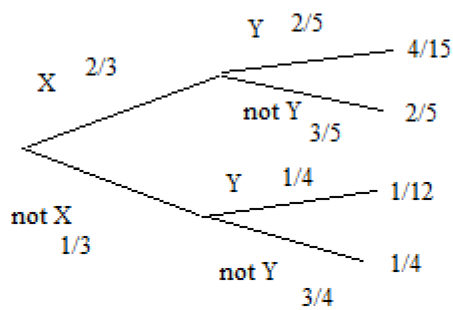
Probability #4 (11/19/2020 - Written Notes)

#6) Binomial Distribution with n = 6, p = .4

Probability hit exactly 4 times: $\binom{6}{4}.4^4.6^2 = 15\left(\frac{16}{625}\times\frac{9}{25}\right) = \frac{3\times16\times9}{3125} = \frac{432}{3125}$

Probability hit first time on third throw = .6 x .6 x .4 = .144

#7) Consider the following probability tree of the situation:



P(Y') = P(X' and Y') + P(X and Y') = 1/4 + 2/5 = (5+8)/20 = 13/20

P(X' or Y') = 1 - P(X and Y) = 1 - 4/15 = 11/15

10) a) 4/6 = 2/3 is probability Jack wins on the first throw

b) p(jack 5 or 6)*p(jill 1,2,3 or 4) = 1/3 * 2/3 = 2/9 is prob Jill wins on first throw

c) Let X be probability Jack wins.

Jack can win in two ways:

1) Win on first turn

2) Have both Jack and Jill not win, and then the game starts again.

X = 2/3 + (1/3)*(1/3)*X

X = 2/3 + X/9

X - X/9 = 2/3

8X/9 = 2/3

X = 3/4

Infinite series: $2/3 + (1/9)(2/3) + (1/9)^2(2/3) + (1/9)^3(2/3) + \ldots$

$$= \frac{\frac{2}{3}}{1-\frac{1}{9}} = \frac{3}{4}$$

11) $\int_0^2 x(\frac{1}{12})(8x - x^3)dx = \frac{1}{12}\int_0^2 (8x^2 - x^4) = \frac{1}{12}(\frac{8}{3}x^3 - \frac{x^5}{5})|_0^2 = \frac{16}{9} - \frac{8}{15} = \frac{56}{45}$

b) $\int_0^m \left(\frac{1}{12}\right)(8x - x^3)dx = .5$

$$\frac{m^2}{3} - \frac{m^4}{48} = \frac{1}{2}$$

$$16m^2 - m^4 = 24$$

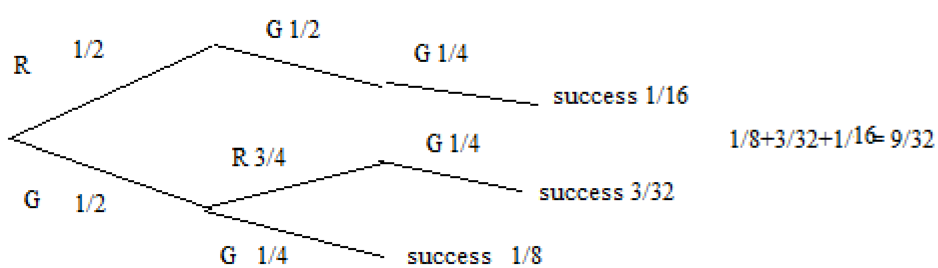
$$m^4 - 16m^2 + 24 = 0$$

c) mode is most common value, so find maximum value of function and what x equals there.

Take derivative: 2/3 - (1/4)x^2= 0 → $x = \frac{2\sqrt{2}}{3}$.

<u>Old AMC/AIME Problems</u>

1) Here is the tree diagram, the result is 9/32



2) Two dice. Labels are [1,2,4,4,5,6] and [1,2,3,3,5,6] P(sum is odd).

Slow way, make a 6 by 6 table and count how many values are odd.

Fast way, just recognize that our dice have [4 even, 2 odd] and [2 even, 4 odd] labels.

# of combinations = 6 x 6 = 36

# of combos that are odd = OE + EO = 2 x 2 + 4 x 4 = 4 + 16 = 20

Probability - 20/36 = **5/9**.

3) Sample space is (11 C 6), the number of ways to choose 6 items out of 11?

How many of these ways have a sum of values that is odd?

1 to 11 = 6 odd numbers, 5 even numbers.

We want combinations of	1 odd 5 even OR	(6 C 1)(5 C 5) = 6
	3 odd 3 even OR	(6 C 3)(5 C 3) = 20 x 10 = 200
	5 odd 1 even	(6 C 5)(5 C 1) = 6 x 5 = 30

Just like the senator question (How many committees of 3 women 5 men?) or the lottery question (how many ways to pick 3 correct #s and 3 incorrect #s?)

