Summation

Prove for all positive integers n, that $\sum_{i=1}^{n}(-1)^{i-1}i^2 = \frac{n(n+1)(-1)^{n-1}}{2}$.

Base case: n = 1, LHS = $\sum_{i=1}^{1}(-1)^{i-1}i^2 = (-1)^0 1^2 = 1$, RHS = $\frac{1(1+1)(-1)^{1-1}}{2} = 1$

The formula holds for n=1.

Inductive Hypothesis: Assume for an arbitrary positive integer n = k, where k is an arbitrarily chosen positive integer 2 or greater, that $\sum_{i=1}^{k}(-1)^{i-1}i^2 = \frac{k(k+1)(-1)^{k-1}}{2}$.

Inductive Step: Prove for n = k+1 that $\sum_{i=1}^{k+1}(-1)^{i-1}i^2 = \frac{(k+1)(k+2)(-1)^{k}}{2}$.

$$\sum_{i=1}^{k+1}(-1)^{i-1}i^2 = (\sum_{i=1}^{k}(-1)^{i-1}i^2) + (-1)^k(k+1)^2$$

$$= \frac{k(k+1)(-1)^{k-1}}{2} + (-1)^k(k+1)^2 \text{, using IH}$$

$$= \frac{k(k+1)(-1)^{k-1}}{2} + \frac{2}{2}(-1)^k(k+1)^2$$

$$= \left(\frac{k+1}{2}\right)[\frac{k(-1)^{k-1}}{1} + \frac{2}{1}(-1)^k(k+1)\quad]$$

$$= \left(\frac{k+1}{2}\right)(-1)^{k-1}[k-2(k+1)]$$

$$= \left(\frac{k+1}{2}\right)(-1)^{k-1}[-k-2]$$

$$= \left(\frac{k+1}{2}\right)(-1)^{k-1}(-1)[k+2]$$

$$= \left(\frac{(k+1)(k+2)}{2}\right)(-1)^k$$

Inequality – emphasis on the freedom in inequality steps and the need to use it

Prove using induction on n that for all positive integers n, $\sum_{i=1}^{n}\frac{i}{i+1} \leq \frac{n^2}{n+1}$.

Base case: n = 1, LHS $=\sum_{i=1}^{1}\frac{i}{i+1} = \frac{1}{2}$, RHS = $\frac{1^2}{1+1} = \frac{1}{2}$, thus the inequality holds for n=1.

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = k that

$$\sum_{i=1}^{k}\frac{i}{i+1} \leq \frac{k^2}{k+1}.$$

Inductive Step: Prove for n = k+1 that

$$\sum_{i=1}^{k+1}\frac{i}{i+1} \leq \frac{(k+1)^2}{k+2}$$

$$\sum_{i=1}^{k+1}\frac{i}{i+1} = \left(\sum_{i=1}^{k}\frac{i}{i+1}\right) + \frac{k+1}{k+2}$$

$$\leq \frac{k^2}{k+1} + \frac{k+1}{k+2}, \text{ using the IH.}$$

$$= \frac{k(k)(k+2) + (k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k^3 + 2k^2 + k^2 + 2k + 1}{(k+1)(k+2)}$$

$$= \frac{k^3 + 3k^2 + 2k + 1}{(k+1)(k+2)}$$

$$\leq \frac{k^3 + 3k^2 + 3k + 1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^3}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+2)}$$

Strong Induction Example

Prove, using strong induction on n with 3 base cases that if and only if 3 | n, then 2 | F_n, where F_n denotes the nth Fibonacci Number. (F_0=0, F_1=1, $F_n = F_{n-1} + F_{n-2}$, for all ints n >= 2) Prove for all non-negative integers n.

Base cases: n=0,1, F_0=0 and is divisible by 3, but F_1 = 1, and isn't divisible by 3 so the base case holds for n=0 and 1.

Inductive hypothesis: Assume all non-negative integers n <= k, where k is an arbitrarily selected positive integer 1 or greater that if and only if 3 | n, then 2 | F_n.

Inductive Step: Prove for n = k+1 that if 3 | (k+1), then 2 | F_{k+1}, otherwise if 3 | (k+1) is false, then 2 | F_{k+1} is false.

