

- 1) Sum Techniques
- 2) Discrete Random Variables
- 3) Avg Case Run Time Analysis

Know: $\sum_{i=a}^b (ci+d)$, a, b, c, d are constants

Note: Arithmetic sequence $x_1 = ca+d$ $x_{b-a+1} = cb+d$

Geometric sequence $\sum_{i=0}^{n-1} a_1 r^i = a_1 + a_1 \cdot r + a_1 \cdot r^2 + \dots + a_1 \cdot r^{n-1}$
 $a_1 = 1^{\text{st}} \text{ term}$ $r = \text{common ratio}$

$$S = \underline{a_1} + \cancel{a_1 r} + \cancel{a_1 r^2} + \dots + \cancel{a_1 r^{n-1}}$$

$$- rS = \cancel{a_1 r} + \cancel{a_1 r^2} + \dots + \cancel{a_1 r^{n-1}} + a_1 r^n$$

$$S - rS = a_1$$

$$S(1-r) = a_1(1-r^n)$$

$$S = \frac{a_1(1-r^n)}{(1-r)} = \frac{a_1(r^n-1)}{(r-1)}, \quad r \neq 1$$

$$S = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \dots$$

$$-rS = \quad a_1 r + a_1 r^2 + a_1 r^3 + \dots$$

$$\sum_{i=0}^{\infty} a_i \cdot r^i$$

$$S - rS = a_1$$

$$S(1-r) = a_1$$

$$S = \frac{a_1}{1-r}, \quad |r| < 1.$$

$$S = \underline{1 \times \frac{1}{2}} + 2 \times \frac{1}{4} + 3 \times \frac{1}{8} + 4 \times \frac{1}{16} + \dots$$

$$-\frac{1}{2}S = \quad 1 \times \frac{1}{4} + 2 \times \frac{1}{8} + 3 \times \frac{1}{16} + \dots$$

$$\frac{1}{2}S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

$$\frac{1}{2}S = \frac{\frac{1}{2}}{1 - \frac{1}{2}}$$

$$\frac{1}{2}S = \frac{1/2}{1/2} = 1$$

$$\boxed{S = 2}$$

Discrete Random Variables

$$X = \begin{cases} 1, w/prob \frac{1}{6} \\ 2, w/prob \frac{1}{6} \\ 3, w/prob \frac{1}{6} \\ 4, w/prob \frac{1}{6} \\ 5, w/prob \frac{1}{6} \\ 6, w/prob \frac{1}{6} \end{cases} \quad E(X) = \sum_{x \in X} x \cdot p_x$$

$$1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{7}{2}$$

$$Y = \begin{cases} 1, w/prob \frac{1}{6} \\ 3, w/prob \frac{1}{3} \\ 12, w/prob \frac{1}{2} \end{cases}$$

$$1 \times \frac{1}{6} + 3 \times \frac{1}{3} + 12 \times \frac{1}{2} = \boxed{7\frac{1}{6}}$$

Linearity of Expectation

$$E(X+Y) = E(X) + E(Y)$$

$$\sum_{x \in X} p_x \sum_{y \in Y} p_y (x+y) = \sum_{x \in X} p_x \sum (x \cdot p_y + y \cdot p_y)$$

ASSUMES
INDEPENDENCE
of X, Y .

$$= \sum_{x \in X} \left(p_x \sum x \cdot p_y \right) + \left(p_x \sum y \cdot p_y \right)$$

$$= \left(\sum_{x \in X} (p_x \cdot x) \right) \left(\sum p_y \right) + \sum_{x \in X} p_x \cdot E(Y)$$

$$= \sum_{x \in X} p_x \cdot x \cdot 1 + E(Y)$$

$$= E(X) + E(Y)$$

To get best case run-time - imagine fastest way alg ends. Worst - imagine longest it can run

AUG CASE - tough (expected run-time of the algorithm where we model all possibilities of run-time w/a Discrete Random Variable.)

