The nth Harmonic number is 1 + 1/2 + 1/3 + … +1/n, in summation, we have $H_n = \sum_{i=1}^{n} \frac{1}{i}$.

1) Prove $\sum_{i=1}^{n} H_i =$ **(n+1)H_n – n, using induction. Note that H_n =** $\sum_{i=1}^{n} \frac{1}{i}$. Prove this for all positive integers n.

Use induction on n>0.

Base case: n=1. LHS = 1/1 = 1
RHS = (1+1)(1/1) – 1 = 1
Thus, the assertion holds for n=1 and the base case is true.

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = k that

$$\sum_{i=1}^{k} H_i = (k+1)H_k - k$$

Inductive step: Under this assumption, we must prove the formula for n = k+1:

$$\sum_{i=1}^{k+1} H_i = (k+2)H_{k+1} - (k+1)$$

$$\sum_{i=1}^{k+1} H_i = \left(\sum_{i=1}^{k} H_i\right) + H_{k+1}$$

= (k+1)H_k – k + H_{k+1}, using inductive hypothesis.

This is a mismatch!!! Nothing to factor out.
We would really love for that H_k to be an H_{k+1}.
But…we can't just randomly change stuff…
Instead, we must investigate the relationship between H_k and H_{k+1}…

H_k = 1 + 1/2 + 1/3 + … + 1/k
H_{k+1} = 1 + 1/2 + 1/3 + … 1/k + 1/(k+1)
H_{k+1} = H_k + 1/(k+1)
H_k = H_{k+1} - 1/(k+1)

= (k+1)(H_{k+1} – 1/(k+1)) – k + H_{k+1}
= (k+1)H_{k+1} – 1 – k + H_{k+1}
= (k+2) H_{k+1} – (1+k), which completes the induction.

Thus, we have shown $\sum_{i=1}^{n} H_i =$ (n+1)H_n – n, for all positive integers n.

<u>**Inequalities (and sums not to n)**</u>

Prove using induction on n, for all positive integers n,

$$\sum_{i=1}^{2^n} log_2 i \leq (n-1)2^n + 1.$$

Base case n=1: LHS = $\sum_{i=1}^{2^1} log_2 i = log_2 1 + log_2 2 = 0 + 1 = 1$
RHS = $(1-1)2^1 + 1 = 0 + 1 = 1.$

Inductive hypothesis: assume for an arbitrarily chosen positive integer n = k that

$$\sum_{i=1}^{2^k} log_2 i \leq (k-1)2^k + 1.$$

Inductive step: Prove for n = k+1 that

$$\sum_{i=1}^{2^{k+1}} log_2 i \leq \left((k+1)-1\right)2^{k+1} + 1 = k2^{k+1} + 1.$$

Couple observations: IS sum has lots more terms than the IH sum…in fact, twice as many…also, the sum on the left is irrational.

$$\sum_{i=1}^{2^{k+1}} log_2 i = (\sum_{i=1}^{2^k} log_2 i) + (\sum_{i=2^k+1}^{2^{k+1}} log_2 i)$$

$$\leq (k-1)2^k + 1 + \sum_{i=2^k+1}^{2^{k+1}} log_2 i$$

$$\leq (k-1)2^k + 1 + \sum_{i=2^k+1}^{2^{k+1}} log_2 2^{k+1}$$

Why can I do this?
Answer for any set of n numbers S, the sum of those numbers is less than or equal to n*max(S). {3, 2, 7, 9, 1} 3+2+7+9+1 <= 9+9+9+9+9 = 5*9. 3+2+7+9+1 <= 1+1+1+1+1, sum(S) >= n*min(S)

So, since log is an increasing function, if I take the log of a list of consecutive numbers, the log of the last number will be the largest.

$$= (k-1)2^k + 1 + \sum_{i=2^k+1}^{2^{k+1}} (k+1)$$

$$= (k-1)2^k + 1 + (k+1)2^k$$

The number of integers from a to b is equal to b - a + 1. For this case, we have $2^{k+1} - (2^k+1) + 1 = 2^{k+1} - 2^k - 1 + 1 = 2^{k+1} - 2^k = 2(2^k) - 2^k = 2^k(2-1) = 2^k$.

$$= (2k)2^k + 1$$
$$= (k)2^{k+1} + 1$$

This completes the inductive step. Thus, we can conclude for all positive integers n that $\sum_{i=1}^{2^n} log_2 i \leq (n-1)2^n + 1$.

