Problem 1.

Use Laws of Logic and Rules of Inference to justify the following arguments.

(a) Section 1.5 #28, 5 pts
\[
\begin{align*}
\forall x (P(x) \lor Q(x)) \\
\forall x ((\neg P(x) \land Q(x)) \rightarrow R(x)) \\
\therefore \forall x (\neg R(x) \rightarrow P(x))
\end{align*}
\]

<table>
<thead>
<tr>
<th>Step</th>
<th>Reason</th>
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<tbody>
<tr>
<td>1.</td>
<td>\forall x (P(x) \lor Q(x))</td>
</tr>
<tr>
<td>2.</td>
<td>P(a) \lor Q(a)</td>
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<tr>
<td>3.</td>
<td>\forall x ((\neg P(x) \land Q(x)) \rightarrow R(x))</td>
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<tr>
<td>4.</td>
<td>\forall x ((\neg P(x) \lor \neg Q(x)) \lor R(x))</td>
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<tr>
<td>5.</td>
<td>\forall x ((P(x) \lor \neg Q(x)) \lor R(x))</td>
</tr>
<tr>
<td>6.</td>
<td>P(a) \lor \neg Q(a) \lor R(a)</td>
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<tr>
<td>7.</td>
<td>(s \lor p) \rightarrow t</td>
</tr>
<tr>
<td>8.</td>
<td>P(a) \lor R(a)</td>
</tr>
<tr>
<td>9.</td>
<td>\neg R(a) \rightarrow P(a)</td>
</tr>
<tr>
<td>10.</td>
<td>\forall x (\neg R(x) \rightarrow P(x))</td>
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(b) 5 pts
\[
\begin{align*}
p \lor q \\
u \land r \\
r \rightarrow \neg t \\
(s \lor p) \rightarrow t
\end{align*}
\]

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<td>r</td>
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<td>q</td>
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</table>
Grading:
1. Full credit if everything follows correctly \textit{with reasons}.
2. $-2$ points each if reasons are missing, or missing more than half of the steps.
3. $-3$ points each if the solution used verbal arguments instead of algebra (logic and inference).
4. Give 0 point each if there exists no procedure but conclusion.

Problem 2.
Prove or disprove each of the following statements.

(a) (Section 1.6 #8, 5 pts) If $n \geq 1$ is a perfect square, then $n + 2$ is not a perfect square.
Proof: Since $n \geq 1$ is a perfect square
There exists an integer $m \geq 1$ so that $n = m^2$
Therefore the smallest perfect square greater than $n$ is $(m + 1)^2$
$(m + 1)^2 - (n + 2) = m^2 + 2m + 1 - (n + 2) = n + 2m + 1 - n - 2 = 2(m - 1) + 1$
Since $m \geq 1$, thus $2(m - 1) + 1 > 0$
Therefore $(m + 1)^2 > (n + 2)$, and $n + 2$ cannot be a perfect square.

(b) (Section 1.7 #10, 5 pts) Consider the following numbers.
\[
\begin{align*}
65 & \quad 1006 \\
1120 & \quad 92399 \\
24491 & \quad 8190 + 71775
\end{align*}
\]
It is possible to select 2 different numbers from the 3 numbers above such that their product is non-negative.
Proof: Of these three numbers, at least two must have the same sign (both non-negative or both negative), since there are only two signs (negative and non-negative) need to be considered.
The product of two with the same sign is non-negative.
It is a non-constructive proof, since we have not identified which product is non-negative.

(c) (Section 1.7 #12, 5 pts) If $a$ and $b$ are rational numbers, then $a^b$ is also rational.
Disproof: Take $a = 2$ and $b = 1/2$,
Then $a^b = 2^{1/2} = \sqrt{2}$
Because we know that $\sqrt{2}$ is not rational, then we disprove the statement.

(d) (Section 1.7 #32, 5 pts) $\sqrt[3]{2}$ is irrational.
Proof: We can prove it by contradiction, which means that $\sqrt[3]{2}$ is rational.
Then we have $\sqrt[3]{2} = a/b$, where $a$ and $b$ are integers without common factors.
It follows that $2 = a^3/b^3$, hence, $2b^3 = a^3$.
By the definition of an even integer it follows that $a^3$ is even, therefore $a$ must also be even.
Furthermore we can let $a = 2c$ for some integer $c$, thus $2b^3 = 8c^3$.
After dividing both sides of this equation by 2 gives $b^3 = 4c^3$,
which means $b^3$ is even, again $b$ must be even as well.
We have now concluded that $a$ and $b$ are both even, thus 2 is a common divisor of $a$ and $b$.
This contradicts the choice of $a/b$.
Therefore the assumption—that $\sqrt[3]{2}$ is rational—is in error, so we have proved that $\sqrt[3]{2}$ is irrational.
Grading:

1. −2 each if answer is correct but justification is incorrect.
2. −3 each if answer is correct but missing justification.
3. Give 0 point if answer is incorrect.

Problem 3.
Let $A$, $B$ and $C$ be sets and let $P(X)$ be the powerset of set $X$. Prove or disprove the following statements.

(a) (5 pts) If $A \subseteq (B \cup C)$, then $A \subseteq B$ or $A \subseteq C$.

Disproof: Take $A = \{1, 2\}$, $B = \{1, 3\}$ and $C = \{2, 4\}$. Then $A \subseteq B \cup C = \{1, 2, 3, 4\}$, but $A$ is neither a subset of $B$, nor a subset of $C$.

(b) (5 pts) $(A - C) \cap (C - B) = \emptyset$.

Proof: We can prove it by contradiction. Assume that $(A - C) \cap (C - B) \neq \emptyset$ to show that it results to contradiction. $(A - C) \cap (C - B) \neq \emptyset$ means that there exists some $x \in (A - C) \cap (C - B)$. By the definition of intersection we can imply, that there exists $x$ for which the following proposition is true: $p = (x \in A - C) \wedge (x \in C - B)$.

Using the definition of set difference we can rewrite $p$ as:

$p = (x \in A) \wedge (x \notin C) \wedge (x \in C) \wedge (x \notin B)$.

But $(x \notin C) \wedge (x \in C) = \text{False}$, so

$p = (x \in A) \wedge [(x \notin C) \wedge (x \in C)] \wedge (x \notin B) = (x \in A) \wedge \text{False} \wedge (x \notin B) = \text{False}.$

Thus, the assumption that intersection $(A - C) \cap (C - B) \neq \emptyset$ is not empty results to contradiction which proves that this assumption is false, i.e. intersection is empty.

(c) (5 pts) $P(A) - P(B) \subseteq P(A - B)$.

Disproof: Take a counterexample: $A = \{1, 2\}$, $B = \{2, 3\}$, $P(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, $P(B) = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$, $P(A) - P(B) = \{\{1\}, \{1, 2\}\}$, $A - B = \{1\}$, $P(A - B) = \{\emptyset, \{1\}\}$

Thus, $\{1, 2\} \in P(A) - P(B)$, but $\{1, 2\} \notin P(A - B)$

so the proposition $P(A) - P(B) \subseteq P(A - B)$ is disproved.

Grading:

1. −2 each if answer is correct but justification is incorrect.
2. −3 each if answer is correct but missing justification.
3. Give 0 point if answer is incorrect.