Derivatives and Averages

Lecture-6

Derivatives and Averages

- Derivative: Rate of change of some quantity
  - Speed is a rate of change of a distance
  - Acceleration is a rate of change of speed
- Average (Mean)
  - The numerical result obtained by dividing the sum of two or more quantities by the number of quantities

Derivative

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x) - f(x - \Delta x)}{\Delta x} = f'(x) = f_x$$

$$v = \frac{ds}{dt} \qquad \text{speed}$$

$$a = \frac{dv}{dt} \qquad \text{acceleration}$$

Examples

$$y = x^2 + x^4 \qquad y = \sin x + e^{-x}$$

$$\frac{dy}{dx} = 2x + 4x^3 \qquad \frac{dy}{dx} = \cos x + (-1)e^{-x}$$

Second Derivative

$$\frac{df_x}{dx} = f''(x) = f_{xx}$$

$$y = x^2 + x^4$$

$$\frac{dy}{dx} = 2x + 4x^3$$

$$\frac{d^2 y}{dx^2} = 2 + 12x^2$$

Discrete Derivative

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x) - f(x - \Delta x)}{\Delta x} = f'(x)$$

$$\frac{df}{dx} = \frac{f(x) - f(x-1)}{1} = f'(x)$$

$$\frac{df}{dx} = f(x) - f(x-1) = f'(x)$$

(Finite Difference)

Discrete Derivative

$$\frac{df}{dx} = f(x) - f(x-1) = f'(x)$$ Left difference

$$\frac{df}{dx} = f(x) - f(x+1) = f'(x)$$ Right difference

$$\frac{df}{dx} = f(x+1) - f(x-1) = f'(x)$$ Center difference

Example

	F(x)=10	10	10	10	20	20	20
Left difference	F'(x)=0	0	0	0	10	0	0
	F''(x)=0	0	0	0	10	-10	0

-1	1		left difference
1	-1		right difference
-1	0	1	center difference

Derivatives in Two Dimensions

$$f(x, y)$$

(partial Derivatives)

$$\frac{\partial f}{\partial x} = f_x = \lim_{\Delta x \to 0} \frac{f(x, y) - f(x - \Delta x, y)}{\Delta x}$$

$$\frac{\partial f}{\partial y} = f_y = \lim_{\Delta y \to 0} \frac{f(x, y) - f(x, y - \Delta y)}{\Delta y}$$

(f_x, f_y) Gradient Vector

$$\text{magnitude} = \sqrt{(f_x^2 + f_y^2)}$$

$$\text{direction} = \theta = \tan^{-1} \frac{f_y}{f_x}$$

$$\nabla^2 f = f_{xx} + f_{yy} = \text{Laplacian}$$

Derivatives of an Image

Derivative & average

$$\begin{matrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{matrix} \qquad \begin{matrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{matrix}$$

$$f_x \qquad\qquad f_y$$

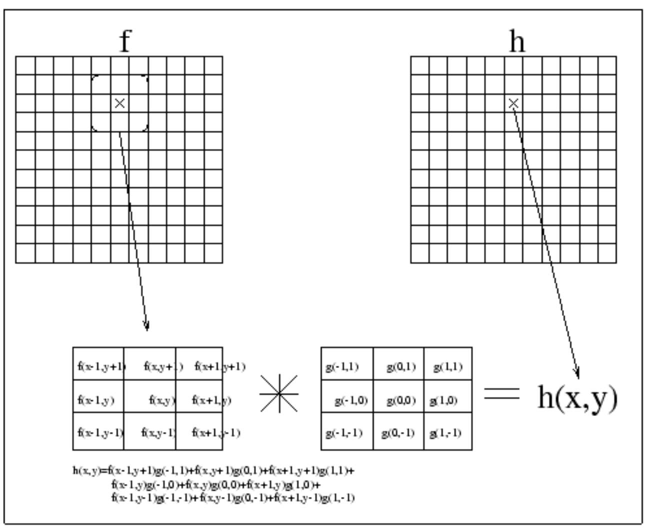
Prewit

$$I(x, y) = \begin{bmatrix} 10 & 10 & 20 & 20 & 20 \\ 10 & 10 & 20 & 20 & 20 \\ 10 & 10 & 20 & 20 & 20 \\ 10 & 10 & 20 & 20 & 20 \\ 10 & 10 & 20 & 20 & 20 \end{bmatrix} \qquad I_x = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 30 & 30 & 0 & 0 \\ 0 & 30 & 30 & 0 & 0 \\ 0 & 30 & 30 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Derivatives of an Image

$$I(x,y) = \begin{bmatrix} 10 & 10 & 20 & 20 & 20 \\ 10 & 10 & 20 & 20 & 20 \\ 10 & 10 & 20 & 20 & 20 \\ 10 & 10 & 20 & 20 & 20 \\ 10 & 10 & 20 & 20 & 20 \end{bmatrix} \qquad I_y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Convolution



