Let a = $\prod_{p_i \in Primes} {p_i}^{a_i}$, and b= $\prod_{p_i \in Primes} {p_i}^{b_i}$.

a = $2^3 3^6 5^2 11$, b = $2^5 3^4 5^7 7$, gcd(a, b) = $2^3 3^4 5^2$

gcd(a, b) = $\prod_{p_i \in Primes} {p_i}^{min(a_i, b_i)}$.

Now, let's consider the least common multiple of a and b.

lcm(a, b) where a and b are positive integers is the smallest positive integer n such that a | n and b | n.

lcm(12, 18) = 36, since 12 | 36, 18 | 36.

A = $2^3 3^6 5^2 11$, B = $2^5 3^4 5^7 7$, lcm(a, b) = $2^5 3^6 5^7 7^1 11^1$

lcm(a, b) = $\prod_{p_i \in Primes} {p_i}^{max(a_i, b_i)}$.

$$\begin{aligned}
\text{gcd(a, b) x lcm(a ,b)} &= \prod_{p_i \in Primes} {p_i}^{min(a_i, b_i)} \times \prod_{p_i \in Primes} {p_i}^{max(a_i, b_i)} \\
&= \prod_{p_i \in Primes} {p_i}^{min(a_i, b_i)} \times {p_i}^{max(a_i, b_i)} \\
&= \prod_{p_i \in Primes} {p_i}^{min(a_i, b_i) + \max(a_i, b_i)} \\
&= \prod_{p_i \in Primes} {p_i}^{a_i + b_i} \\
&= \prod_{p_i \in Primes} {p_i}^{a_i} \times {p_i}^{b_i} \\
&= \prod_{p_i \in Primes} {p_i}^{a_i} \times \prod_{p_i \in Primes} {p_i}^{b_i} \\
&= ab
\end{aligned}$$

lcm(a,b) = $\frac{ab}{\gcd(a,b)}$

Proof of an Infinite # of Primes

Proof by Contradiction.

Assume that there are a finite # of primes.
If there are, we can list them all. Let the list be p_1, p_2,…, p_n, where n is some fixed integer.

Consider the integer $X = p_1p_2p_3\ldots p_n + 1$. Either this number is prime, or it is the product of prime factors.

Is this integer divisible by p_1? Remainder when X is divided by p_1 is 1, so it is not divisible.

Does this argument work for all the other primes on the list? Yes…

So, we have created a new integer that isn't divisible by any of the primes on our list. But, either the number itself is prime or there is a prime we didn't put on our list that divides evenly into X. Either way, we have contradicted the fact that my list was complete.

Counting the Number of Divisors of an integer via its Prime Factorization

$n = 2^3 3^7 5^1 11^1$

We know any divisor must take the form $2^a 3^b 5^c 11^d$, where

$0 \le a \le 3$, $0 \le b \le 7$, $0 \le c \le 1$, $0 \le d \le 1$.

How many different solutions (a, b, c, d) of integers satisfy the requirements above?

Well, a can be one of 4 values (0, 1, 2, 3)
B can be one of 8 values (0, 1, 2, 3, 4, 5, 6, 7)
C can be one of 2 values (0 ,1)

D can be one of 2 values (0, 1)

$2^0 3^0 5^0 11^0$, $2^0 3^0 5^0 11^1$, $2^0 3^0 5^1 11^0$, $2^0 3^0 5^1 11^1$, (all combos a=0,b=0)

Total number of solutions is just 4 x 8 x 2 x 2 = 128

More generally, given an integer in its prime factorization:

Let a = $\prod_{p_i \in Primes} {p_i}^{a_i}$,

The number of divisors a has is $\prod_{p_i \in Primes}(a_i + 1)$.

One other fact about divisors

Divisors come in pairs, for any divisor k of a positive integer n, n/k is also a divisor. The special case is when k = n/k, which means n = k^2, the the pair isn't a pair but the same number twice.

For all integers that are NOT perfect squares, they have an even number of divisors, which come in pairs, where one number is less than the square root of the number and the other number is greater than the square root.

For perfect squares, they have an odd number of divisors, which their square root doesn't have a different matching pair, but the rest of the divisors do.

72 = 1 x 72, 2 x 36, 3 x 24, 4 x 18, 6 x 12, 8 x 9

36 = 1 x 36, 2 x 18, 3 x 12, 4 x 9, 6 x 6

In order for the expression $\prod_{p_i \in Primes}(a_i + 1)$ to be odd, each term in the product must be odd, which means that each term a_i must be even. By definition, if each a_i is even, the number is a perfect square.

Primality Checking
A prime number is a positive integer greater than 1 divisible by only 1 and itself.

By Definition: A way to check if n is prime is trial division, see if any integer from 2 to n-1 divides evenly into n. If none do, it's prime.

Using the fact that divisors come in pairs, we see that all composite numbers must have a divisor less than or equal to square root of n, the number we are checking for primality.

Theorem: For all composite integers n, they have at least one divisor greater than 1 and less than or equal to sqrt(n).

