Important Fall 2020 COT 3100 Final Exam Information

Exam Date and Time

Date: Tuesday, December 8, 2020
Time: 1:00 pm – 3:50 pm **Location: Your House**

NOTE THE EARLIER START TIME!!!

Exam Aids:

Open Book Exam, but don't look up anything outside the course web page during the exam. Due to the circumstances, most of the points will be assigned to the thinking *communicated on the files you submit* and very little points will be awarded for answers. You can use a calculator, but in your write up, you must state every time you used the calculator and what you put in to it.

Exam Format

All Free Response

Four Parts

Part A 1:00 pm - 1:40 pm, late turn in 1:50 pm
Part B 1:45 pm - 2:20 pm, late turn in 2:30 pm
Part C 2:25 pm - 3:05 pm, late turn in 3:15 pm
Part D 3:10 pm - 3:50 pm, late turn in 4:00 pm

Part A - Recitation Material

1) Root of Polynomials
2) D = RT
3) Logs
4) Arith/Geo Series
5) Factorization
6) Average Problems
7) Random Algebra

Part B - Logic/Sets/Relations/Functions

1) Logic
2) Sets
3) Relations
4) Functions

Part C - Number Theory/Induction

1) Number Theory
2) Induction

Part D - Counting/Probability

1) Counting
2) Probability

If it doesn't come in my the late turn in, it's a 0.

My recommendation is to type. Scanning takes time that you lose from solving problems.

Accepted file types: doc, docx, txt, pdf

Fall 2019 Final Exam - Strategies to Answer the Question

1) (8 pts) A swimming pool is the shape of a rectangular prism with a length of 12 feet, a width of 10 feet and a depth of 8 feet. The pool is full at Tuesday at 8 am but springs a leak from the bottom of the pool surface that leaks 1 cubic inch of water per second into the ground. (This means that slowly, the water level in the pool decreases.) How much lower (in inches) is the water level at Wednesday morning at 8 am as compared to Tuesday at 8 am when the pool was full? Which piece of information given in the problem is mostly irrelevant?

Solution

Cross Sectional Area = 144 in x 120 in

If the pool leaks 1 cubic inch per second, then it takes 144 x 120 seconds to lower the level of the pool by 1 inch.

Total time = 24 hours = 24 x 60 x 60 seconds

$$\frac{24 \times 60 \times 60\ sec}{12 \times 12 \times 12 \times 10\ sec/in} = 5\ in$$

How deep because the leak from the bottom and unless the whole thing leaked, we never need to know it is 8 feet deep.

2) (8 pts) Let the roots of the quadratic equation $f(x) = x^2 + ax + b$ be r_1 and r_2. What is the quadratic equation with leading coefficient 1 that has roots r_1^2 and r_2^2? Please give your answer in terms of a and b only.

Solution

$r_1 + r_2 = -a$

$r_1r_2 = b$

$(r_1 + r_2)^2 = (-a)^2$

$r_1^2 + 2r_1r_2 + r_2^2 = a^2$

$r_1^2 + 2b + r_2^2 = a^2$

$r_1^2 + r_2^2 = a^2 - 2b$, so the sum of the roots of the new equation is $a^2 - 2b$

$r_1^2r_2^2 = (r_1r_2)^2 = b^2$, so the product of the roots of the new equation is b^2.

The corresponding quadratic is $x^2 - (a^2 - 2b)x + b^2$.

3) (8 pts) Consider the following premises involving Boolean variables p, q, r, and s:

$$p \rightarrow (q \wedge r)$$
$$q \rightarrow s$$
$$\bar{r}$$

Can we conclude $\bar{s}$? If so, prove this conclusion via the rules of inference. If not, show a single truth setting such that the three given premises are true but the conclusion $\bar{s}$ is false.

Solution

The conclusion does NOT always follow. Consider the following counter-example:

s = true
r = false
p = false
q = false

With this truth setting, the first two implications are true because p and q are both false and the third premise is also true since we set r to false. But not(s) is false so this conclusion does NOT follow.

4) (8 pts) Bytelandia has coins with the following denominations: 1 cent, 8 cents, 32 cents and 44 cents. Bytesar has coins that add up to exactly 1239 cents. What is the minimum number of 1 cent coins he could have? Give a set of coins that adds up to 1239 cents which achieves this minimum number of 1 cent coins and prove that it's impossible for another combination to have fewer 1 cent coins.