Convolution (contd)

$$h(x,y) = \sum_{i=-1}^{1} \sum_{j=-1}^{1} f(x+i, y+j) g(i,j)$$

$$h(x,y) = f(x,y) * g(x,y)$$

Average

$$I = \frac{I_1 + I_2 + \ldots I_n}{n} = \frac{\sum_{i=1}^{n} I_i}{n}$$

Weighted Average

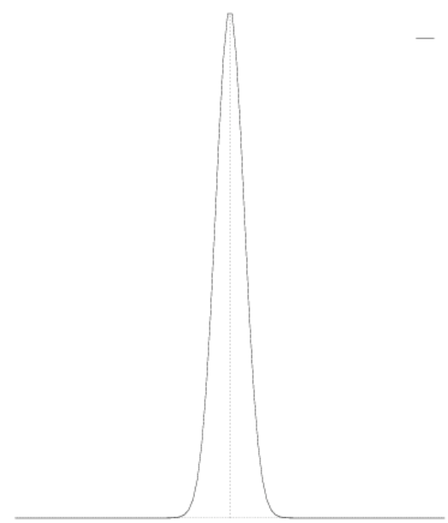
$$I = \frac{w_1 I_1 + w_2 I_2 + \ldots + w_n I_n}{n} = \frac{\sum_{i=1}^{n} w_i I_i}{n}$$

Gaussian

$$g(x) = e^{\frac{-x^2}{2\sigma^2}}$$

Standard deviation

x	-3	-2	-1	0	1	2	3
g(x)	.011	.13	.6	1	.6	.13	.011



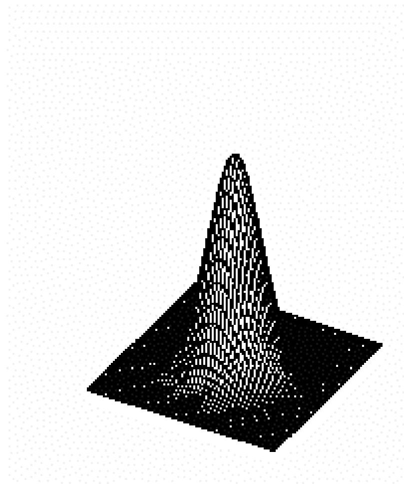
2-D Gaussian

$$g(x,y) = e^{\frac{-(x^2+y^2)}{2\sigma^2}}$$

0	0	0	0	1	2	2	2	1	0	0	0	0
0	0	1	3	6	9	11	9	6	3	1	0	0
0	1	4	11	20	30	34	30	20	11	4	1	0
0	3	11	26	50	73	82	73	50	26	11	3	0
1	6	20	50	93	136	154	136	93	50	20	6	1
2	9	30	73	136	198	225	198	136	73	30	9	2
2	11	34	82	154	225	255	225	154	82	34	11	2
2	9	30	73	136	198	225	198	136	73	30	9	2
1	6	20	50	93	136	154	136	93	50	20	6	1
0	3	11	26	50	73	82	73	50	26	11	3	0
0	1	4	11	20	30	34	30	20	11	4	1	0
0	0	1	3	6	9	11	9	6	3	1	0	0
0	0	0	0	1	2	2	2	1	0	0	0	0

$\sigma = 2$

2-D Gaussian



Gaussian

- Most natural phenomenon can be modeled by Gaussian.
- Take a bunch of random variables of any distribution, find the mean, the mean will approach to Gaussian distribution.
- Gaussian is very smooth function, it has infinite no of derivatives.

Gaussian

- Fourier Transform of Gaussian is Gaussian.
- If you convolve Gaussian with itself, it is again Gaussian.
- There are cells in human brain which perform Gaussian filtering.
  - Laplacian of Gaussian edge detector

Carl F. Gauss

- Born to a peasant family in a small town in Germany.
- Learned counting before he could talk.
- Contributed to Physics, Mathematics, Astronomy,…
- Discovered most methods in modern mathematics, when he was a teenager.

Carl F. Gauss

- Some contributions
  - Gaussian elimination for solving linear systems
  - Gauss-Seidel method for solving sparse systems
  - Gaussian curvature
  - Gaussian qudrature

Noise

- Image contains noise due to
  - Lighting variations
  - Lens de-focus
  - Camera electronics
  - Surface reflectance
- Remove noise
  - Averaging
  - Weighted averaging

Example

F(x)=	10	10	10	10	20	20	20
n(x)=	0	5	0	0	3	0	0
F~(x)=	10	15	10	10	23	20	20
H(x)=	10	12	12	14	17	21	20

Edge Detection

- Find edges in the image
- Edges are locations where intensity changes the most
- Edges can be used to represent a shape of an object

Edge Detection

- Images contain noise, need to remove noise by averaging, or weighted averaging
- To detect edged compute derivative of an image (gradient)
- If gradient magnitude is high at pixel, intensity change is maximum, that is an edge pixel
- If Laplacian (second derivative) is zero then at that point the first derivative is maximum, that point is an edge pixel.

Edge Detectors

- Prewit
- Sobel
- Roberts
- Marr-Hildreth (Laplacian of Gaussian)
- Canny (Gradient of Gaussian)
- Haralick (Facet Model)

Derivatives of an Image

P
Sobel

$$\begin{matrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{matrix} \qquad \begin{matrix} -1 & -2 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{matrix}$$

$$f_x \qquad\qquad f_y$$

Roberts

$$\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \qquad \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}$$

$$f_x \qquad\qquad f_y$$

Laplacian of Gaussian

- Filter the image by weighted averaging (Gaussian)
- Find Laplacian of image
- Detect zero-crossings

$$\nabla^2 f = f_{xx} + f_{yy} = \text{Laplacian}$$

Canny Edge Detector

- Filter the image with Gaussian
- Find the gradient magnitude
- Edges are maxima of gradient magnitude

Haralick's facet Model based Edge detector

- Fit a bi-cubic polynomial to a local neighborhood of a pixel
- If the second derivative is zero, and the third derivative is negative, then that point is an edge point.