COP 3502 Recitation Sheet: Recurrence Relations Solutions

1) Use the iteration technique to solve the following recurrence relation in terms of n:

$$T(n) = 2T(n/2) + 1, for\ all\ integers\ n > 1$$
$$T(1) = 1$$

Find a tight Big-Oh answer.

Solution

$$T(n) = 2T\left(\frac{n}{2}\right) + 1$$
$$T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{4}\right) + 1$$
$$T(n) = 2\left(2T\left(\frac{n}{4}\right) + 1\right) + 1$$
$$T(n) = 4T\left(\frac{n}{4}\right) + 2 + 1$$
$$T(n) = 4T\left(\frac{n}{4}\right) + 3$$
$$T\left(\frac{n}{4}\right) = 2T\left(\frac{n}{8}\right) + 1$$
$$T(n) = 4\left(2T(\frac{n}{8}\right) + 1) + 3$$
$$T(n) = 8T(\frac{n}{8}) + 4 + 3$$
$$T(n) = 8T(\frac{n}{8}) + 7$$

Based on these three iterations, we see that after k iterations, the recurrence is

$$T(n) = 2^k T(\frac{n}{2^k}) + (2^k - 1)$$

Plug in the value of k such that $\frac{n}{2^k} = 1$ to this recurrence. This means that $2^k = n$. Substituting, we get:

$$T(n) = nT(1) + (n - 1)$$
$$T(n) = n + (n - 1)$$
$$T(n) = 2n - 1$$

It follows that T(n) = O(n).

2) What is the closed form solution to the following recurrence relation? Please use the iteration technique, show all of your work and provide your final answer in Big-Oh notation.

T(1) = 1
T(n) = 2T(n/4) + 1

<u>Solution</u>

Iterate the recurrence three times:

$T(n) = 2T\left(\frac{n}{4}\right) + 1$ (one iteration)

$T(n) = 2(2T\left(\frac{n}{16}\right) + 1) + 1$

$T(n) = 4T\left(\frac{n}{16}\right) + 3$ (two iterations)

$T(n) = 4(2T\left(\frac{n}{64}\right) + 1) + 3$

$T(n) = 8T\left(\frac{n}{64}\right) + 7$ (three iterations)

Now, let's make a guess as to the form of the recurrence after iterating k times based on the first three iterations:

$$T(n) = 2^k T\left(\frac{n}{4^k}\right) + (2^k - 1)$$

Since we know T(1), we want to plug in the value of k for which $\frac{n}{4^k} = 1$, in for k. Solving, we find that $n = 4^k$. Taking the square root of both sides, we find $\sqrt{n} = \sqrt{4^k} = \sqrt{2^{2k}} = (2^{2k})^{\frac{1}{2}} = 2^k$. Substituting for both 4^k and 2^k, in the right hand of the recurrence, we get:

$$T(n) = \sqrt{n}T\left(\frac{4^k}{4^k}\right) + \left(\sqrt{n} - 1\right) = \sqrt{n}T(1) + \left(\sqrt{n} - 1\right) = \sqrt{n} + \sqrt{n} - 1 \in \boldsymbol{O}(\sqrt{\boldsymbol{n}})$$

3) Use the iteration technique to determine a close form solution for the recurrence relation $T(n)$ defined below. Note: due to the nature of this recurrence, it's possible to get an exact solution for $T(n)$, so please try to do that instead of just getting a Big-Oh bound.

$$T(n) = 2T(n-1) + 2^n$$
$$T(1) = 2$$

Solution

Iterate as follows:

$$\begin{aligned} T(n) &= 2T(n-1) + 2^n \\ &= 2(2T(n-2) + 2^{n-1}) + 2^n \\ &= 4T(n-2) + 2^n + 2^n \\ &= 4T(n-2) + 2(2^n) \\ &= 4(2T(n-3) + 2^{n-2}) + 2(2^n) \\ &= 8T(n-3) + 2^n + 2(2^n) \\ &= 8T(n-3) + 3(2^n) \end{aligned}$$

In general, after k iterations, we'll have:

$$= 2^k T(n-k) + k(2^n)$$

Plug in k = n-1 to obtain

$$\begin{aligned} T(n) &= 2^{n-1} T(n-(n-1)) + (n-1)(2^n) \\ &= 2^{n-1} T(1) + (n-1)(2^n) \\ &= 2^{n-1}(2) + (n-1)(2^n) \\ &= 2^n + (n-1)(2^n) \\ &= \boldsymbol{n2^n} \end{aligned}$$

4) Using the iteration technique, find a Big-Oh bound for the following recurrence relation, in terms of n:

$$T(n) = 2T\left(\frac{n}{2}\right) + n^2, \text{ for } n > 1$$
$$T(1) = 1$$

Solution

$$T(n) = 2T\left(\frac{n}{2}\right) + n^2$$
$$T(n) = 2[2T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^2] + n^2$$
$$T(n) = 4T\left(\frac{n}{4}\right) + 2 \times \frac{n^2}{4} + n^2$$
$$T(n) = 4T\left(\frac{n}{4}\right) + \frac{3n^2}{2}$$
$$T(n) = 4[2T\left(\frac{n}{8}\right) + (\frac{n}{4})^2] + \frac{3n^2}{2}$$
$$T(n) = 8T\left(\frac{n}{8}\right) + 4 \times \frac{n^2}{16} + \frac{3n^2}{2}$$
$$T(n) = 8T\left(\frac{n}{8}\right) + \frac{7n^2}{4}$$

In general, after k steps, we see that our formula will iterate to:

$$T(n) = 2^k T\left(\frac{n}{2^k}\right) + \frac{(2^k - 1)n^2}{2^{k-1}}$$

Plug in $\frac{n}{2^k} = 1$, so let $n = 2^k$ and $k = log_2 n$ to obtain:

$$T(n) = nT(1) + \frac{(n-1)n^2}{{}^{n}/_{2}}$$
$$T(n) = n + 2n(n-1)$$
$$T(n) = 2n^2 - n = \boldsymbol{O(n^2)}$$

5) Use the iteration technique to determine a Big-Oh solution for the following recurrence relation:

$$T(n) = 4T\left(\frac{n}{2}\right) + n^2, T(1) = 1$$

Solution

$$T(n) = 4T\left(\frac{n}{2}\right) + n^2$$

$$T(n) = 4(4T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^2) + n^2$$

$$T(n) = 16T\left(\frac{n}{4}\right) + 4(\frac{n^2}{4}) + n^2$$

$$T(n) = 16T\left(\frac{n}{4}\right) + 2n^2$$

$$T(n) = 16(4T\left(\frac{n}{8}\right) + \left(\frac{n}{4}\right)^2) + 2n^2$$

$$T(n) = 64T\left(\frac{n}{8}\right) + 16(\frac{n^2}{16}) + 2n^2$$

$$T(n) = 64T\left(\frac{n}{8}\right) + 3n^2$$

After k iterations, we guess the form of our recurrence to be:

$$T(n) = 4^kT\left(\frac{n}{2^k}\right) + kn^2$$

We plug in a value of k such that $\frac{n}{2^k} = 1$. Namely, $n = 2^k$ and $k = log_2 n$. Note that $4^k = (2^2)^k = (2^k)^2 = n^2$:

$$T(n) = n^2T(1) + (log_2 n)n^2$$

Plugging in T(1) = 1, we have

$$T(n) = n^2 + (log_2 n)n^2 = \boldsymbol{O(n^2 lgn)}$$