Solution: 7.9.1.2

For the system of Figure 1 we have

\[ G_p = \frac{10}{(s + 3)(s + 30)} \quad \text{and} \quad G_c = \frac{K(s + b)}{s + 0.1} \]

Figure 2 shows the root locus for the plant \( G_p \). The break out point is far to the left. Simple gain compensation yields a closed loop system with poles at 

\[ s = -16.5 \pm j16.5. \]

The gain required to place the closed loop poles at this location is

\[ K = \frac{13.5^2 + 13.5^2}{10} = 45.45. \]

Then

\[ K_p = \lim_{s \to 0} \frac{10 \times 45.45}{(s + 3)(s + 30)} = 5.05, \]

and

\[ \epsilon_{ss} = \frac{1}{1 + 5.05} = 0.165. \]

Thus we do not meet the specification on steady state error. If we add the compensator, then we have the situation shown in Figure 3. For the root locus to be on the desired line of constant damping ratio, we must have

\[ \alpha - \theta_1 - \theta_2 - \theta_3 = -180^\circ. \]

The following program searches along the line of constant damping ratio \( \zeta = 1/\sqrt{2} \), computing \( a, K, K_p, \) and \( \epsilon_{ss} \).
Figure 2: Root Locus for Gain Compensation of $G_p$

Figure 3: Satisfaction of Angle Condition Along Line of Constant Damping Ratio
x = 2.0
dx = 0.1
i = 0
while x < 16
s = -x + j*x;
i = i + 1;
alpha(i) = angle(s + 0.1) + angle(s + 3) + angle(s + 30) - pi;
a(i) = x + (x / tan(alpha(i)));
xp(i) = x;
k(i) = (abs(s + 0.1)*abs(s + 3)*abs(s + 30))/(10*abs(s + a(i)));
kp(i) = (10*k(i)*a(i))/(0.1*30*3);
ess(i) = 1 / (1 + kp(i));
x = x + dx;
end
figure(1)
subplot(2,1,1), plot(xp,ess)
subplot(2,1,2), plot(xp,k,'r:',xp,kp,'k')
print -deps lag2a.eps
figure(2)
plotyy(xp,ess,'k',xp,a,'k')

Figure 4 is a graph of \(e_{ss}\) and \(a\) versus \(x\) the magnitude of the real and imaginary parts of the complex poles. The step response is shown in Figure 5. As can be seen, the steady state error is minimized for

\[10 < x < 12.\]

We see that we can get one percent steady state error with a very modest gain. For instance, if we place the dominant poles at

\[s = -4 \pm j4\]

Then

\[a = 5.693\text{ and } K = 13.95\]

Thus, a very good design is

\[G_c(s) = \frac{13.95(s + 5.693)}{s + 0.1}.\]

The step response is shown in Figure 5.
Figure 4: $e_{ss}$ and $a$ as a function of $|\text{Re}(x)|$
Figure 5: Step Response of Compensated System
Solution: 7.9.1.5

For the system of Figure 1 we have

\[ G_p = \frac{1}{(s + 1)(s + 10)} \quad \text{and} \quad G_c = \frac{K(s + a)}{s + 0.1} \]

Figure 2 shows the root locus for the plant \(G_p\). The break out point is far to the left. Simple gain compensation yields a closed loop system with poles at

\[ s = -5.5 \pm j4.125 \]

The gain required to place the closed loop poles at this location is

\[ K = \frac{4.5^2 + 4.125^2}{1} = 37.266. \]

Then

\[ K_p = \lim_{s \to 0} \frac{37.266}{(s + 1)(s + 10)} = 3.73, \]

and

\[ \varepsilon_{ss} = \frac{1}{1 + 3.73} \quad 0.2116 \]

If we add the compensator, then we have the situation shown in Figure 3. For the root locus to be on the desired line of constant damping ratio, we must have

\[ \alpha - \theta_1 - \theta_2 - \theta_3 = -180^\circ. \]