Solution

Let x be the # of 1 cent coins, y be the # of 8 cent coins, z be the # of 32 cent coins and w be the # of 44 cent coins:

$x + 8y + 32z + 44w = 1239$

Consider the equation mod 4:

$x + 0 + 0 + 0 \equiv 3 \pmod 4$

The number of pennies is equivalent to 3 (mod 4).

Minimum # of pennies is 3 pennies, we know it can't be lower due to the mod argument above and here is one combination that works:

3 pennies, 1 44 cent coin, 149 8 cent coins.

5) (8 pts) Prove or disprove for finite sets A, B and C: if $A \cap B = C \cap B$, then $A = C$.

Solution
This is false. Consider the following counter example:

B = {}
A = {1}
C = {2}

In this example $A \cap B = C \cap B$, as both are empty, but A and C are not equal sets.

6) (8 pts) Prove or disprove for finite sets A, B and C: if $A \subseteq C$, then $A \cap B \subseteq C \cap B$.

Solution
This is true.
Let's use direct proof to prove it.
Let x be an arbitrary member of $A \cap B$. We must show that x also belongs to $C \cap B$.

By definition of intersection x belongs to A and x belongs to B.
By definition of subset and the fact that x belongs to A and $A \subseteq C$, x belongs to C.
By definition of intersection, since x belongs to A and x belongs to C, x belongs to $C \cap B$.

7) (15 pts) Find all integer solutions for x and y to the equation 297x + 234y = 36.

Solution
Find gcd(297, 234):

297 = 1 x 234 + 63
234 = 3 x 63 + 45
63 = 1 x 45 + 18
45 = 2 x 18 + 9
18 = 2 x 9, thus the desired gcd is 9.

Divide the given equation through by 9 to get: 33x + 26y = 4. Run the Extended Euclidean for 33 and 26:

33 = 1 x 26 + 7
26 = 3 x 7 + 5
7 = 1 x 5 + 2
5 = 2 x 2 + 1

5 – 2 x 2 = 1
5 – 2(7 – 5) = 1
5 – 2 x 7 + 2 x 5 = 1
3 x 5 – 2 x 7 = 1
3(26 – 3 x 7) – 2 x 7 = 1

$3 \times 26 - 9 \times 7 - 2 \times 7 = 1$
$3 \times 26 - 11 \times 7 = 1$
$3 \times 26 - 11(33 - 26) = 1$
$3 \times 26 - 11 \times 33 + 11 \times 26 = 1$
$14 \times 26 - 11 \times 33 = 1$

Multiply this equation through by 4:

$4(14 \times 26 - 11 \times 33) = 4$
$(4 \times 14) \times 26 - (4 \times 11) \times 33 = 4$
$56 \times 26 - 44 \times 33 = 4$

It follows that one integer solution to $33x + 26y = 4$ is $(-44, 56)$. Given an arbitrary solution to the equation (a, b), another solution is $(a + 26, b - 33)$. It follows that all solutions are the set:

$\{ (-44+26a, 56 - 33a) \mid a \in Z \}$

Another way to represent this set using a different initial solution is to plug in $a = 2$ above and obtain the set $\{ (8+26a, -10 - 33a) \mid a \in Z \}$

8) (10 pts) The student government at a particular school has 1 president, 2 vice presidents, and a council of 5 students. If the school has 100 students, how many possible combinations of students can be elected to the student government? Two governments are different if they either have different presidents, or if one of the vice presidents between the governments differs, or if one of the council members between the two governments differs. (For example, let ten of the students be A, B, C, D, E, F, G, H, I and J. The government of president = A, VPs = C, E, Council = F, G, H, I, J is different than president = A, VPs = C, E, and Council = D, F, G, H and I. But, the government of president = A, VPs = E, C, Council = J, I, G, H, and F is the same as the first government listed.)

<u>Solution</u>
100 choices for the president.
$\binom{99}{2}$ choices for the vice presidents.
$\binom{97}{5}$ choices for the council.

Final answer $100\binom{99}{2}\binom{97}{5}$, since we can pair up any president with any combo of VPs and any combo of councils.

9) (12 pts) Joanna has 12 chocolates that she would like to eat within a 40 day time span. To make sure that she doesn't eat too many all at once, she is restricting herself to eat no more than 1 a day, and also to never eat chocolates on consecutive days. An example of a valid schedule of eating the chocolates is to have a chocolate on the following days 3, 6, 8, 10, 15, 17, 19, 22, 29, 33, 36, and 40. How many valid schedules of eating the chocolates are possible, given Joanna's restrictions? (Two schedules are different if one schedule has her eating a chocolate on a day that the other schedule doesn't.)

