Possible Sets of Values for a Universe

Real Numbers - $\mathcal{R}$

Integers - Z

Positive Integers - Z^+ (1, 2, 3, 4, 0 is not positive, 0 is not negative, 0 is an integer)

Natural Numbers - N (0, 1, 2, 3, …)

Rational Numbers - Q (Int/Int, but 0 can't be in the denominator)

Complex Numbers - $\mathbb{C}$ (a+bi, where a, b are real and I is the square root of -1.)

q(x,y): x is a prime number that divides into y evenly AND is less than or equal to y.

∀y∃x [q(x,y)]

Is true using the universe of all positive integers 2 or greater.

We say, " x | y", which is read as "y is divisible by x", and formally, this means the following:

$$\exists c \in Z | y = cx.$$

So the vertical bar has multiple meanings. It can mean divisibility or it can translate to "such that"

∀y∃x [q(x,y)]

$$\forall y \in (Z^+ - \{1\})[\exists x \in Z|\ (x\ |y)]$$

For all y that are real numbers greater than or equal to 1, …

Difference between

"For all y, there exists a x"

"there exists an x, for all y"

q(x,y): x is a prime number that divides into y evenly AND is less than or equal to y.

X Y	2	3	4	5	6	7
2	T	F	T	F	T	F
3	F	T	F	F	T	F
4	F	F	F	F	F	F
5	F	F	F	T	F	F
6	F	F	F	F	F	F
7	F	F	F	F	F	T

For all y, means, "for all columns of this chart", "there exists an x" means' for all columns of this chart, there is at least one True in it.

There exists an x for which all y…**"This means, there is one ROW, for which, all of the entries are true."**

My favorite example…

"For all people x, in the world, there exists some person y, who is their soulmate".

Though many of you are cynics, this is at least plausible…

"There exists a person y, such that for all people x, person y is their soulmate"
Suggestions: Ghengis Khan and Keanu Reeves, Ringo Starr. Megan Fox, and the list goes on…

When someone says a statement is "vacuously true" or "trivially true", what they mean is that the if portion is always false.

<u>Key ideas</u>

To prove a there exists, just find one value that makes the item true.
To disprove a for all, just find one value that makes the item false.
To prove a for all, it's not quite as easy.
To disprove a there exists, it's also not quite as easy.

Make sure you memorize the meanings of contrapositive, inverse and converse.

A statement and its contrapositive are logically equivalent.
A statement's inverse and a statement's converse are logically equivalent.

"If it is raining, the ground is wet." Is equivalent to
"If the ground is dry, it is not raining" (contrapositive)

The converse is "if the ground is wet, it is raining" (not always true)
Inverse, "if it is not raining, then the ground is dry."

One takeaway: if I want to prove an if-then statement, I can also prove its contrapositive.

Universe is {1,2,3}
P(1) is true, P(2), P(3) are false
Q(1) is false, Q(2) is true, Q(3) false
In this situation $\exists x p(x)$ (1 makes this true) is true $\exists x q(x)$ (2 makes this true) is true.
But, $\exists x(p(x) \wedge q(x))$ is false.

But, if we have
P(1) is true, q(1) is true, p(2), q(2), p(3), q(3) is false, then
$\exists x(p(x) \wedge q(x))$ is true because 1 makes both true.
If there is a number that makes both true, we can deduce that there is a number that make p true.
We can also deduce that there is a number that makes q true.
Quick Counterexample to #4 backwards is p(x) = x is even, q(x) = x is odd, and the universe is integers.

<u>Definition of Even</u>

An integer n is even if and only if there exists an integer c, such that n = 2c. For example, 0 is even because 0 = 2 x 0.

Even integers are integers divisible by 2.

Odd integer definition: An integer n is odd if and only if there exists an integer c, such that n = 2c + 1.

How do we prove a for all statement???

Two rules:

Universal Specification - if I already know that a for all rule is true, if I can find some item in its universe, I can plug that in to obtain a true statement.

Proven fact: "All mammals have hair", "Cats are mammals"

Deduction using universal rule of specification: "cats have hair"

Universal Generalization Rule: In order to prove a for all statement, I must prove it for an arbitrarily chosen item of the universe.

If I want to prove something for all UCF students, I can't say, "Derek is a UCF student, …"

Instead I do this:

"Let x be an arbitrarily chosen UCF student, …"

1. For all integers n, prove that n(n+1) is even.

Proof by Cases:

For all integers n, n is either even or odd. We will prove that the claim is true for both cases.

Case 1: n is even.

There exists an integer c such that n = 2c.

n(n+1) = 2c(2c+1) = 2((c)(2c+1), since we've expressed the quantity as 2 times in integer (since c is an integer, it follows that c(2c+1) is also an integer), it follows that n(n+1) is even.

Case 2: n is odd.

There exists an integer c such that n = 2c+1

N(n+1) = (2c+1)(2c+2) = 2((2c+1)(c+1)), since we've expressed n(n+1) as 2 times an integer (recall that c is an integer so (2c+1)(c+1) is as well), n(n+1) is even.

Symbolically what we did is this:

$\forall \boldsymbol{x} \in \boldsymbol{Odd}, \boldsymbol{p}(\boldsymbol{x})$, $\forall \boldsymbol{x} \in \boldsymbol{Even}, \boldsymbol{p}(\boldsymbol{x})$,

$\forall \boldsymbol{x} \in \boldsymbol{Z}[\boldsymbol{x} \in \boldsymbol{Odd} \vee \boldsymbol{x} \in \boldsymbol{Even}]$

So, if we show all of these three things, then we've shown:

If $\boldsymbol{n} \in \boldsymbol{Z} \rightarrow \boldsymbol{n}(\boldsymbol{n}+\mathbf{1})\ \boldsymbol{is\ even}$

2. For all odd integers n, prove that 8 | (n^2 - 1)

If we want to prove an if then, one technique is direct proof.

Here is structurally how you do direct proof:

1. ASSUME THE IF PART.

Let n be an arbitrarily chosen odd integer. There exists some integer c such that n = 2c+1.

$n^2 - 1 = (2c+1)^2 - 1$

$= 4c^2 + 4c + 1 - 1$

$= 4c(c+1)$

Previously, we had shown that for any integer n, n(n+1) is even. This means that since c is an integer, c(c+1) is even and if c(c+1) is even, there exists an integer d such that c(c+1) = 2d, now substitute:

$= 4(2d)$

$= 8d$

Since d is an integer, we can conclude that $8 \mid (n^2 - 1)$

Try it…

N=13 $n^2 - 1 = 13^2 - 1 = 168 = 8 \times 21$… (not a proof, but it's an example of the idea in the proof.)