The following program searches along the line of constant damping ratio \(\zeta = 1/\sqrt{2}\), computing \(a, K, K_p, \) and \(\varepsilon_{ss}\).
Figure 2: Root Locus For Gain Compensation

Figure 3: Angle Contributions Along $\zeta = 0.8$
\[ p_1 = 1 \]
\[ p_2 = 10 \]
\[ plag=0.1 \]
\[ kplant=1 \]
\[ omegan = 1 \]
\[ dw = 0.05 \]
\[ i = 0 \]
while omegan < 6
\[ s = -omegan*0.8 + j*omegan*0.6; \]
\[ x = omegan*0.8; \]
\[ i = i + 1; \]
\[ alpha(i) = \text{angle}(s + plag) + \text{angle}(s + p1) + \text{angle}(s + p2) - \pi; \]
\[ a(i) = x + (x / \tan(alpha(i))); \]
\[ xp(i) = x; \]
\[ k(i) = (\text{abs}(s + plag)*\text{abs}(s + p1)*\text{abs}(s + p2)) / (kplant*\text{abs}(s + a(i))); \]
\[ kp(i) = (kplant*k(i)*a(i))/(plag*pi*p2); \]
\[ ess(i) = 1 / (1 + kp(i)); \]
\[ omegan = omegan + dw; \]
end
subplot(2,1,1), plot(xp,ess)
subplot(2,1,2), plot(xp,k,'r:','xp,kp','k')
print -deps lag2a.eps

Figure 4 is a graph of \( \epsilon_{ss} \), \( K_p \), and \( K \) versus \( x \) the magnitude of the real part of the complex poles.

Figure 5 shows the zero location (absolute value of zero) as a function of the real part of the dominant poles. For the three zero locations specified the real part is between 3.7 and 4.6. Exact correlations are:

\[ a = 0.5 \quad K = 34 \quad x = 4.65 \]
\[ a = 1.0 \quad K = 35.6 \quad x = 4.28 \]
\[ a = 1.5 \quad K = 34.5 \quad x = 3.75 \]

Then for \( a = 0.5 \),
\[ G_c = \frac{34(s + 0.5)}{s + 0.1}, \]
while for \( a = 1.0 \)
\[ G_c = \frac{34.6(s + 1)}{s + 0.1}, \]
and for \( a = 1.5 \)
\[ G_c = \frac{34.5(s + 1.5)}{s + 0.1}. \]
Figure 4: $\varepsilon_{ss}$ and $K_p$ and $K$ as a function of $z$
Figure 5: $a$ as a function of $x$
Figure 6: Step Responses For Three Values of $a$

The step responses for the three choices of $a$ are shown in Figure 6. The best choice is clearly $a = 1.0$. 
Solution: 7.9.3.2

For the system of Figure 1 we have

\[ G_p = \frac{10}{s(s + 0.5)(s + 20)}, \quad \text{and} \quad G_c = \frac{K(s + b)}{s + 10}. \]

Figure 2 shows the vector evaluation of \( G_c G_p \) at \( s = -3 + j3 \). To get the root locus to pass through this point we must have

\[ \angle G_c G_p (s) \bigg|_{s=-3+j3} = -180^\circ. \]

Each of the vectors, as labeled in Figure 2, is the polar representation of one of the factors in \( G_c G_p \) evaluated at \( s = -3 + j3 \). Thus

\[ G_c G_p \bigg|_{s=-3+j3} = \frac{K|s + b|\angle \alpha}{|s||s + 0.5||s + 10||s + 20|} \frac{\angle \theta_1}{\angle \theta_2} \frac{\angle \theta_3}{\angle \theta_4} \]

The evaluation of \( G_c G_p \) at \( s = -3 + j3 \) has now been broken down into a composite magnitude and a composite angle. For the root locus to pass through \( s = -3 + j3 \) we must have

\[ \alpha - \theta_1 - \theta_2 - \theta_3 - \theta_4 = -180^\circ. \]

The gain that places a closed loop pole at \( s = -3 + j3 \), and another at \( s = -3 - j3 \) is obtained by solving

\[ \frac{10K|s + b|}{|s||s + 0.5||s + 10||s + 20|} = 1, \]  \hspace{1cm} (1)

or

\[ K = \frac{|s||s + 0.5||s + 10||s + 20|}{10|s + b|}. \]  \hspace{1cm} (2)
Figure 2: Cascade Compensation with Unity Feedback

It should be clear that it is the angle condition that drives this whole business. The angle condition will be used to find $b$. Once $b$ is determined then $K$ can easily be calculated.

