Solution 9.10.3.3

\[ S_a^G = \frac{a}{G} \frac{\partial G}{\partial a} \]
\[ = \frac{a}{s(s + a)} \frac{\partial}{\partial a} \frac{a}{s(s + a)} \]
\[ = \frac{s(s + a)}{1} \left[ \frac{-as}{s^2(s + a)^2} + \frac{1}{s(s + a)} \right] \]
\[ = s(s + a) \left[ \frac{-as + s^2 + as}{s^2(s + a)^2} \right] \]
\[ = \frac{s^2}{s(s + a)} \]
\[ = \frac{s}{s + a} \]

\[ S_a^{T_c} = \frac{a}{T_c} \frac{\partial T_c}{\partial a} \]
\[ = \frac{a}{s^2 + as + a} \frac{\partial}{\partial a} \frac{a}{s^2 + as + a} \]
\[ = \frac{s^2 + as + a}{1} \left[ \frac{-a(s + 1)}{(s^2 + as + a)^2} + \frac{1}{s^2 + as + a} \right] \]
\[ = (s^2 + as + a) \left[ \frac{-as - a + s^2 + as + a}{(s^2 + as + a)^2} \right] \]
\[ = \frac{s^2}{(s^2 + as + a)} \]

\[ S_K^G = S_K^{T_c} = 0 \text{ since there is no dependence on } K. \]
Solution 9.10.4.2

Figure 1: Unity Feedback with Disturbance at the Output

For the system of Figure 1,

\[ G_p(s) = \frac{2}{s(s+2)} \]

and we want

\[ T_c(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

With \( \zeta = 0.8 \) and \( \omega_n = 10 \).

Then

\[ T_c(s) = \frac{100}{(s+8-j6)(s+8+j6)} \]

To find \( G_c \) we set \( D = 0 \) and note that

\[ T_c = \frac{G_c G_p}{1 + G_c G_p} \]

Solving first for \( G_c G_p \), we have

\[ G_c G_p = \frac{T_c}{1 - T_c} \]

Then

\[ G_c = \frac{1}{G_p} \frac{T_c}{1 - T_c} \]

In the present case

\[ \frac{T_c}{1 - T_c} = \frac{100}{(s+8-j6)(s+8+j6)} - \frac{100}{1 - \frac{100}{(s+8-j6)(s+8+j6)}} \]
\[ = \frac{100}{s^2 + 16s + 100 - 100} \]
\[ = \frac{100}{s(s + 16)}. \]

Then
\[ G_c(s) = \frac{100}{s(s + 16)} \frac{s(s + 2)}{2} \]
\[ = \frac{50(s + 2)}{(s + 16)}. \]

Thus, \( G_c \) simply cancels the pole at \( s = -2 \) and replaces it by another pole at \( s = -16 \).

The closed loop transfer function between \( D \) and \( C \) with \( R = 0 \), can be found as follows. With \( R = 0 \),
\[ C_d = D - C_dG, \]
which can be rearranged as
\[ C_d = \frac{D}{1 + G}. \]

or
\[ T_d = \frac{C_d}{D} \]
\[ = \frac{1}{1 + G} \]
\[ = \frac{1}{1 + G_cG_p} \]
\[ = \frac{1}{1 + \frac{T_c}{1 - T_c}} \]
\[ = 1 - T_c. \]

We thus have two expressions for \( T_d \):
\[ T_d = \frac{1}{1 + G_cG_p} = 1 - T_c. \]

We now compute \( T_d \) by both methods.
\[ T_d(s) = \frac{1}{1 + G_cG_p} \]
Figure 2: Frequency Response of $T_d(j\omega)$

\[
\frac{1}{1 + \frac{50(s+2)(2)}{s(s+2)(s+16)}} = \frac{s(s+16)}{s^2 + 16s + 100} = 1 - T_c
\]

\[
1 - \frac{100}{(s + 8 - j6)(s + 8 + j6)} = \frac{s^2 + 16s + 100 - 100}{(s + 8 - j6)(s + 8 + j6)} = \frac{s(s+16)}{(s + 8 - j6)(s + 8 + j6)}
\]

b.