Final answer = 236/($_{11}C_6$) = 236/462 = **118/231**

4) Sample space = $_{10}C_6$, Count # of combos with 3 as the second smallest:

2 choices for smallest (1 or 2)

1 choice for 3

$_7C_4$ Choices for 4 numbers out of remaining 7

Total # of ways to do this is 2 x $_7C_4$= 2 x 35 = 70

$_{10}C_6$ = 210, so the desired probability is 70/210 = **1/3.**

5) Exponentiate 3… 3, 9, 7, 1, 3, 9, 7, 1, repeats every 4

Exponentiate 7…7, 9, 3, 1, 7, 9, 3, 1, repeats every 4

Sample space is 100 x 100, but really, this is 625 copies of the same 4 x 4 square since

The pattern repeats every 4. Let's just make this 4 by 4 square:

3^a 7^b	7	9	3	1
3	0	2	6	4
9	6	8	2	0
7	4	6	0	8
1	8	0	4	2

Answer = 3/16

6) Using the binomial distribution, what we want is

$$\sum_{k=0}^{25} \binom{50}{2k} \left(\frac{2}{3}\right)^{2k} (\frac{1}{3})^{50-2k}$$

This is an ugly sum ☹

Well, let's consider something that might look similar but different:

$$1 = (\frac{2}{3} + \frac{1}{3})^{50} = \sum_{k=0}^{50} \binom{50}{k} \left(\frac{2}{3}\right)^{k} (\frac{1}{3})^{50-k}$$

Even terms are identical in the two sums. Odd terms are missing in the first but evident in the second. What if we could subtract out the odd terms?

$$(\frac{1}{3})^{50} = (\frac{2}{3} - \frac{1}{3})^{50} = \sum_{k=0}^{50} \binom{50}{k} \left(\frac{2}{3}\right)^{k} (-\frac{1}{3})^{50-k}$$

In this different sum, the odd terms are subtracted and the even terms are added.

So, if we add both of these sums, the odd terms disappear and the even terms show up twice:

2S = $1 + (\frac{1}{3})^{50}$, where S is the desired probability.

Thus, the answer to the question is $\mathbf{\frac{1+(\frac{1}{3})^{50}}{2}}$**.**

7) What is the probability it will take more than 4 draws to remove all the shiny pennies?

Answer the opposite and subtract from 1? What is the probability all the shiny pennies will be removed within the first 4 draws?

SSSDDDD

Sample Space: # of permutatation SSSDDDD = 7!/(3!4!) = 35

SSSDDDD

DSSSDDD

SDSSDDD

SSDSDDD

All the ways all 3 S's are in the first four slots. So probability all shiny are gone by turn 4 is 4/35, so the answer to the question is 1 - 4/35 = **31/35.**

A more elegant way is to see that we can choose 3 slots out of the first four for the three S's in 4 C 3 ways.

8) {1,2,3,4,5} choose a,b,c randomly with replacement. What is the probability ab+c is even.

Sample space = 5 x 5 x 5 = 125

Ab = even AND c = even	(a=even and	b=even and	c = even,	2 x 2 x 2 = 8
	a=even,	b = odd	c = even	2 x 3 x 2 = 12
	a=odd	b= even	c= even)	3 x 2 x 2 = 12

Other way to do this question is for a and b we have 25 - 9 = 16 ways and 2 ways for c, so we get 16 x2 = 32 ways.

Ab = odd and c = odd (a = odd, b = odd, c = odd) 3 x 3 x 3 = 27

32 + 27 = 59 ways

So probability is 59/125.

9) Let A be a randomly chosen divisor of 10^{99}. What is the probability that A is a multiple of 10^{88}?

299599and has 100 x 100 = 10,000 divisors.

How many values of a, the exponent to 2, are 88 <= a <= 99 AND

How many values of b, the exponent to 5, are 88 <= b <= 99

So a can be one of 12 values and b can be one of 12 values. So there are 12 x 12 = 144 divisors of 10^{99} that are multiples of 10^{88}. Thus, our desired probability is 144/10000 = 9/625

10) Let p equal probability of getting heads on a single toss.

Probability of getting 1 H in 5 tosses : $5p(1-p)^4$, using binomial formula

Probability of getting 2 H in 5 tosses : $10p^2(1-p)^3$, using binomial formula

$5p(1-p)^4 = 10p^2(1-p)^3$

1-p = 2p

3p = 1, so p =1/3

Question: what is the probability of getting 3 H in 5 tosses:

$10(1/3)^3(2/3)^2= 40/243$

11) Sample space is 2^{10}. Now, count how many sequences of H's and T's have no consecutive H's.

Let f(n) = # of sequences of length n that have no consecutive H's.

F(1) = 2 (H, T)

F(2) = 3 (TH, TT, HT)

F(3) = 5 (THT, TTT, HTT, HTH, TTH)

F(4) = 8, F(5) = 13, F(6) = 21, F(7) = 34, F(8) = 55, F(9) = 89, F(10) = 144.

Now, consider F(n) for n > 2.

If I start with a valid sequence, I can always end it with TH. So, take all sequences of length n-2 and add TH to them, and this gives us sequences of length n that end in H.

If I start with a valid sequence, I can always end it with T. So take all sequences of length n-1 and add T to then, and this gives us sequences of length n that end in T.

F(n) = F(n-1) + F(n-2)

The desired probability is 144/1024 = **9/64.**

Longer way = count # of sequences with 0Hs, 1 H, 2 Hs, 3 Hs, 4Hs and 5 Hs.