Sample Assignment # 1.1

1. Prove that, for sets A and B, 
   \( A = B \) if and only if \( (A \cap \sim B) \cup (\sim A \cap B) = \emptyset \), 
   where \( \sim S \) is the complement of \( S \)

Part 1) Prove if \( A = B \), then \( (A \cap \sim B) \cup (\sim A \cap B) = \emptyset \)

   Assume \( A = B \) then \( (A \cap \sim B) \cup (\sim A \cap B) = (A \cap A) \cup (\sim A \cap A) \)

   Now, any set intersected with its complement must be empty by the definition of complement, so \( (A \cap A) = \emptyset \) and \( (\sim A \cap A) = \emptyset \) and thus their union is also empty, proving that \( A = B \) implies \( (A \cap \sim B) \cup (\sim A \cap B) = \emptyset \). 

Part 2) Prove if \( (A \cap \sim B) \cup (\sim A \cap B) = \emptyset \), then \( A = B \)

   Assume \( (A \cap \sim B) \cup (\sim A \cap B) = \emptyset \) then, by definition of union, 
   \( (A \cap \sim B) = \emptyset \) and \( (\sim A \cap B) = \emptyset \) else the union would have at least one element in it. This in turn implies that no element of \( A \) is in the complement of \( B \) and no element of \( B \) is in the complement of \( A \). Thus, all elements of \( A \) are in \( B \) and all elements of \( B \) are in \( A \). Stated more formally, \( A \subseteq B \) and \( B \subseteq A \). But, mutual containment is the definition of set equality and so \( A = B \). proving that \( (A \cap \sim B) \cup (\sim A \cap B) = \emptyset \) implies \( A = B \).
Sample Assignment # 1.2

2. Prove, If S is any finite set with \(|S| = n\), then \(|S \times S \times S \times S| \leq |P(S)|\), for all \(n \geq N\), where \(N\) is some constant, the minimum value of which you must discover and use as the basis for your proof.

Proof:

(This is the same as showing \(n^4 \leq 2^n\), for all \(n \geq N\). We shall show this is true when \(N=16\).)

Basis: \(16^4 = (2^4)^4 = 2^{16} \leq 2^{16}\). This proves the base case. Note: that

\(15^4 = 50625\) and \(2^{15} = 32768\) and so \(N=15\) fails.

I.H. Assume for some \(K\), \(K \geq 16\), \(K^4 \leq 2^K\).

I.S.(K+1) : \((K+1)^4 = K^4 + 4K^3 + 6K^2 + 4K + 1\)

\[\leq K^4 + 4K^3 + 6K^3 + 4K^3 + K^3 \text{ since } K \geq 1\]

\[= K^4 + 15K^3 \leq K^4 + K^4 \text{ since } K \geq 16\]

\[\leq 2^K + 2^K \text{ by IH}\]

\[= 2^{K+1}\]

Thus, \((K+1)^4 \leq 2^{K+1}\) and the I.S. is proven.
3. Consider the function \( \text{pair}: \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) defined by \( \text{pair}(x,y) = 2^x \cdot (2y + 1) - 1 \)

Show that \( \text{pair} \) is a surjection (onto \( \mathbb{N} \)).

Note: It’s actually a bijection (1-1 onto \( \mathbb{N} \)), but I am not asking you to show that.

Proof:

Case 1: All even numbers are in range.
Let \( x=0 \). Then \( 2^x \cdot (2y + 1) - 1 = 2y + 1 - 1 = 2y \) where \( y \geq 0 \)
Since \( y \) ranges over the natural numbers, \( 2y \) ranges over all even numbers and case 1 is shown.

Case 2: All odd numbers are in range.
Let \( x>0 \). Odd numbers are all those of the form \( 2z-1, z>0 \). That is, they have a non-trivial even factor and an odd factor that could be just 1. Essentially, then, every odd number is 1 less than some non-zero even number. But, every non-zero even number has a factorization that is of the form \( 2^x \cdot (2y + 1) \), where \( x>0 \) and \( y \geq 0 \). This shows that \( 2^x \cdot (2y + 1) - 1 \) ranges over all odd numbers, when \( x>0 \) and case 2 is shown.
3. Consider the function $\text{pair}: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ defined by $\text{pair}(x,y) = 2^x (2y + 1) - 1$.

Show that $\text{pair}$ is a surjection (onto $\mathbb{N}$).

Note: It’s actually a bijection (1-1 onto $\mathbb{N}$), but I am not asking you to show that.

Simpler Proof Informed by case 2 on previous page:

All natural numbers are in range of $2^x (2y + 1) - 1$.

We show this by proving that all positive natural numbers are in the range of $2^x (2y + 1)$. We note that every non-zero natural number has a unique factorization that is of the form $2^x (2y + 1)$, where the first term captures the number’s even part (or $x=0$ if not even) and the second part captures its odd part. This shows that $2^x (2y + 1) - 1$ ranges over all natural numbers and so the range of $\text{pair}$ is the set of all natural numbers. Note, we actually show uniqueness here based on the unique prime factorization theorem and so $\text{pair}$ is not just a surjection; it is a bijection.