All of the angles in equation (1) are known except $\alpha$. So we can write

$$\alpha = \theta_1 + \theta_2 + \theta_3 + \theta_4 - 180^\circ$$

$$= [180^\circ - \tan^{-1}(3/3)] + [180^\circ - \tan^{-1}(3/2.5)][\tan^{-1}(3/17)] + [\tan^{-1}(3/7)] - 180^\circ$$

$$= 135^\circ + 129.81^\circ + 10.01^\circ + 23.2^\circ - 180^\circ$$

$$= 118^\circ$$

We now use simple trigonometry to find

$$-b = -3 + \frac{3}{\tan(180^\circ - 118^\circ)}$$

$$= -3 + \frac{3}{\tan(62^\circ)}$$

$$= -3 + 1.6$$

$$= -1.4$$

We can now find the gain using equation (2).

$$K = \frac{|s||s + 0.5||s + 10||s + 20|}{|s + 1.4|}$$

$$= \frac{4.243 \times 3.905 \times 7.616 \times 17.263}{10 \times 3.4}$$

$$= 49.2 \rightarrow 49$$
Thus the complete compensator is

\[ G_c = \frac{49(s + 1.4)}{s + 10}. \]

The following Matlab dialogue leads to the step response shown in Figure 3

\texttt{EDU}\textless{}g = \texttt{zpk([-1.4],[0 -0.5 -10 -20],492)}

\texttt{Zero/pole/gain:}
\begin{center}
\begin{tabular}{c}
492 (s+1.4) \\
\hline
s (s+0.5) (s+10) (s+20)
\end{tabular}
\end{center}

\texttt{EDU}\textless{}h = 1

\texttt{h =}
\begin{center}
1
\end{center}

\texttt{EDU}\textless{}tc = \texttt{feedback(g,h)}

\texttt{Zero/pole/gain:}
\begin{center}
\begin{tabular}{c}
492 (s+1.4) \\
\hline
(s+21.83) (s+4.562) (s^2 + 4.112s + 6.918)
\end{tabular}
\end{center}

\texttt{EDU}\textless{}step(tc,4)
\texttt{EDU}\textless{}print -deps leadiastep.eps
\texttt{EDU}\textgreater
Figure 3: Step Response of Compensated System
Solution: 7.9.3.4

Figure 1: Cascade Compensation with Unity Feedback

For the system of Figure 1 we have let

\[ G_p = \frac{5}{s(s+1)(s+10)}, \quad \text{and} \quad G_c = \frac{K_c(s+a)}{s+b}. \]

Part a.

To achieve \( t_p = 0.785 \) s we solve

\[ t_p = \frac{\pi}{\omega_d} \]

for \( \omega_d \). That is

\[ \omega_d = \frac{\pi}{t_p} = \frac{3.14159}{0.785} = 4 \text{ rad/s}. \]

Since \( \omega_d = \omega_n \sqrt{1-\zeta^2} \), and \( \zeta = 0.8 \), we have

\[ \omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \frac{4.0}{\sqrt{0.36}} = 6.667 \text{ rad/s}. \]

Then the real part of the complex roots is

\[ -\zeta \omega_n = -0.8 \times 6.667 = -5.33. \]

Part b.

