Proof Techniques, Set #1

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Proof by Contradiction

Prove that the square root of 2 is irrational.
A rational # is one that can be expressed as an integer divided by an integer. And all rational numbers can be expressed in lowest terms, where the numerator and denominator don't share any common factor greater than 1.

Step 1: Assume the opposite of what we want to prove
Step 2: Make logical deductions from that assumption and other known facts.
Step 3: Arrive at some piece of information that is absurd (like 0 > 1…) which is called a contradiction…

Idea is…we did everything right, except for our initial assumption which we guessed without proof…if we get an absurd conclusion, there must have been an error in the proof, so that error had to be the initial assumption…if that initial assumption is wrong, we've actually proven what we set out to prove.

We use proof by contraction.
Assume the opposite that the square root of 2 is rational.
It follows that there exist integers p and q such that sqrt(2) = 1.414213562373095 = p/q, where gcd(p, q) = 1, (greatest common divisor), fraction is in lowest terms:

sqrt(2) = p/q
q(sqrt(2)) = p
$2q^2 = p^2$, note that the left hand side is even, so the right hand side has to be even also. Since odd x odd = odd, p must be even.
Since p is even by definition of even numbers there exists an integer a, such that p = 2a, now substitute:

$2q^2 = (2a)^2$
$2q^2 = 4a^2$

$q^2 = 2a^2$, note that the right hand side is even, so the left hand side has to be even as well. What does this mean about q?

q must be even as well.

Do we have a problem?

Yes, we have a problem. We have proven that both p and q are divisible 2 and share a common factor of 2, but we know that p and q don't share any common factors. This is a contradiction!!!

So there is something wrong with this proof. But, all the steps were valid except for our initial assumption that sqrt(2) is rational. It follows that is incorrect, which means the sqrt(2) must be irrational, as desired.

Direct Proof
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if n is even, prove that $16 \mid n^4$. (a | b is read as "b is divisible a" and this is true iff there exists an integer c such that b = ac.)

1. Assume the if portion is true.
2. Take logical steps.
3. Arrive at the conclusion.

We use direct proof. Let n be an arbitrary even integer. By defintion of even integers, there exists some integer c such that n = 2c.

$n^4 = (2c)^4 = 16c^4$, since c is an integer, it follows that c^4 is an integer and we can conclude that $16 \mid n^4$.

Proof of the Contrapositive

The contrapositive of p -->q is not(q) --> not(p), both are equivalent statements. So, if you want, you can take an if-then, and turn it into its equivalent contrapositive statement, and prove that instead.

Sets

Subset examples

A = {1, 2, 3}

Here are some subsets of A:

{1}, {1, 3}, {2, 3}, {1, 2, 3}

For all of these sets, S, above, if an element belongs to S, then it also belongs to A.

Here are some sets that aren't subsets of A:

{4}, {2, 3, 17}, {1, 5}

In order for a set S to NOT be a subset of a set A,
there must exist some element x in S such that x is NOT in A.

$$\exists x \mid x \in S \land S \notin A$$

Proper Subset is different from subset in that the set itself is NOT a proper subset. For A to be a proper subset of B, A must be a subset of B and also not equal to B.

Empty Set is a set with no elements!
Commonly denoted with a circle with a line through it.
It can also be denoted this way: {}

Set Operator Examples

A = {1, 3, 5, 6, 8, 9}
B = {2, 3, 4, 6, 7}

A U B = {1, 2, 3, 4, 5, 6, 7, 8, 9}
A intersect B = {3, 6}
complement(A) = {all integers except 1,3,5,6,8 and 9}

A intersect B = {3, 6}
complement(A) = {all integers except 1,3,5,6,8 and 9}
A - B = {1,5,8,9}
B - A = {2,4,7}

Symmetric Difference

A (triangle) B = (A U B) - (A intersect B)
= (A - B) U (B - A)

How to Prove that Two sets are equal:

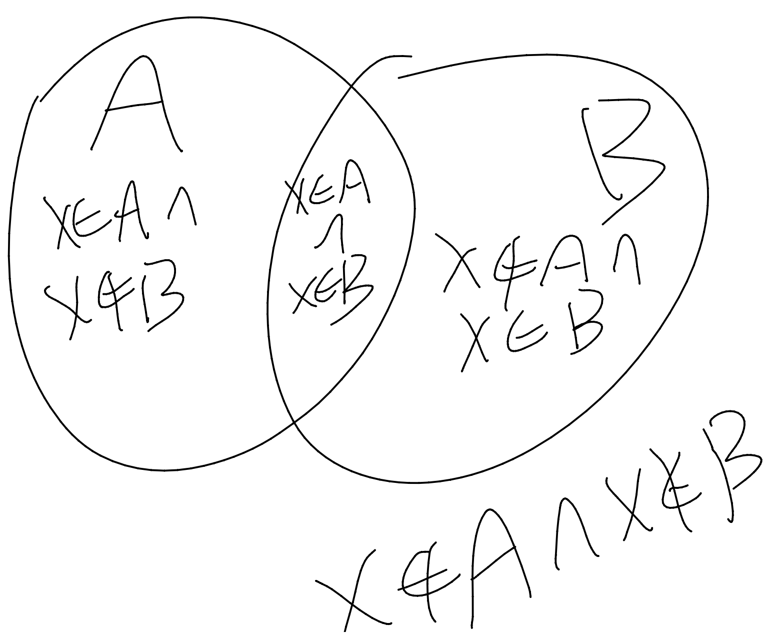
1. Set laws
2. Table Method
3. Direct Proof (A = B)
   a. A is a subset of B
   b. B is a subset of A

To show that some set A is a subset of a set B, we must take an arbitrarily chosen element A and then use logical step to show it also belongs to B.

What the table method really does is an exhaustive proof by cases.

Consider the first proof we looked at.

For all items in our universe there are four cases of where they might lie:



$x \notin A$

Those are our four cases:
for an arbitrary element x it must lie in one of these four places:

1. not in A, not in B
2. not in A, in B
3. in A, not in B
4. in A, in B

If two columns are identical in a set table, we have proven that both sets are equivalent via an exhaustive search where we should that membership is identical for all arbitrarily chosen elements by breaking up our search into lots of cases.

Cartesian Product

A x B = { (a, b) | a is an element of A and b is an element of B}

One way to think about this is a value meal…
choose 1 appetizer (nachos, churros, fries)
choose 1 entrée (taco, burrito, quesadilla, mexican pizza)

A meal is any choice of app and entrée. What are all of the possible meals?

	N	C	F
T	(T,N)	(T,C)	(T,F)
B	(B,N)	(B,C)	(B,F)
Q	(Q,N)	(Q,C)	(Q,F)
M	(M,N)	(M,C)	(M,F)

M	(M,N)	(M,C)	(M,F)

A x B = { (T,N), (T,C), (T, F), (B, N), (B,C), (B,F), (Q,N), (Q,C), (Q,F), (M, N), (M,C), (M,F) }

| A x B | = |A| x |B|

A x Empty Set = Empty Set, imagine making the table I just made above. It would have 0 columns. And if there 0 columns, there are 0 entries.

A Cartesian Product is a set of ordered pairs, not a set of individual items per se.

Definition: Power Set:

P(A) = the set of all subsets of A

The types of objects in a power set are sets themselves.

A = {1, 2, 3}
P(A) { {}, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3} }

The empty set is a subset of all sets because the following statement is vacuously true:

For all elements x that belong to the empty set, x belongs to the set A.

Again, the key issue is that the type of object that belongs to a power set is a set itself and that's different than other sets.

To me type matching is really important.