Inequalities (and sums not to n) - Example 2

Use induction on n to prove the following inequality for all positive integers n:

$$\sum_{i=1}^{n^2} \sqrt{i} \leq \frac{n(n+1)(4n-1)}{6}$$

Base Case: n=1. LHS =$\sum_{=1}^{1^2} \sqrt{i}$= √(1) = 1
RHS = 1(1+1)(4(1) – 1)/6 = 1
So the assertion is true for n = 1 and the base case holds.

Inductive hypothesis: Assume for an arbitrary positive integer n=k that

$$\sum_{i=1}^{k^2} \sqrt{i} \leq \frac{k(k+1)(4k-1)}{6}$$

Inductive Step: Under this assumption, prove for n=k+1 that

$$\sum_{i=1}^{(k+1)^2} \sqrt{i} \leq \frac{(k+1)((k+1)+1)(4(k+1)-1)}{6} = \frac{(k+1)(k+2)(4k+3)}{6}$$

Note: it's very important to plug in the k+1 in parentheses, as this example shows.

$$\sum_{i=1}^{(k+1)^2} \sqrt{i} = \sum_{i=1}^{k^2} \sqrt{i} + \sum_{i=k^2+1}^{(k+1)^2} \sqrt{i}$$

We stop the sum at k^2 because that is what your IH applies to. But, this means there are quite a few terms leftover that we didn't express, so we have to put those in a second sum. The next integer after k^2 is k^2+1, so this is where that second sum must start.

$$\leq \frac{k(k+1)(4k-1)}{6} + \sum_{i=k^2+1}^{(k+1)^2} \sqrt{i}$$ **, using the inductive hypothesis.**

$$\leq \frac{k(k+1)(4k-1)}{6} + \sum_{i=k^2+1}^{(k+1)^2} \sqrt{(k+1)^2}$$ **,**

because each term in the summation is less than or equal to that last term when i = (k+1)^2.

$$= \frac{k(k+1)(4k-1)}{6} + \sum_{i=k^2+1}^{(k+1)^2} (k+1)$$

$$= \frac{k(k+1)(4k-1)}{6} + [(k+1)^2 - k^2](k+1)$$ **,**

because the first summation from the IH contains k^2 terms while the summation from the IS contains (k+1)^2, leaving the difference for this summation.

$$= \frac{k(k+1)(4k-1)}{6} + (2k+1)(k+1)$$

$$= \frac{(k+1)}{6}\left(k(4k-1) + 6(2k+1)\right)$$

$$= \frac{(k+1)}{6}\left(4k^2 - k + 12k + 6\right)$$

$$= \frac{(k+1)}{6}\left(4k^2 + 11k + 6\right)$$

$$= \frac{(k+1)(k+2)(4k+3)}{6}$$

Divisibility

Prove using induction on n, for all non-negative integers n, that

$$9 \mid (2^{2n} + 6n - 1).$$

Base case: n=0, $2^{2(0)} + 6(0) - 1 = 1 + 0 - 1 = 0$, $0 = 9 \times 0$, so $9 \mid 0$ and the base case is proven. Thus the assertion holds for n = 0.

Inductive hypothesis: assume for an arbitrarily chosen non-negative integer n = k that

$$9 \mid (2^{2k} + 6k - 1)$$

This means we are assuming that there is an integer c such that

$$2^{2k} + 6k - 1 = 9c$$

Inductive step: Prove for n = k+1 that

$$9 \mid (2^{2(k+1)} + 6(k+1) - 1)$$

We must prove there exists some integer d such that

$$2^{2(k+1)} + 6(k+1) - 1 = 9d$$

$2^{2(k+1)} + 6(k+1) - 1 = 2^{2k+2} + 6k + 6 - 1$

$= 2^2 2^{2k} + 6k + 6 - 1$

$= 4(2^{2k}) + 6k + 6 - 1$

$= 4(2^{2k}) + 6k + 6 - 1 + 18k - 18k$

$= 4(2^{2k}) + 24k + 6 - 1 - 18k - 4 + 4$

$= 4(2^{2k}) + 24k - 4 + (6 - 1 - 18k + 4)$

$= 4[(2^{2k}) + 6k - 1] + (9 - 18k)$

$= 4[9c] + (9 - 18k)$, using the IH

$= 9[4c + 1 - 2k]$,

Since c and k are integers, it follows that 4c + 1 - 2k is an integer as well, so we've proven that $9 \mid 2^{2(k+1)} + 6(k+1) - 1$, proving the inductive step. It follows that the given assertion is true for all non-negative integers n.

