frac{169}{16} - \frac{88}{16} = \frac{81}{16}$$

$$\sqrt{a^2+b^2+c^2} = \boxed{\frac{9}{4}}$$

$$\begin{array}{r} 169 \\ 88 \\ \hline 81 \end{array}$$

For how many ordered pairs of integers (a,b) does the polynomial x^3+ax^2+bx+6 have 3 distinct integer roots?

$$(x-r_1)(x-r_2)(x-r_3)$$

$$x^3-(r_1+r_2+r_3)x^2+(r_1r_2+r_1r_3+r_2r_3)x-r_1r_2r_3$$

$6 = -r_1r_2r_3$

$b = r_1r_2+r_1r_3+r_2r_3$

$a = -(r_1+r_2+r_3)$

$\pm 1, \pm 2, \pm 3$

$\pm 1, \pm 1, \pm 6$

Can have

$1,2,-3$ ~~$1,1,-6$~~

$1,-2,3$ $-1,1,6$

$-1,2,3$ ~~$-1,-1,6$~~

$-1,-2,-3$

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Problem

Flora the frog starts at 0 on the number line and makes a sequence of jumps to the right. In any one jump, independent of previous jumps, Flora leaps a positive integer distance m with probability $\frac{1}{2^m}$.

What is the probability that Flora will eventually land at 10?

(A) $\frac{5}{512}$ (B) $\frac{45}{1024}$ (C) $\frac{127}{1024}$ (D) $\frac{511}{1024}$ (E) $\frac{1}{2}$

A rectangular box P has distinct edge lengths a, b, and c. The sum of the lengths of all 12 edges of P is 13, the sum of the areas of all 6 faces of P is $\frac{11}{2}$, and the volume of P is $\frac{1}{2}$. What is the length of the longest interior diagonal connecting two vertices of P?

(A) 2 (B) $\frac{3}{8}$ (C) $\frac{9}{8}$ (D) $\frac{9}{4}$ (E) $\frac{3}{2}$

Problem 14

For how many ordered pairs (a, b) of integers does the polynomial $x^3 + ax^2 + bx + 6$ have 3 distinct integer roots?

(A) 5 (B) 6 (C) 8 (D) 7 (E) 4

Pf irr$^{\text{irr}}$ could = rat

Prime div

4) (10 pts) With proof, prove or disprove the following statement:

$$\forall y \in Z^+ \exists x \in Q^+ [\frac{x}{y} = x^2 - x]$$

Note: The set Z^+ is the set of positive integers, and the set Q^+ is the set of positive rational numbers.

1) (12 pts) Prove or disprove the following statement over the universe of all real numbers for x and y: $\exists x \forall y [y^2 - 6y + x \geq 0]$.

4) (5 pts) For an open statement P(x, y), it is known that $\exists x[\forall y(P(x,y)]$. Is it necessarily true that $\forall y[\exists x(P(x,y)]$? If it is the case, prove it, if the assertion is false, create a specific open statement P(x, y) for which $\exists x[\forall y(P(x,y)]$ is true and $\forall y[\exists x(P(x,y)]$ is false and explain why the first is true and the second is false.

3) (5 pts) Prove or disprove the following assertion over the universe of integers:

$$\exists x \forall y [xy = 0]$$

Please clearly note whether the assertion is true or not, followed a justification of your answer. Most of the points are awarded for the justification.

4) (5 pts) Prove or disprove the following assertion over the universe of real numbers:

$$\forall x \exists y [xy = 1]$$