Solution

There are 28 days she does NOT eat chocolate. Let these 28 days be separators:

__ NC __ NC __ … __ NC __

There are 29 slots in between the no chocolate days where we can choose to eat chocolate.

We can choose these slots in $\binom{29}{12}$ ways. Each unique set of choices corresponds to a different valid schedule. Since there is a one to one correspondence between the ways to choose these slots and valid schedules, the answer to the question is just $\binom{29}{12}$.

10 days 3 chocolates:

__ NC C NC __ NC __ NC C NC __ NC C NC:

Schedule: 2, 6, 9

Charles's way:

__ C __ C __ C __ C __ C … __ C __

11 baskets, give 1…28 - 11 = 17 free days to distribute amongst 13 slots:

x1 + x2 + x3 + … + x13 = 17

Combinations with repetition…

$$\binom{17+13-1}{13-1} = \binom{29}{12}$$

10) (8 pts) 1% of the population has a genetic disease. Of the people with the disease, 96% of them have long ears while of the people without the disease, only 6% have long ears. A person with long ears is chosen at random. What is the probability she has the genetic disease? Express your answer as a fraction in lowest terms.

Here is the tree diagram:

```
                                         .96 (long ears)
.01 (disease)                         /------------------- (.0096 disease and long ears)
-------------------------------/   .04 (reg ears)
\                                     \------------------- (.0004 disease and reg ears)
  \                                     .06 (long ears)
   \                                  /-------------------  (.0594 no disease and long ears)
    \---------------------------/  .94 (reg ears)
        .99 (no disease           \-------------------- (.9306 no disease and reg ears)
```

We want p(disease | long ears).

p(long ears) .0594 + .0096 = .0690
p(disease and long ears) = .0096
p(disease | long ears) = .0096/.0690 = 96/690 = 48/345 = **16/115**

11) (8 pts) A bag of skittles has 10 green skittles and 20 red skittles. Johnny randomly grabs 8 skittles from the bag without looking. What is the probability that he grabs exactly 3 green skittles and 5 red skittles?

Solution

Sample space = $\binom{30}{8}$. How many ways to select 3G, 5R? $\binom{10}{3}\binom{20}{5}$. Answer is $\frac{\binom{10}{3}\binom{20}{5}}{\binom{30}{8}}$.

12) (10 pts) Let f(x) = $\frac{a}{x+a}$, where a is a positive constant with a domain of all real x except x= -a. For this domain, determine $f^{-1}(x)$. What are the domain and range of $f^{-1}(x)$?

Solution

So we exchange x and y and solve for y:

$$x = \frac{a}{y+a}$$

$$y + a = \frac{a}{x}$$

$$f^{-1}(x) = \frac{a}{x} - a$$

Domain: all real x except for x = 0
Range: all real y except for y = -a.

13) (3 pts) What is the remainder when 3^{10003} is divided by 10?

Investigate 3^a mod 10 for small values of a:

$3^0 \equiv 1 \bmod 10$
$3^1 \equiv 3 \bmod 10$
$3^2 \equiv 9 \bmod 10$
$3^3 \equiv 27 \equiv 7 \bmod 10$
$3^4 \equiv 81 \equiv 1 \bmod 10$

Thus, these values repeat every four. We can see from the last statement that $3^{4a} \equiv 1 \bmod 10$, for all non-negative integers a. It follows that $3^{10000} \equiv 1 \bmod 10$. Finally, we can solve for the desired value:

$3^{10003} \equiv 3^{10000}3^3 \equiv 1(3^3) \equiv 27 \equiv$ **7 (mod 10).**

It follows that the desired remainder is **7.**

14) (10 pts) Define the following relation R over the set of positive integers:

$R = \{ (a, b) \mid \gcd(a, b) > 1 \}$

With proof, determine if R is (a) reflexive, (b) irreflexive, (c) symmetric, (d) anti-symmetric and (e) transitive.

Solution

a) Not reflexive, because (1, 1) is NOT in the relation because gcd(1, 1) = 1.
b) Not irreflexive, because (2, 2) is in the relation because gcd(2, 2) = 2 > 1.
c) Symmetric, We must prove if (a, b) is in R, then (b, a) is also in R.
Let (a, b) be an arbitrary element of R.
Then, gcd(a, b) > 1, gcd(a, b) = gcd(b, a) > 1, so (b,a) is also in the relation.
d) Not antisymmetric: (2, 4) is in the relation (4, 2) is in the relation, and $2 \neq 4$.
e) Not transitive: (3, 15) is in R, (15, 5) is in R, but (3, 5) is NOT in R.