Assume the opposite that there is some integer n that is composite but has no divisor less than sqrt(n) except for 1.

n = ab, sicne it's composite we can break it down into two ints a>1, b>1
 $> \sqrt{n}\sqrt{n}$
 = n

It's impossible for n to be greater than n, so there must be an error somewhere, but the only place that could have happened is our assumption. Thus, at least one of the divisors of n has to be less than or equal to the square root.

Better Algorithm: Stop trial division when you reach the square root of the number being checked for primality.

Prime Sieve (Sieve of Eratosthenes)

Goal is to list all primes from 2 to some positive integer n.

1	2	3	~~4~~	5
~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~
~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~

Sum of Divisors

We know the # of divisors of an integer. Let's see if we can derive a formula for the sum of those divisors.

$n = 2^4 3^3 5^2$

$n' = (1 + 2 + 2^2 + 2^3 + 2^4)(1 + 3 + 3^2 + 3^3)(1 + 5 + 5^2)$

To do this multiplication, we "FOIL" everything, which really means selecting each possible combination of term from the first sum, term from the second sum and term from the last sum.

This expansion will have 5 x 4 x 3 = 60 terms.

Each of these 60 terms is unique. Each are divisors of n. n has 60 divisors and this has 60 unique terms, which means that this expression is nothing but the sum of the divisors of the number.

$$\sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1}$$

Given an integer a = $\prod_{p_i \in Primes} p_i^{a_i}$, the sum of its divisors is:

$$\prod_{p_i \in Primes} \frac{{p_i}^{a_i+1} - 1}{p_i - 1}$$

<u>Number of Times that n! is divisible by a prime number p</u>

n = 12, p = 2

n! = 12! = 1 x 2 x 3 x 4 x 5 x 6 x 7 x 8 x 9 x 10 x 11 x 12

1 x	~~2~~	x 3 x	~~4~~	x 5 x	~~6~~	x 7 x	~~8~~	x 9 x	~~10~~	x 11 x	~~12~~
	1		2		3		4		5		6

So far we've divided 2 into 12! 6 times. This is 12/2 using integer division.

1 x	~~2~~	x 3 x	~~4~~	x 5 x	~~6~~	x 7 x	~~8~~	x 9 x	~~10~~	x 11 x	~~12~~
	1		~~2~~		3		~~4~~		5		~~6~~
			1				2				3

Now I've divided out an addition 3 copies of 2. This is 6/2 using integer division.

1 x	~~2~~	x 3 x	~~4~~	x 5 x	~~6~~	x 7 x	~~8~~	x 9 x	~~10~~	x 11 x	~~12~~
	1		~~2~~		3		~~4~~		5		~~6~~
			1				~~2~~				3
							1				

So, we have now canceled 1 additional copy 2, for a total of 6 + 3 + 1 = 10.

2 | 12
2 | 6 6 + 3 + 1

2 | 3
2 | 1
    0

$$\sum_{i=1}^{n}\left\lfloor\frac{n}{p^i}\right\rfloor$$

Is the number of times a prime number p divides evenly into n!

By hand I did 12/2 + 6/2 + 3/2 (int dive)
The formula does 12/2 + 12/4 + 12/8 (int div)

One Last Fun Problem
This problem is from the very first USA Mathematical Olympiad in 1972 (first problem)

Let (a,b,c) denote the gcd of a, b and c. Let [a,b,c] denote the lcm of a, b and c. (Similarly let (a,b) denote the gcd a, b. [a,b] denote lcm of a and b.)

Prove $\frac{[a,b,c]^2}{[a,b][a,c][b,c]} = \frac{(a,b,c)^2}{(a,b)(a,c)(b,c)}$

Previously, we had proven that [a,b](a,b) = ab…

Let a = $\prod_{p_i \in Primes} {p_i}^{a_i}$, and b= $\prod_{p_i \in Primes} {p_i}^{b_i}$, c= $\prod_{p_i \in Primes} {p_i}^{c_i}$

$$\frac{[a,b,c]^2}{[a,b][a,c][b,c]} = \frac{\prod_{p_i \in Primes} {p_i}^{2\max(a_i,b_i,c_i)}}{\prod_{p_i \in Primes} {p_i}^{\max(a_i,b_i)+\max(a_i,c_i)+\max(b_i,c_i)}}$$

$$= \prod_{p_i \in Primes} {p_i}^{2\max(a_i,b_i,c_i)-\max(a_i,b_i)-\max(a_i,c_i)-\max(b_i,c_i)}$$

$$= \prod_{p_i \in Primes} p_i^{-\operatorname{middle}(a_i, b_i, c_i)}$$

$$\frac{(a,b,c)^2}{(a,b)(a,c)(b,c)} = \frac{\prod_{p_i \in Primes} p_i^{2\min(a_i,b_i,c_i)}}{\prod_{p_i \in Primes} p_i^{\min(a_i,b_i)+\min(a_i,c_i)+\min(b_i,c_i)}}$$

$$= \prod_{p_i \in Primes} p_i^{2\min(a_i,b_i,c_i)-\min(a_i,b_i)-\min(a_i,c_i)-\min(b_i,c_i)}$$

$$= \prod_{p_i \in Primes} p_i^{-\operatorname{middle}(a_i, b_i, c_i)}$$

Thus, both sides equal the same thing, proving the desired equation.