Induction

Inequality example…

Before we get started… To prove A = B we do

A = A1

= A2

= A3

…

= B

Every step is an equals step.

To prove A >= B, we can do

A = A1

= A2

>= A3

…

= B (we are allowed both = steps and >= and >, but if we have the last one we've proved A>B)

Pretend I want to prove that I am a better basketball player than you!

I am just as good at basketball as Samantha

Samatha is either as good or better than Tom

Tom is the same level that you are.

What you can't do is an opposite direction step.

You are worse than Sarah.

A = B >= C = D < E there is no way to relate E to either A, B or C.

Replace Sarah with Lebron James, and hopefully you see my point!

Prove for all integers n >= 2 that $\sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}$.

Base case n = 2: LHS = $\sum_{i=1}^{2} \frac{1}{i^2} = 1 + \frac{1}{4} = \frac{5}{4}$, RHS = $2 - \frac{1}{2} = \frac{3}{2}$. Since 5/4 < 3/2, the base case holds and the given assertion is true for n = 2.

Inductive hypothesis: Assume for an arbitrarily chosen integer n = k, where k >= 2 that

$$\sum_{i=1}^{k} \frac{1}{i^2} < 2 - \frac{1}{k}.$$

Inductive step: Prove for n=k+1 that

$$\sum_{i=1}^{k+1} \frac{1}{i^2} < 2 - \frac{1}{k+1}.$$

$$\sum_{i=1}^{k+1} \frac{1}{i^2} = \left(\sum_{i=1}^{k} \frac{1}{i^2}\right) + \frac{1}{(k+1)^2}$$

$$< 2 - \frac{1}{k} + \frac{1}{(k+1)^2}, \text{ using IH}$$

$$= 2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2}\right)$$

$$= 2 - \left(\frac{(k+1)^2 - k}{k(k+1)^2}\right)$$

$$= 2 - \left(\frac{k^2 + 2k + 1 - k}{k(k+1)^2}\right)$$

$$= 2 - \left(\frac{k^2 + k + 1}{k(k+1)^2}\right)$$

$$< 2 - \left(\frac{k^2 + k}{k(k+1)^2}\right)$$

$$= 2 - \left(\frac{k(k+1)}{k(k+1)^2}\right)$$

$$= 2 - \frac{1}{k+1}$$

Let M = $\begin{bmatrix} a & 0 \\ 1 & 1 \end{bmatrix}$, where a is a positive constant not equal to 1. Prove, using induction on n, for all positive integers n that $M^n = \begin{bmatrix} a^n & 0 \\ \frac{a^n-1}{a-1} & 1 \end{bmatrix}$.

Base case: n = 1, LHS = $\begin{bmatrix} a & 0 \\ 1 & 1 \end{bmatrix}$, RHS = $\begin{bmatrix} a^1 & 0 \\ \frac{a^1-1}{a-1} & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 1 & 1 \end{bmatrix}$. So the equation holds for n=1 and the base case is true.

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = k that:

$M^k = \begin{bmatrix} a^k & 0 \\ \frac{a^k-1}{a-1} & 1 \end{bmatrix}$.

Inductive step: Prove for n=k+1 that:

$M^{k+1} = \begin{bmatrix} a^{k+1} & 0 \\ \frac{a^{k+1}-1}{a-1} & 1 \end{bmatrix}$.

$$\begin{aligned} M^{k+1} &= M^k M \\ &= \begin{bmatrix} a^k & 0 \\ \frac{a^k-1}{a-1} & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 1 & 1 \end{bmatrix}, \text{ using IH} \\ &= \begin{bmatrix} a^k(a) + 0(1) & 0(a^k) + 0(1) \\ a\left(\frac{a^k-1}{a-1}\right) + 1 & 0\left(\frac{a^k-1}{a-1}\right) + 1 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} a^{k+1} & 0 \\ \left(\frac{a^{k+1}-a}{a-1}\right) + \frac{a-1}{a-1} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^{k+1} & 0 \\ \left(\frac{a^{k+1}-1}{a-1}\right) & 1 \end{bmatrix}$$

This completes the inductive step. Thus we can conclude for all positive integers n that

$$M^n = \begin{bmatrix} a^n & 0 \\ \frac{a^n-1}{a-1} & 1 \end{bmatrix}.$$

Prove DeMoivre's Thrm for non-negative integers n: $(cos\theta + isin\theta)^n = \cos(n\theta) + isin(n\theta)$.