- ① Binary Search
- ② Counter (Binary)
- ③ Quick Sort
- ④ Quick Select

$$E(X+Y) = \sum_{x \in X} \sum_{y \in Y} p_{x,y} \cdot (x+y)$$

$$= \sum_{x \in X} \left(\sum_{y \in Y} x \cdot p_{x,y} + \sum_{y \in Y} y \cdot p_{x,y} \right)$$

$$= \left(\sum_{x \in X} \sum_{y \in Y} x \cdot p_{x,y} \right) + \left(\sum_{x \in X} \sum_{y \in Y} y \cdot p_{x,y} \right)$$

$$= \left(\sum_{x \in X} x \sum_{y \in Y} p_{x,y} \right) + \left(\sum_{y \in Y} y \sum_{x \in X} p_{x,y} \right)$$

over all y , must add to p_x .

over all x must add to p_y

$$= \sum_{x \in X} x \cdot p_x + \sum_{y \in Y} y \cdot p_y$$

$$= E(X) + E(Y)$$

Proof that works w/o
assuming independence of
 X, Y .

Binary Search $n = 2^k - 1$

2	1	2
---	---	---

3	2	3	1	3	2	3	3	2	1
---	---	---	---	---	---	---	--------------	--------------	--------------

$$1 \text{ Step} = \frac{1}{n} \text{ prob}$$

$$2 \text{ Steps} = \frac{2}{n} \text{ prob}$$

$$3 \text{ Steps} = \frac{4}{n} \text{ prob}$$

$$k \text{ Steps} = \frac{2^{k-1}}{n}$$

$$S = 1 \times \frac{1}{n} + 2 \times \frac{2}{n} + 3 \times \frac{4}{n} + \dots + k \times \frac{2^{k-1}}{n}$$

$$S = \frac{1}{n} (1 \times 1 + 2 \times 2 + 3 \times 4 + \dots + k \times 2^{k-1})$$

$$- 2S = \frac{1}{n} (1 \times 2 + 2 \times 4 + 3 \times 8 + \dots + (k-1) \times 2^{k-1} + k \times 2^k)$$

$$- S = \frac{1}{n} (1 + 2 + 4 + 8 + \dots + 2^{k-1} - k \cdot 2^k)$$

$$- S = \frac{1}{n} (2^k - 1 - k \cdot 2^k)$$

$$S = \frac{1}{n} (k \cdot 2^k - 2^k + 1)$$

$$S = \frac{1}{n} (k \cdot 2^k - 2^k + 1)$$

$$S = \frac{1}{n} ((n+1)(\log_2(n+1) - 1) + 1)$$

$$O(\lg n)$$

Binary Counter

		n bit counter	
\$2	0000		total # of bits flips
2	0001	→ 1 bit	
4	0010	→ 2 bits	avg # of bits flipped
6	0011	→ 1 bit	per operation
8	0100	→ 3 bits	

1111 → 1 bit
flips every time

$2^n - 1$ flips
 $2^{n-1} - 1$ flips
 $2^{n-2} - 1$ flips
 $2^1 - 1$ flips

$$\text{Avg} = \frac{2^{n+1} - n - 2}{2^n - 1}$$

$$\sim \boxed{2}$$

$$\boxed{2^{n+1} - 2} \quad -n$$

General idea behind amortized analysis: pay some "fixed" cost for each operation + prove you'll always have enough money saved up for costly operations.

Avg Case Run-Time Quick Sort

Let $T(n)$ = expected run-time of
Q.S. of n elements

Partition and there are n possibilities,
each equally likely:

<u>Option</u>	<u>Split</u>	<u>Prob</u>
1	0, $n-1$	$\frac{1}{n}$
2	1, $n-2$	$\frac{1}{n}$
3	2, $n-3$	$\frac{1}{n}$
$\vdots$		$\vdots$
$n-1$	$n-2, 1$	$\frac{1}{n}$
n	$n-1, 0$	$\frac{1}{n}$

$$T(n) = \frac{1}{n} (T(0) + T(n-1)) + \frac{1}{n} (T(1) + T(n-2)) + \dots + \frac{1}{n} (T(n-2) + T(1)) + \frac{1}{n} (T(n-1) + T(0)) + n$$

