Sum Rule

if $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B|$

Product Rule

$|A \times B| = |A||B|$

The number of ordered pairs (a, b), where a is selected from A and b is selected from B is simply the product of the sizes of the sets

	Taco	Burrito	Quesadilla	Mexican Pizza
Nachos	(N, T)	(N, B)	(N, Q)	(N, M)
Churro	(C, T)	(C, B)	(C, Q)	(C, M)
fries	(F, T)	(F, B)	(F, Q)	(F, M)

How many 3 letter strings with 2 consonants and 1 vowel

3 x 21 x 21 x 5

C C V

C V C

V C C 3 represents the position of the vowel.

For each vowel position, there are 5 choices for the vowel, 21 choices for the first consonant and 21 choices for the last consonant

Subtraction Principle

Count everything that you're NOT supposed to count and subtract that out of the total. Sometimes this is easier!

Division Principle

If you've counted everything you wanted to count k times exactly, then take your answer and divide by k.

Imagine counting how many handshakes there were at a party…

Ask each person, how many people's hands did you shake?

Add these numbers up.

A = 3   B = 2   C = 2   D = 4   E = 1

3 + 2 + 2 + 4 + 1 = 12

Each handshake involved 2 people. So, in the sum above, a single handshake gets represented with the number 2, because each person added 1 to their count.

Thus, the total number of handshakes is 12/2 = 6

Since D shook 4 hands

DA, DB, DC, DE, so E is done.

A shook 2 more hands, B and C shook 1 more hand

AB, AC

Whenever you do this, the count you get will be even…this is known as the handshaking lemma. Also, in a graph, if you ever sum the degrees of the vertices, you will always get an even number. And the edges represent handshakes…

Permutations

We can permute n distinct objects in n(n-1)(n-2)…(2)(1) = n! ways using the product rule.

If we are only permuting k of those objects, we can do this in

n(n-1)(n-2)…(n-k+1) ways.

$$\frac{n(n-1)(n-2)\ldots(n-k+1) * (n-k)(n-k-1)\ldots*1}{(n-k)(n-k-1)\ldots*1} = \frac{n!}{(n-k)!} = {}_nP_k$$

Permutation with a Repeated Character

If I have a repeated letter (BEER) example, then there are two permutations which are really the same: B E1 E2 R and B E2 E1 R are counted differently in the 4!, but should not be. In fact, this is true for 12 pairs of permutations.

More generally, let's say we had some letter A appear k times out of n letters and the rest of the letters were distinct.

How many times did we count this: L R Q A1 G A2 U … Ak?

We can reorder all of the As and each one of those is counted. The number of orderings of the A's is k!

Basically, we've overcounted every permutation k! times. So, just take our answer and divide by k! We can repeat this argument for each repeated letter.

If we have f_1 of L_1, f_2 of $L_2 \ldots$, f_k of L_k, then the number of permutations is

$$\frac{(\sum_{i=1}^{k} f_i)!}{\prod_{i=1}^{k} f_i!}$$

Combinations

# of subsets of a set of size n is 2^n, because we can take each of the n items, and for each we have 2 choices: in the set or not in the set:

A	B	C	
0	0	0	empty
0	0	1	{C}
0	1	0	{B}
1	0	0	{A}
0	1	1	{B,C}
1	0	1	{A,C}
1	1	0	{A,B}
1	1	1	{A, B, C}

How many subsets are of size k of a set of size n?

Imagine picking 5 items of 8, say A, B, C, D, E, F, G, H

We 8 choices for the first, 7 choices for the second, 6 choices for the third, 5 choices for the fourth, 4 choices for the fifth: BHAEC

But, this is overcounting, because we could select the same subset in the order CEHBA. In fact, we could select the same subset in exactly 5! ways, any of the orderings of the items.

So the answer is $\frac{8\times7\times6\times5\times4}{5!} = \frac{8\times7\times6\times5\times4\times3!}{5!3!} = \frac{8!}{5!3!} = \binom{8}{5}$

More generally, we have $\frac{n\times(n-1)\ldots\times(n-k+1)}{k!} = \frac{n\times(n-1)\ldots\times(n-k+1)\times(n-k)!}{k!(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k} = {}_nC_k$

How many permutations of TENNESSEE are there?

9 letters (4 Es, 2 Ns, 2S, 1 T) = $\frac{9!}{4!2!2!}$

How many permutations of TENNESSEE start and end with a vowel?

E __ __ __ __ __ __ __ E (7 letters, 2 Es, 2S, 1T, 2 Ns) = $\frac{7!}{2!2!2!}$

How many permutations of TENNESSEE start and end with a consonant?

Case 1: N … T, T … N, S… T, T..S (7 letters, 4 Es, 2 of 1 Let, 1 of 1 Let) = $\frac{7!}{4!2!}$

Case 2: N … N, S … S (7 letters, 4 Es, 1 T, 2 of 1 Let) = $\frac{7!}{4!2!}$

Case 3: S… N, N..S, (7 letters, 4 Es, 1 N, 1S, 1T) = $\frac{7!}{4!}$

Final answer = $6 \times \frac{7!}{4!2!} + 2 \times \frac{7!}{4!} = \frac{7!}{4!}(3 + 2) = 5 \times 5 \times 6 \times 7 = 1050$

How many permutations do NOT have consecutive vowels?

__ T __ N __ N __ S __ S __

1. Choose 4 slots out of 6 slots for the vowels. We can do this in $\binom{6}{4}$ ways.

2. Permute the consonants in $\frac{5!}{2!2!}$ ways.

We can pair any of the vowel slots with any of the consonant permutations, so we multiply to get

$$\binom{6}{4} \times \frac{5!}{2!\,2!} = 15 \times 30 = 450$$

Word: CONTINUOUSLY (5 vowels 2Os, 2Us 1I, 7 consonants (C, 2Ns, T,S, L, Y)

1. With 7 consonants there are 8 slots to choose to place the 5 vowels, we can do this $\binom{8}{5}$ ways.

__ C __ N __ N __ T __ S __ L __ Y __

2. We can permute the vowels in $\frac{5!}{2!2!}$ ways.

3. Then I can permute the consonants in $\frac{7!}{2!}$ ways.

Final Answer = $\binom{8}{5} \times \frac{5!}{2!2!} \times \frac{7!}{2!}$.

We must choose 7 out of 10. We must chosoe at least 4 out of the first 6.

Subtraction Principle: How many ways can we select 3 or fewer of the first 6.

<u>Subtraction</u>

Can we select only 2 of the first 6? **NO - because we need 7 out of 10.**

Choose 3 out of 6 in $\binom{6}{3}$ ways…the rest are forced!!!

So the final answer is simply $\binom{10}{7} - \binom{6}{3} = \frac{10\times9\times8}{6} - \frac{6\times5\times4}{6} = 120 - 20 = 100$

<u>Addition</u>

Choose 4 of 6 and 3 of 4	$\binom{6}{4}\binom{4}{3} = 15 \times 4 = 60$
Choose 5 of 6 and 2 of 4	$\binom{6}{5}\binom{4}{2} = 6 \times 6 = 36$
Choose 6 of 6 and 1 of 4	$\binom{6}{6}\binom{4}{1} = 1 \times 4 = 4$

60 + 36 + 4 = 100