$F_{k+1} = F_k + F_{k-1}$

Cases: (1) k+1 is divisible by 3	(2), k+1 = 1 mod 3	(3) k+1 = 2 mod 3
This means k, k-1 are	k = 0 mod 3	k = 1 mod 3
Not divisible by 3	k-1 = 2 mod 3	k-1 = 0 mod 3
By the IH, F_k is NOT even	F_k = even, using IH	F_k odd using IH
F_{k-1} is also not even.	F_{k-1} = odd, using IH	F_{k-1} even using IH
Since both are odd	F_k = 2a, F_{k-1} = 2b+1	F_k = 2a+1, F_{k-1} = 2b
Let F_k = 2a+1,	for ints a, b	for ints a,b
F_{k-1} = 2b+1 for some		
Integers a and b		
= (2a+1)+(2b+1)	= 2a + 2b + 1	= 2a + 1 + 2b
= 2(a+b+1)	= 2(a+b) + 1	= 2(a+b) + 1
Since a,b are ints, a+b+1 is	Since a,b are ints	Since a and b are ints
An int and F_{k+1} is even	F_{k+1} is odd as desired	F_{k+1} is odd as desired.

Note: Steps in blue are why strong induction with 2 base cases is necessary.

Strong Induction Example: NIM

2 players, 2 piles of stones. On a single turn a player picks one pile and takes 1 or more stones from that pile. The winner is the last person to take a stone.

A	B	
3	2	Player Melia takes 2 stones from pile A
1	2	Player Sandra takes 1 stone from pile B
1	1	Player Melia takes 1 stone from pile B
1	0	Player Sandra takes 1 stone from pile A
0	0	Melia can't play, so Sandra wins!

Using strong induction on n, prove that if 2 players play NIM with two piles initially with n stones each and both players play optimally, player 2 will win.

Base case: n=1, Player 1 is forced to take 1 stone from either pile, so player 2 can respond by taking 1 stone from the other pile for the win.

Inductive hypothesis: Assume for all integers n <= k, where k is an arbitrarily chosen positive integer, that if 2 players play NIM with n stones in each pile, player 2 will win.

Inductive step: Prove for n=k+1 that if 2 players play NIM with k+1 stones in each pile, player 2 will win.

Player 1 goes and there are k+1 stones in both piles. No matter which pile she takes stones from, one pile will remain with k+1 stones and the other pile will have k+1-a stones, where a is in between 1 and k+1, inclusive. Now, player 2 will just take a stones from the larger pile, leaving the game with k+1-a stones in both piles. This is now a game of NIM for which the inductive hypothesis applies, and Player 2 wins this game according to the hypothesis. This means that the inductive step is proven!

11	11
5	11
5	5 (Player 2 can always mirror the move!!!)

Who wins if the piles are unequal at the beginning? Player 1→ they will just get the piles to be equal and Player 2 will be forced to go next!

Incidentally, if there are more than 2 piles, the second player wins if and only if the bitwise XOR of the number of stones in each pile is 0.

Last Problem – Binary Billy

Binary Billy's punishment is to write all the binary numbers upto n bits. So if $n = 3$, he has to write: 0, 1, 10, 11, 100, 101, 110, 111.

Let $B(n)$ represent the number of bits he has to write if he writes all the numbers upto a full n bit representation. Using induction on n, prove that $B(n) = (n-1)2^n + 2$ for all positive integers n.

Base case: $n = 1$, When $n=1$, binary billy writes 0, 1, which is 2 bits. So $B(1) = 2$

$$\text{RHS} = (1-1)2^1 + 2 = 2$$

Inductive Hypothesis: Assume for an arbitrarily chosen integer $n = k$ that

$$B(k) = (k-1)2^k + 2$$

Inductive step: Prove for $n = k+1$ that $B(k+1) = ((k+1)-1)2^{k+1} + 2 = k2^{k+1} + 2$

Try calculating $B(k+1)$…

Step 1: first Binary billy writes all the numbers upto k bits.

Step 2:

Then he writes

100000000 (1 followed by k 0s)

100000001, etc.

111111111 (last number being k+1 1s)

In all, he writes 2^k numbers, each of which have k+1 bits in them.

By the inductive hypothesis, he writes $(k-1)2^k + 2$ bits in step 1. And we know he writes $2^k(k+1)$ bits in step 2.

In total he writes $(k-1)2^k + 2 + 2^k(k+1) = 2^k(k-1+k+1) + 2 = 2^k(2k) + 2 = 2^{k+1}k + 2$. This is what we were trying to prove, so the inductive step is finished.

$$\begin{aligned}
B(k+1) &= B(k) + 2^k(k+1) \\
&= (k-1)2^k + 2 + 2^k(k+1), \text{ using IH} \\
&= 2^k(k-1+k+1) + 2 \\
&= 2^k(2k) + 2 \\
&= 2^{k+1}k + 2
\end{aligned}$$