Random Problem
Prove for all positive integers n, that a $2^n \times 2^n$ region with one unit square removed can be tiled with trominos. (A tromino is an L shaped tile of three unit squares.)

Base case: n = 1: Consider tiling a $2^1 \times 2^1$ region with one unit square remove. This will always have 3 squares (4 - 1) that are in an L shape, so we can fit exactly one tromino into the design.

Inductive hypothesis: Assume for an arbitrary positive integer n=k that we can tile a $2^k \times 2^k$ region with one unit square missing with trominos.

Inductive step: Prove for n = k+1 that we can tile a 2^{k+1} x 2^{k+1} region with one square missing.

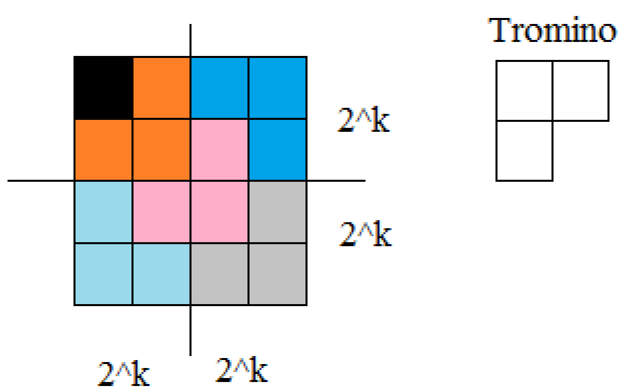
So, we can always split our picture into four squares of size 2^k x 2^k.
One of these four quadrants will have a hole in it. So, we use the inductive hypothesis to tile that quadrant.

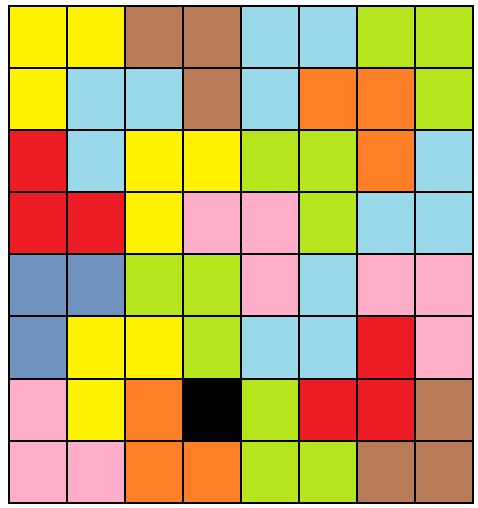
BUT WE HAVE A PROBLEM! WE HAVE THREE UNTILED QUADRANTS, NONE OF WHICH HAVE HOLES!

So, what we'll do is place a single tromino in the center, which is guaranteed to have one "covered" square and 3 that haven't been taken care of!

So now, all of the other three quadrants have a "hole" (a tile), so we can apply the inductive hypothesis to tile all three of those quadrants and we're done!

Here are two examples of tilings using this proof, the hole is the single black square in each example:





<u>Strong Induction</u>

Key differences:

1) We usually have more than 1 base case.

2) Instead of assuming that the formula is true for an arbitrary integer n = k, we assume that the formula is true **<u>FOR ALL</u>** integers n <= k, where k is an arbitrarily chosen integer.

We try to prove the following:

$$f(1) \wedge f(2) \wedge f(3) \wedge f(4) \wedge \ldots \wedge f(k) \rightarrow f(k+1)$$

It turns out that strong induction is logically equivalent to regular induction… The reason we usually need to use strong induction is that when we prove the IS, we end up having to assume that the statement is true for a few previous cases, not just one.

Chicken Nugget Problem - Strong Induction

Prove for all integers $n \geq 12$, using strong induction with 4 base cases, that if you can buy 4 packs of chicken nuggets and 5 packs of chicken nuggets that you can buy exactly n chicken nuggets.

Base case(s):

n = 12, buy three 4 packs

n = 13, buy two 4 packs, one 5 pack

n = 14, buy one 4 pack, two 5 packs

n = 15, buy three 5 packs

So, the statement is true for n = 12, 13, 14, 15

Inductive hypothesis: Assume for all n, $12 \leq n \leq k$, where k is an arbitrarily chosen integer that is at least 15, that you can always buy exactly n chicken nuggets using 4 and 5 packs.

Inductive step: Prove, for n = k+1 that you can buy exactly n chicken nuggets.

We are trying to buy k+1 nuggets, go ahead and buy one four pack, now we need k-3 nuggets. Since k >= 15, k-3 >= 12, and we can apply the inductive hypothesis and buy exactly k-3 nuggets and we have proven the inductive step.