Solution for Assignment #1

Khurram Hassan-Shafique

To prove statements of Q.2, I have first proved that a matrix A is a rotation matrix if and only if it is orthonormal and $\det A = 1$ (There may be other ways to prove the same statements). The definitions and other properties used in the proofs are given below:

Definition: An $n \times n$ matrix A over R is said to be orthonormal iff $A^t A = I$. Equivalently, $A^{-1} = A^t$.

Lemma: The following are equivalent.

- (i) A is orthonormal.
- (ii) $Ax.Ay = x.y$ for all vectors x and y.
- (iii) The columns of A are mutually orthogonal unit vectors.

Proof. The equivalence of (i) and (iii) is trivial. To prove that (i) implies (ii), assume that A is orthonormal then

$$\begin{aligned} Ax.Ay &= (Ax)^t (Ay) \\ &= x^t A^t A y \\ &= x^t I y \\ &= x^t y \\ &= x.y \end{aligned}$$

To prove that (ii) implies (i), assume that $Ax.Ay = x.y$ for all x and y, then we have

$$\begin{aligned} e_i.e_j &= e_i.e_j \\ \Rightarrow e_i^t e_j &= Ae_i.Ae_j \\ \Rightarrow e_i^t I e_j &= e_i^t A^t A e_j \\ \Rightarrow e_i^t I e_j - e_i^t A^t A e_j &= 0 \\ \Rightarrow e_i^t \left(I - A^t A\right) e_j &= 0 \\ \Rightarrow I - A^t A &= 0 \\ \Rightarrow A^t A &= I \end{aligned}$$

Therefore A is orthonormal.

Definition: Let SO_n be the set of all $n \times n$ orthonormal matrices A such that $\det A = 1$.

Theorem: A is a 2 dimensional rotation matrix if and only if $A \in SO_2$.

Proof: $\Longrightarrow$ Let A be a 2-D rotation matrix, then

$$A^t A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus A is orthonormal.

Now,

$$\det A = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

Hence, $A \in SO_2$.

$\Longleftarrow$ Let $A \in SO_2$ and write $A = [v_1\ v_2]$. Let the vector v_1 makes an angle θ with canonical basis vector e_1. Let R be the rotation matrix through angle θ, then we have $Re_1 = v_1$. From necessity part of the proof, $R \in SO_2$ and thus is invertible and $R^{-1} \in SO_2$. Since SO_2 is closed with respect to matrix multiplication, we have $R^{-1}A \in SO_2$. Therefore, $R^{-1}Ae_1 = R^{-1}v_1 = e_1$. Since $R^{-1}A$ is orthonormal, $R^{-1}Ae_2 = e_2$ or $-e_2$. But $\det R^{-1}A = 1 \Rightarrow R^{-1}Ae_2 = e_2$. Therefore, $R^{-1}A = I$ and hence, $A = R$.

Definition: A rotation in R^3 is a rigid motion fixing the origin and fix some vector v and acting like a 2-D rotation in the plane orthogonal to v.

Lemma: If $A \in SO_n$ and n is odd then 1 is an eigen value of A.

Proof: Recall that Λ is an eigen value of A iff $\det(A - \Lambda I) = 0$. Now,

$$\begin{aligned}
\det(A-I) &= \det A^t \det(A-I) \quad \text{Since } \det A^t = 1 \\
&= \det A^t(A-I) \\
&= \det(A^tA - A^t) \\
&= \det(I - A^t) \\
&= \det(I-A)^t \\
&= \det(I-A) \\
&= \det(-(A-I)) \\
&= (-1)^n \det(A-I) \\
&= -\det(A-I) \qquad \text{Since } n \text{ is odd} \\
\Rightarrow \det(A-I) &= -\det(A-I) \\
\Rightarrow \det(A-I) &= 0
\end{aligned}$$

Theorem: A is a 3D rotation matrix if and only if $A \in SO_3$.

Proof: $\Longrightarrow$ Let A be a 3D rotation matrix. Since A is a rigid motion, we have, $\forall x, y$,

$$\begin{aligned}
|Ax - Ay| &= |x-y| \\
\Rightarrow (Ax-Ay).(Ax-Ay) &= (x-y).(x-y)
\end{aligned}$$

In particular, for y=0, we have $Ax.Ax = x.x$. Now,

$$\begin{aligned}
(Ax-Ay).(Ax-Ay) &= (x-y).(x-y) \\
\Rightarrow Ax.Ax - 2Ax.Ay + Ay.Ay &= x.x - 2x.y + y.y \\
\Rightarrow Ax.Ay = x.y &
\end{aligned}$$

So A preserves dot product and hence is an orthonormal matrix. Determinant of an orthonormal matrix is either 1 or -1. But since, rotation is orientation preserving, we have $\det A = 1$. Thus $A \in SO_3$.

