Problem 1. (25 points) Consider a mechanical system shown in the figure:

Its differential equation is given by

\[
\begin{align*}
  m_1 \frac{d^2 y_1(t)}{dt^2} + (k_1 + k_2) y_1(t) - k_2 y_2(t) &= u_1(t) \\
  m_2 \frac{d^2 y_2(t)}{dt^2} - k_2 y_1(t) + (k_1 + k_2) y_2(t) &= u_2(t)
\end{align*}
\]

Express the system in the state space representation, considering \(y_1(t)\) and \(y_2(t)\) to be the output and \(u_1(t)\) and \(u_2(t)\) to be the input (note that this is a two-input two-output system).

Solution: Let

\[
X = \begin{bmatrix} y_1 \\ y_2 \\ y_1' \\ y_2' \end{bmatrix}, \quad U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}
\]

\[
\begin{align*}
  \dot{X}_1 &= X_2 \\
  \dot{X}_2 &= -\frac{k_1+k_2}{m_1} y_1(t) + \frac{k_2}{m_1} y_2(t) + \frac{u_1}{m_1} \\
  \dot{X}_3 &= X_4 \\
  \dot{X}_4 &= \frac{k_2}{m_2} y_1(t) - \frac{k_1+k_2}{m_2} y_2(t) + \frac{u_2}{m_2}
\end{align*}
\]

\[
X = \begin{bmatrix}
  0 & 1 & 0 & 0 \\
  -\frac{k_1+k_2}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\
  0 & 0 & 0 & 1 \\
  \frac{k_2}{m_2} & 0 & -\frac{k_1+k_2}{m_2} & 0
\end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ \frac{1}{m_2} \end{bmatrix} U
\]

\[
Y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} X.
\]
Problem 2. (25 points) Suppose $\mathbf{x}$ is a column vector

$$\mathbf{x} = \begin{bmatrix} 7 \\ 2 \\ 4 \end{bmatrix}.$$  

Then with respect to the standard basis $(e_1, e_2, e_3)$ the representation is just $\mathbf{x} = [7 \ 2 \ 4]^T$. Suppose we take as a basis the vectors

$$p_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad p_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad p_3 = \begin{bmatrix} 7 \\ 8 \end{bmatrix},$$

What is the representation of $\mathbf{x}$ in the new basis?

Solution: \[\mathbf{x} = a_1 \mathbf{p}_1 + a_2 \mathbf{p}_2 + a_3 \mathbf{p}_3\]

$$\begin{bmatrix} 7 \\ 2 \\ 4 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + a_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + a_3 \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

$$\Rightarrow a_1 = -16, \quad a_2 = 18, \quad a_3 = -7$$

So the representation of $\mathbf{x}$ in the basis $\{\mathbf{p}_i\}$ is

$$\begin{bmatrix} -16 \\ 18 \\ -7 \end{bmatrix}.$$
Problem 3. (25 points) For the following system
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 \\
1 & 2 & 0 \\
0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\]
find its eigenvalues, eigenvectors, and then diagonalize it.

\[A = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 2 \\
0 & 1 & 1
\end{bmatrix}\]

Its characteristic polynomial is
\[
\Delta(\lambda) = \det(\lambda I - A) = \det
\begin{bmatrix}
\lambda & 0 & 0 \\
-1 & \lambda & -2 \\
0 & -1 & \lambda - 1
\end{bmatrix}
= \lambda[\lambda(\lambda - 1) - 2] = (\lambda - 2)(\lambda - 1)^2
\]

Thus A has eigenvalues 2, -1, and 0. The eigenvector associated with \(\lambda = 2\) is any nonzero solution of
\[(A - 2I)q_1 = \begin{bmatrix}
-2 & 0 & 0 \\
1 & -2 & 2 \\
0 & 1 & -1
\end{bmatrix} q_1 = 0\]

Thus \(q_1 = [0 \ 1 \ 1]'\) is an eigenvector associated with \(\lambda = 2\). Note that the eigenvector is not unique, \([0 \ \alpha \ \alpha]'\) for any nonzero real \(\alpha\) can also be chosen as an eigenvector. The eigenvector associated with \(\lambda = -1\) is any nonzero solution of
\[(A - (-1)I)q_2 = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 2 \\
0 & 1 & 2
\end{bmatrix} q_2 = 0\]

which yields \(q_2 = [0 \ -2 \ 1]'\). Similarly, the eigenvector associated with \(\lambda = 0\) can be computed as \(q_3 = [2 \ 1 \ -1]'\). Thus the representation of A with respect to \(\{q_1, q_2, q_3\}\) is
\[
\hat{A} = \begin{bmatrix}
2 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

It is a diagonal matrix with eigenvalues on the diagonal. This matrix can also be obtained by computing
\[\hat{A} = Q^{-1}AQ\]

with
\[Q = [q_1 \ q_2 \ q_3] = \begin{bmatrix}
0 & 0 & 2 \\
1 & -2 & 1 \\
1 & 1 & -1
\end{bmatrix}
\]
Problem 4. (25 points) For the following system
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
1 \\
1
\end{bmatrix} u,
\]
is it controllable?

Solution. Controllability matrix

\[ P = [B \ AB] = \begin{bmatrix}
1 & -1 \\
1 & -1
\end{bmatrix} \]

\[ \text{rank}(P) = 1 \]

Not controllable.