Quiz: Week 2

1. Prove by induction that $\sum_{i=1}^{n} i^3 = \left( \sum_{i=1}^{n} i \right)^2$ for all $n \geq 1$

Ans: Step I: Statement $\sum_{i=1}^{n} i^3 = \left( \sum_{i=1}^{n} i \right)^2$ is true for $n=1$. Because $1^3 = (1)^2$

Step 2: Assume that $\sum_{i=1}^{k} i^3 = \left( \sum_{i=1}^{k} i \right)^2$ is true for $k=1$ to $n$.

We have to show that $\sum_{i=1}^{n+1} i^3 = \left( \sum_{i=1}^{n+1} i \right)^2$ is true.

We know that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^3 = \sum_{i=1}^{n} i^3 + (n+1)^3$. Expanding the right hand side of the last equation we get

$$\sum_{i=1}^{n+1} i^3 = \left( \sum_{i=1}^{n} i \right)^2 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3$$

$$= (n+1)^2 \left( \frac{n^2}{4} + (n+1) \right) = (n+1)^2 \left( \frac{(n+2)^2}{4} \right) = \left( \sum_{i=1}^{n+1} i \right)^2$$

2. Compute the asymptotic growth time for an extendable table for which the array size is increased from $N$ to the following possible values:
   a. $N + \left\lfloor \sqrt{N} \right\rfloor$
   b. $N + \left\lceil \log N \right\rceil$

Ans: Let us assume the array is full (i.e. array size = $N$) and we wish to insert items to the table.

In either case we have to create a new array of appropriate size. We will copy $N$ items from the original array to the new array.

Let us consider case (a).

So for the insertion of the 1st item the cost is $N+1$.

for the next $\left\lfloor \sqrt{N} \right\rfloor -1$ items the cost will be $\left\lfloor \sqrt{N} \right\rfloor -1$
Thus total cost for insertion of $\sqrt{N}$ items is $N + \sqrt{N}$.

So the average cost for insertion of 1 item is $\frac{N + \sqrt{N}}{\sqrt{N}} \leq \sqrt{N} + 1$.

Then cost for insertion of $N$ items is $\leq N\sqrt{N} + N$ or $O(\sqrt{N})$.

Let us consider case (b). So for the insertion of the 1st item the cost is $N + 1$.

for the next $\left\lfloor \log N \right\rfloor - 1$ items the cost will be $\left\lfloor \log N \right\rfloor - 1$.

Thus total cost for insertion of $\left\lfloor \log N \right\rfloor$ items is $N + \left\lfloor \log N \right\rfloor$.

So the average cost for insertion of 1 item is $\frac{N + \left\lfloor \log N \right\rfloor}{\left\lfloor \log N \right\rfloor} = \frac{N}{\left\lfloor \log N \right\rfloor} + 1$.

Then cost for insertion of $N$ items is $\frac{N^2}{\left\lfloor \log N \right\rfloor} + N$ or $O \left( \frac{N^2}{\left\lfloor \log N \right\rfloor} \right)$.