$\Longleftarrow$ Let $A \in SO_3$, then by above Lemma, we have that A has 1 as an eigen value. Hence, there exists a unit vector v_1 such that $Av_1 = v_1$. Consider an orthonormal basis $\{v_1, v_2, v_3\}$ of R^3 and let $P = [v_1\ v_2\ v_3]$ such that

$\det P = 1$. Note that $P \in SO_3$, hence $P^{-1} \in SO_3$ and therefore the matrix $A' = P^{-1} * A * P \in SO_3$. Now,

$$A'e_1 = P^{-1}APe_1 = P^{-1}Av_1 = P^{-1}v_1 = e_1$$

Therefore,

$$A' = \begin{bmatrix} 1 & 0_{1\times 2} \\ 0_{2\times 1} & R_{2\times 2} \end{bmatrix}$$

where $R \in SO_2$ and is thus a rotation matrix in 2D. Hence A' and also A have the effect of rotating a vector in a plane orthogonal to a fixed vector v.

Q.2 (a) Prove that the product of two 3D rotation matrices is also a 3D rotation matrix.

Proof: Let R_1 and R_2 be two rotation matrices and $A = R_1R_2$. Since $R_1, R_2 \in SO_3$, $A \in SO_3$ and hence by above theorem is a rotation matrix.

Q.2 (b) Prove that the rank of a 3D rotation matrix has rank 3.

Proof: Since every 3D rotation matrix $R \in SO_3$ and orthonormal matrices are invertible, rank of a 3D rotation matrix is 3.

Q.2 (c) Prove that the inverse of a 3D rotation matrix is its transpose.

Proof: Since every 3D rotation matrix R is orthonormal, by definition of orthonormal matrix, its inverse is its transpose.

Q.2 (d) Prove that the product of two matrices associated with rigid transformations is a matrix associated with some rigid transformation.

Proof: Let D_1 and D_2 be the two matrices associated with rigid transformations. Then, we have

$$\begin{aligned} D &= D_1D_2 \\ &= \begin{bmatrix} R_1 & T_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2 & T_2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} R_1R_2 + T_1(0) & R_1T_2 + T_1(1) \\ 0(R_2) + 1(0) & 0(T_2) + 1(1) \end{bmatrix} \\ &= \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \end{aligned}$$

where $R = R_1R_2$ and $T = R_1T_2 + T_1$.

Since R is the product of two rotation matrices, by Q.2 (a), R is also a rotation matrix. Hence D is a matrix associated with some rigid transformation.

Q.2 Prove that e. The change of coordinates associated with a rigid transformation preserves distances and angles.

Proof: Note that this is not a proof because by my definition of rotation matrices, they preserve distances and angles and the property that a matrix is rotation matrix if and only if it is orthonormal and its determinant is 1 is proved by using this definition.

Let p and q be two points in R^3, and let $DX = RX + T$ be a rigid transformation, then we have,

$$
\begin{aligned}
|Dp - Dq| &= |(Rp + T) - (Rq + T)| \\
&= |Rp + T - Rq - T| \\
&= |Rp - Rq| \\
&= (Rp - Rq) \,.\, (Rp - Rq) \\
&= R\,(p - q)\,.R\,(p - q) \\
&= (p - q) \,.\, (p - q) \qquad\qquad \text{Since } R \text{ is orthonormal} \\
&= |p - q|
\end{aligned}
$$

Hence, D preserves distances.

To show that D preserves angles, let p, q, and r be three points in R^3. Then the angle $\angle pqr$ is defined as

$$
\angle pqr = \cos^{-1}\left(\frac{(q - p)\,.\,(r - q)}{|q - p|\,|r - q|}\right)
$$

Now, the cosine of the angle after rigid transformation $\angle DpDqDr$ is given as

$$
\begin{aligned}
\cos\angle DpDqDr &= \left(\tfrac{(Dq - Dp).(Dr - Dq)}{|Dq - Dp||Dr - Dq|}\right) \\
&= \left(\tfrac{(Rq + T - Rp - T).(Rr + T - Rq - T)}{|q - p||r - q|}\right) \quad \text{Since } |Dp - Dq| = |p - q| \\
&= \left(\tfrac{(Rq - Rp).(Rr - Rq)}{|q - p||r - q|}\right) \\
&= \left(\tfrac{R(q - p).R(r - q)}{|q - p||r - q|}\right) \\
&= \left(\tfrac{(q - p).(r - q)}{|q - p||r - q|}\right) \qquad\qquad \text{Since } R \text{ is orthonormal} \\
&= \cos\angle pqr
\end{aligned}
$$

Hence, D preserves the angles.