Block-Diagram Reduction

(a) Cascaded subsystems;
(b) Equivalent transfer function.

Block-Diagram Reduction

(a) Parallel subsystems;
(b) Equivalent transfer function.
Feedback Control System and Simplified Model

 Equivalent Transfer Function

\[ R(s) \quad \frac{G(s)}{1 + G(s)H(s)} \quad C(s) \]

Input \quad \frac{G(s)}{1 + G(s)H(s)} \quad Output
Convert a block diagram to a signal-flow graph

Step 1: signal nodes:

\[ R(s) \quad G_1(s) \quad G_2(s) \quad G_3(s) \quad H_1(s) \quad H_2(s) \quad H_3(s) \quad C(s) \]

\[ V_{d_1}(s) \quad V_{d_2}(s) \quad V_{d_3}(s) \quad V_{d_4}(s) \quad V_{d_5}(s) \]

\[ V_{y_1}(s) \quad V_{y_2}(s) \]

Step 2: Signal-flow graph

\[ R(s) \quad G_1(s) \quad G_2(s) \quad G_3(s) \quad H_1(s) \quad H_2(s) \quad H_3(s) \quad C(s) \]

\[ V_{d_1}(s) \quad V_{d_2}(s) \quad V_{d_3}(s) \quad V_{d_4}(s) \quad V_{d_5}(s) \]

\[ V_{y_1}(s) \quad V_{y_2}(s) \]

\[ H_1(s) \quad H_2(s) \quad H_3(s) \]
Step 3: Simplified signal-flow graph
Mason's Rule

Earlier in this chapter, we showed how to reduce block diagrams to single transfer functions. Now we are ready to show a technique for reducing signal-flow graphs to single transfer functions that relate the output of a system to its input.

The block diagram reduction technique we studied in Section 5.2 requires successive application of fundamental relationships in order to arrive at the system transfer function. On the other hand, Mason’s rule for reducing a signal-flow graph to a single transfer function requires the application of one formula. The formula was derived by S. J. Mason when he related the signal-flow graph to the simultaneous equations that can be written from the graph (Mason, 1953).

In general, it can be complicated to implement the formula without making mistakes. Specifically, the existence of what we will later call nontouching loops increases the complexity of the formula. However, many systems do not have nontouching loops. For these systems, you may find Mason’s rule easier to use than block diagram reduction.

Mason’s formula has several components that must be evaluated. First, we must be sure that the definitions of the components are well understood. Then we must exert care in evaluating the components. To that end, we discuss some basic definitions applicable to signal-flow graphs; then we state Mason’s rule and do an example.

Definitions

Loop gain: The product of branch gains found by traversing a path that starts at a node and ends at the same node without passing through any other node more than once and following the direction of the signal flow. For examples of loop gains, see Figure 5.20. There are four loop gains:

1. \(G_2(s)H_1(s)\)  
2. \(G_4(s)H_2(s)\)  
3. \(G_4(s)G_5(s)H_3(s)\)  
4. \(G_4(s)G_6(s)H_3(s)\)

\[ \text{Forward-path gain: The product of gains found by traversing a path from the input node to the output node of the signal-flow graph in the direction of signal} \]

\[ \text{Example:} \]
flow. Examples of forward-path gains are also shown in Figure 5.20. There are two forward-path gains:

1. \( G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)G_7(s) \)  
2. \( G_1(s)G_2(s)G_3(s)G_4(s)G_6(s)G_7(s) \)  

**Nontouching loops:** Loops that do not have any nodes in common. In Figure 5.20, loop \( G_3(s)H_1(s) \) does not touch loops \( G_4(s)H_2(s) \), \( G_4(s)G_5(s)H_3(s) \), and \( G_4(s)G_6(s)H_3(s) \).

**Nontouching-loop gain:** The product of loop gains from nontouching loops taken two, three, four, etc., at a time. In Figure 5.20, the product of loop gain \( G_2(s)H_1(s) \) and loop gain \( G_4(s)H_2(s) \) is a nontouching-loop gain taken two at a time. In summary, all three of the nontouching-loop gains taken two at a time are:

1. \([G_2(s)H_1(s)][G_4(s)H_2(s)]\)  
2. \([G_2(s)H_1(s)][G_4(s)G_5(s)H_3(s)]\)  
3. \([G_2(s)H_1(s)][G_4(s)G_6(s)H_3(s)]\)  

The product of loop gains \([G_4(s)G_5(s)H_3(s)][G_4(s)G_6(s)H_3(s)]\) is not a nontouching-loop gain since these two loops have nodes in common. In our example there are no nontouching-loop gains taken three at a time since three nontouching loops do not exist in the example.

We are now ready to state Mason’s rule.