$$T(n) = \left(\frac{1}{n} \sum_{i=0}^{n-1} 2T(i) \right) + n$$

$$\text{Let } S(n) = nT(n) = \sum_{i=0}^{n-1} 2T(i) + n^2$$

$$- S(n-1) = (n-1)T(n-1) = \sum_{i=0}^{n-2} 2T(i) + (n-1)^2$$

$$nT(n) - (n-1)T(n-1) = 2T(n-1) + (2n-1)$$

$$\frac{nT(n)}{n(n+1)} = \frac{(n+1)T(n-1)}{n(n+1)} + \frac{(2n-1)}{n(n+1)}$$

Let $U(n) = \frac{T(n)}{n+1}$

$$T(n) = (n+1)U(n) = O(n \lg n)$$

$$\frac{T(n)}{n+1} = \frac{T(n-1)}{n} + \frac{(2n-1)}{n(n+1)}$$

$$U(n) = U(n-1) + \frac{(2n-1)}{n(n+1)}$$

$$= \underbrace{U(n-2)}_{\uparrow \text{peels off terms}} + \frac{(2n-3)}{(n-1)(n)} + \frac{(2n-1)}{n(n+1)}$$

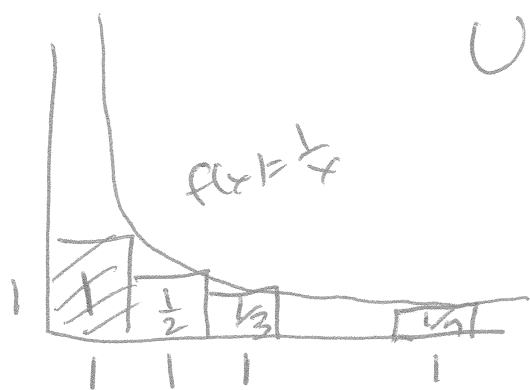
↑
peels off terms

$$U(n) = \sum_{i=1}^n \frac{2i-1}{i(i+1)} \leq \sum_{i=1}^n \frac{2i}{i(i+1)}$$

$$= O(\lg n) = \sum_{i=1}^n \frac{2}{i+1}$$

$$\sim 2H_n$$

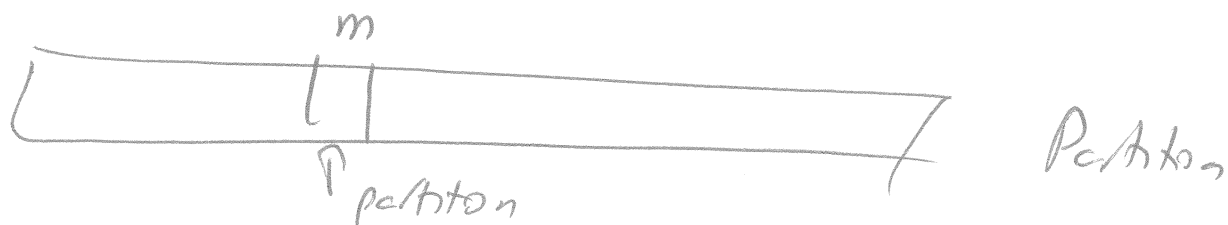
↓
nth harmonic



$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \sim \ln n$$

$$H_n \leq 1 + \int_1^n \frac{1}{x} dx = \ln x \Big|_1^n + 1 = 1 + \ln n$$

Problem: Given an unsorted array of n items and an integer k , $1 \leq k \leq n$, find the k^{th} smallest value.



$m > k-1 \rightarrow$ On left

$m = k-1$ DONE!!!

$m < k-1 \rightarrow$ Goto right

Only 1 recursive call to appropriate side.

Partition	Prob	Work Left
$0, n-1$	$1/n$	$\frac{1}{n} T(1)$ $\left(\frac{n-1}{n}\right) \underline{\underline{T(n-1)}}$
$1, n-2$		$\frac{1}{n} T(1) + \frac{n-2}{n} T(n-2)$
$2, n-3$		$\frac{2}{n} T(2) + \frac{n-3}{n} T(n-3)$
$3, n-4$		
$k, n-k-1$		$\frac{k}{n} T(k) + \frac{n-k-1}{n} T(n-k-1)$
$n-1, 0$		