The frequency response of $T_d(j\omega)$ is shown in Figure 2. The disturbance rejection at 10 r./s. is very poor, about 0 db. To see how to improve it we
generalize the analysis somewhat. We have chosen

\[ T_c(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \]

Then

\[ \frac{T_c}{1 - T_c} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]
\[ = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s}. \]

Then

\[ G_c(s) = \frac{T_c}{1 - T_c} \frac{1}{G_p} \]
\[ = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s} \frac{s(s + 2)}{2} \]
\[ = \frac{(\omega_n^2/2)(s + 2)}{s + 2\zeta\omega_n}. \]

\[ T_d(s) = 1 - T_c(s) \]
\[ = 1 - \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]
\[ = \frac{s(s + 2\zeta\omega_n)}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \]

The situation should now be clear. We can get any noise attenuation we want at 10 rad/s by merely increasing \( \omega_n \). The price we will pay is increased gain in our compensator, since the gain of the compensator is \( \omega_n/2 \). As long as gain is not a problem, noise suppression is not a problem. If, however, there is a limitation on gain, then there will be a limitation on noise suppression.
Solution 9.10.4.3

Figure 1: Unity Feedback with Disturbance at the Output

For the system of Figure 1,

\[ G_p(s) = \frac{4}{(s + 4)^2} \]

and we want

\[ T_c(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \]

To find \( G_c \) we set \( D = 0 \) and note that

\[ T_c = \frac{G_c G_p}{1 + G_c G_p}. \]

Solving first for \( G_c G_p \), we have

\[ G_c G_p = \frac{T_c}{1 - T_c}. \]

Then

\[ G_c = \frac{1}{G_p} \frac{T_c}{1 - T_c}. \]

The closed loop transfer function between \( D \) and \( C \) with \( R = 0 \), can be found as follows. With \( R = 0 \),

\[ C_d = D - C_d G, \]

which can be rearranged as

\[ C_d = \frac{D}{1 + G}, \]
or

\[ T_d = \frac{C_d}{D} = \frac{1}{1+G} = \frac{1}{1+G_cG_p} = \frac{1}{1+ \frac{T_c}{1-T_c}} = 1 - T_c. \]

We thus have two expressions for \( T_d \):

\[ T_d = \frac{1}{1+G_cG_p} = 1 - T_c. \]

Since

\[ T_c(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \]

Then

\[ \frac{T_c}{1-T_c} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s}. \]

Then

\[ G_c(s) = \frac{T_c}{1-T_c G_p} = \frac{1}{\omega_n^2} \left( \frac{(s+4)^2}{s^2 + 2\zeta\omega_n s + 4} \right) = \frac{(\omega_n^2/4)(s+4)^2}{s(s+2\zeta\omega_n)}. \]

Thus, we see that the compensator cancels the two poles of the plant and replaces them with poles at \( s = 0 \) and \( s = -2\zeta\omega_n \).
We now find

\[ T_d(s) = 1 - T_c(s) \]

\[ = 1 - \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]

\[ = \frac{s(s + 2\zeta\omega_n)}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \]

If we choose \( \zeta = 0.8 \) and \( \omega_n = 5 \text{ rad/s} \), then

\[ T_d(s) = \frac{s(s + 2\zeta\omega_n)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]

\[ = \frac{s(s + 2(0.8)(5))}{s^2 + 2(0.8)(5)s + 5^2} \]

\[ = \frac{s(s + 8)}{s^2 + 8s + 5^2} \]

\[ = \frac{s(s + 8)}{(s + 4 - j3)(s + 4 + j3)} \]

\[ G_c(s) = \frac{(5^2/4)(s + 4)^2}{s[s + 2(0.8)(5)]} \]

\[ = \frac{(25/4)(s + 4)^2}{s(s + 8)}. \]

b.

The frequency response of \( T_d(j\omega) \) is shown in Figure 2. The disturbance rejection at 10 r./s. is very poor, about 0 dB. We can improve the situation by increasing \( \omega_n \). In fact, we can get any noise attenuation we want at 10 rad/s by merely increasing \( \omega_n \). The price we will pay is increased gain in our compensator, since the gain of the compensator is \( \omega_n/2 \). As long as gain is not a problem, noise suppression is not a problem. If, however, there is a limitation on gain, then there will be a limitation on noise suppression. For instance, suppose we let \( \zeta = 0.8 \) and \( \omega_n = 1000 \text{ rad/s} \). Then

\[ T_d(s) = \frac{s(s + 2\zeta\omega_n)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]

\[ = \frac{s(s + 2(0.8)(1000))}{s^2 + 2(0.8)(1000)s + 1000^2} \]

\[ = \frac{s(s + 1600)}{s^2 + 1600ss + 10^6}. \]
Figure 2: Frequency Response of $T_d(j\omega)$
Figure 3: Frequency Response of $T_d(j\omega)$

$$G_c(s) = \frac{s(s+8)}{(s+800-j600)(s+800+j600)}$$

The Bode magnitude plot of $T_d$ is shown in Figure 3. We have plenty of noise suppression around 10 rad/s, but we need a gain of 250,000 in our compensator. If we use a more realistic value of $\omega_n = 100$ rad/s, then we see from Figure 4 that we can achieve about 15 dB of noise suppression. This is not great, but it is definitely an improvement. The gain requirement is a more realistic 2500.
Figure 4: Frequency Response of $T_d(j\omega)$