**Mason’s Rule**

The transfer function, \( C(s)/R(s) \), of a system represented by a signal-flow graph is given by:

\[
G(s) = \frac{C(s)}{R(s)} = \frac{\sum T_k \Delta_k}{\Delta}
\]

where

- \( k = \) number of forward paths
- \( T_k = \) the \( k \)th forward-path gain
- \( \Delta = 1 - \sum \) loop gains + \( \sum \) nontouching-loop gains taken two at a time - \( \sum \) nontouching-loop gains taken three at a time + \( \sum \) nontouching-loop gains taken four at a time - \cdots
\[ \Delta_k = \Delta - \sum \text{loop gain terms in } \Delta \text{ that touch the } k\text{th forward path}. \] In other words, \( \Delta_k \) is formed by eliminating from \( \Delta \) those loop gains that touch the \( k\)th forward path.

Notice the alternating signs for the components of \( \Delta \). The following example will help clarify Mason’s rule.

**Example 5.7**

**Transfer function via Mason’s rule**

**Problem** Find the transfer function, \( C(s)/R(s) \), for the signal-flow graph in Figure 5.21.

![Signal-flow graph for Example 5.7](image)

**Solution** First, identify the forward-path gains. In this example there is only one:

\[ G_1(s)G_2(s)G_3(s)G_4(s)G_6(s) \] (5.29)

Second, identify the loop gains. There are four, as follows:

1. \( G_2(s)H_1(s) \) (5.30a)
2. \( G_4(s)H_2(s) \) (5.30b)
3. \( G_7(s)H_4(s) \) (5.30c)
4. \( G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s) \) (5.30d)

Third, identify the nontouching loops taken two at a time. From Eqs. (5.30) and Figure 5.21, we can see that loop 1 does not touch loop 2, loop 1 does not touch loop 3, and loop 2 does not touch loop 3. Notice that loops 1, 2, and 3 all touch loop 4. Thus, the combinations of nontouching loops taken two at a time are as follows:

\[ \text{Loop 1 and loop 2: } G_2(s)H_1(s)G_4(s)H_2(s) \] (5.31a)
\[ \text{Loop 1 and loop 3: } G_2(s)H_1(s)G_7(s)H_4(s) \] (5.31b)
\[ \text{Loop 2 and loop 3: } G_4(s)H_2(s)G_7(s)H_4(s) \] (5.31c)

Finally, the nontouching loops taken three at a time are as follows:

\[ \text{Loops 1, 2, and 3: } G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s) \] (5.32)
Now, from Eq. (5.28) and its definitions, we form $\Delta$ and $\Delta_k$. Hence
\[
\Delta = 1 - [G_2(s)H_1(s) + G_4(s)H_2(s) + G_7(s)H_4(s) \\
+ G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_9(s)] \\
+ [G_2(s)H_1(s)G_4(s)H_2(s) + G_2(s)H_1(s)G_7(s)H_4(s) \\
+ G_4(s)H_2(s)G_7(s)H_4(s)] \\
- [G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)]
\] (5)

We form $\Delta_k$ by eliminating from $\Delta$ the loop gains that touch the $k$th forward path:
\[
\Delta_1 = 1 - G_7(s)H_4(s)
\] (5)

Expressions (5.29), (5.33), and (5.34), are now substituted into Eq. (5.32) yielding the transfer function:
\[
G(s) = \frac{T_1 \Delta_1}{\Delta} = \frac{[G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)]}[1 - G_7(s)H_4(s)]
\] (5)

Since there is only one forward path, $G(s)$ consists of only one term rather than the sum of terms, each coming from a forward path.
Example: Use Mason's rule to get transfer function \( G(s) = \frac{C(s)}{R(s)} \).

**Problem** Reduce the system shown in Figure 5.11 to a single transfer function.

![Block Diagram](image)

**Converting a block diagram to a signal-flow graph**

**Problem** Convert the block diagram of Figure 5.11 to a signal-flow graph.

**Figure 5.19**
Signal-flow graph development:
- a. signal nodes;
- b. signal-flow graph;
- c. simplified signal flow graph

![Signal Flow Graph](image)
Forward-path gains are \( G_1 G_2 G_3 \) and \( G_1 G_3 \).

Loop gains are \(-G_1 G_2 H_1\), \(-G_2 H_2\), and \(-G_3 H_3\).

Nontouching loops are

\[-G_1 G_2 H_1 \| -G_3 H_3] = G_1 G_2 G_3 H_1 H_3\]

and

\[-G_2 H_2 \| -G_3 H_3] = G_2 G_3 H_2 H_3\].

Also, \( \Delta = 1 + G_1 G_2 H_1 + G_2 H_2 + G_3 H_3 + G_1 G_2 G_3 H_1 H_3 + G_2 G_3 H_2 H_3\).

Finally, \( \Delta_1 = 1 \) and \( \Delta_2 = 1 \).

Substituting these values into

\[ T(s) = \frac{C(s)}{R(s)} = \frac{\sum \Delta_k}{\Delta} \]

yields

\[ T(s) = \frac{G_1(s) G_2(s) [1 + G_1(s)]}{[1 + G_2(s) H_2(s) + G_1(s) G_2(s) H_1(s)][1 + G_3(s) H_3(s)]} \]