We will prove the assertion by doing induction on n. (Note that we do NOT do induction on theta and just treat theta as an arbitrary angle the whole time.)

Base case: n=0, LHS = $(cos\theta + isin\theta)^0 = 1$, RHS = $\cos(0 * \theta) + isin(0 * \theta) = 1 + 0i = 1$, so the formula is true for n=0.

Inductive hypothesis: Assume for an arbitrarily chosen non-negative integer n=k that

$$(cos\theta + isin\theta)^k = \cos(k\theta) + isin(k\theta).$$

Inductive step: Prove for n=k+1 that

$$(cos\theta + isin\theta)^{k+1} = \cos((k+1)\theta) + isin((k+1)\theta).$$

$$(cos\theta + isin\theta)^{k+1} = (cos\theta + isin\theta)^k(cos\theta + isin\theta)$$

$$= (\cos(k\theta) + isin(k\theta))(cos\theta + isin\theta)$$

$$= \cos(k\theta)\, cos\theta + isin\theta \cos(k\theta) + icos\theta \sin(k\theta) - sin\theta \sin(k\theta)$$

$$= (\cos(k\theta)\, cos\theta - \sin(k\theta)\, sin\theta) + i(sin\theta \cos(k\theta) + cos\theta \sin(k\theta))$$

$$= \cos(k\theta + \theta) + isin(k\theta + \theta)$$

$$= \cos((k+1)\theta) + isin((k+1)\theta)$$

Thus, we've proved the inductive step and can conclude for all non-negative integers n that

$$(cos\theta + isin\theta)^n = \cos(n\theta) + isin(n\theta).$$

Define the sequence T(n) as follows:

T(1) = 2, for all integers n > 1, T(n) = 2nT(n-1). Prove, using induction on n that for all positive integers n, $T(n) = 2^n n!$

Base case: n=1, LHS = T(1) = 2, RHS = $2^1 1! = 2$, so the statement is true for n=1 and the base case holds.

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n=k that $T(k) = 2^k k!$

Inductive step: Prove for n=k+1 that $T(k+1) = 2^{k+1}(k+1)!$

T(k+1) = 2(k+1)T(k), using the definition of T.

= $2(k+1)2^k k!$, using IH

= $2^{k+1}(k+1)!$

This proves the inductive step. We can conclude for all positive integers n that T(n) = $2^n n!$

Using the product rule and the fact that the derivative of f(x) = 1 is 0 and the derivative of f(x) = x is 1, prove for all positive integers n that $\frac{d}{dx}x^n = nx^{n-1}$.

Base case: n=1, LHS = $\frac{d}{dx}x^1 = 1$, RHS = $1x^{1-1} = 1(1) = 1$. Thus, the base case holds.

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n=k that $\frac{d}{dx}x^k = kx^{k-1}$.

Inductive step: Prove for n=k+1 that $\frac{d}{dx}x^{k+1} = (k+1)x^{k+1-1} = (k+1)x^k$.

$$\frac{d}{dx}x^{k+1} = \frac{d}{dx}(x^k x)$$

$$= x^k\left(\frac{d}{dx}x\right) + x(\frac{d}{dx}x^k)$$

$= x^k(1) + x(kx^{k-1})$, using the IH

$$= x^k(1) + (kx^k)$$

$$= x^k(1) + (kx^k)$$

$$= (k+1)x^k$$

This proves the inductive step. We can conclude for all positive integers n that $\frac{d}{dx}x^n = nx^{n-1}$.