Figure 2 shows the vector evaluation of \( G_cG_p \) at \( s = -5.33 + j4 \). To get the root locus to pass through this point we must have

\[ \angle G_cG_p(s) \bigg|_{s=-5.33+j4} = -180^\circ. \]

Each of the vectors in Figure 2 is the polar representation of one of the factors in \( G_cG_p \) evaluated at \( s = -5.33 + j4 \). That is the vector \( V_1 \) is the polar representation of the factor \( s \) in the denominator of \( G_cG_p \), the vector \( V_2 \) the polar representation of the factor \( s + 1 \), the vector \( V_3 \) the polar representation of the factor \( s + 10 \), the vector \( V_4 \) the polar representation of the factor \( s + b \), and the vector \( V_5 \) the polar representation of the vector \( s + a \). Thus

1
\[ G_cG_p|_{s=-5.33+j4} = \frac{5K_c|V_5|\angle \alpha}{|V_1|\angle \theta_1|V_2|\angle \theta_2|V_3|\angle \theta_3|V_4|\angle \beta} = \frac{5K_c|V_5|}{|V_1||V_2||V_3||V_4|} \angle (\alpha - \beta - \theta_1 - \theta_2 - \theta_3) \]

The evaluation of \( G_cG_p \) at \( s = -5.33 + j4 \) has now been broken down into a composite magnitude and a composite angle. For the root locus to pass through \( s = -5.33 + j4 \) we must have

\[ \alpha - \beta - \theta_1 - \theta_2 - \theta_3 = -180^\circ. \]

The gain that places a closed loop poles at \( s = -5.33 \pm j4 \) is obtained by solving

\[ \frac{5K_c|V_5|}{|V_1||V_2||V_3||V_4|} = 1, \quad (1) \]

or

\[ K_c = \frac{|V_1||V_2||V_3||V_4|}{5|V_5|}. \quad (2) \]

It should be clear that it is the angle condition that drives this whole business. The angle condition will be used to find \( b \). Once \( b \) is determined then \( K \) can easily be calculated.

The angle contribution of the plant is

\[ \angle G_p = -[\theta_1 + \theta_2 + \theta_3] \]

\[ = -\{[180^\circ - \tan^{-1}(4/5.33)] + [180^\circ - \tan^{-1}(4/4.33) + \tan^{-1}(4/4.67)]\} \]

\[ = -\{143.11^\circ + 137.27^\circ + 40.58^\circ\} \]

\[ = -321^\circ \]

Thus

\[ \alpha - \beta = \theta_1 + \theta_2 + \theta_3 - 180^\circ = 321^\circ - 180^\circ = 141.06^\circ. \]

For \( \beta > 0 \) we must have \( \alpha > 141^\circ \). This places the zero of the lead compensator between the pole at \( s = -1 \) and the pole at the origin. We can be more precise by noting that

\[ \tan(180^\circ - 39^\circ) = \tan(39^\circ) = \frac{4}{5.33 - a}. \]
Figure 3: Determinining Location of Zero of Lead

This can be rewritten as

\[ a = 5.33 - \frac{4}{\tan(39^\circ)} = 0.38 \]

So \(-\alpha\) has to be to the right of \(s = -0.38\).

**Part c.**

If the pole of the lead compensator is placed at \(s = -186\), then

\[ \beta = \tan^{-1}(4/180.67) = 1.27^\circ. \]

Then

\[ \alpha = 141.06^\circ + \beta = 142.33^\circ. \]

We now use simple trigonometry to find \(a\). Refering to Figure 3

\[ a = 5.33 - \frac{4}{\tan(180^\circ - 14.33^\circ)} \]
\[ = 5.33 - \frac{4}{\tan(37.67^\circ)} \]
\[ = 0.15 \]

We can now find the gain using equation(2).

\[ K_c = \frac{|V_1||V_2||V_3||V_4|}{|V_5|} = \frac{6.67 \times 5.89 \times 6.14 \times 180.7}{5 \times 6.528} = 1334.5 \]

Thus the complete compensator is

\[ G_c = \frac{1334.5(s + 0.15)}{s + 186} \]

**Part d.**
Figure 4: Calculation of $K$ Near Zero

<table>
<thead>
<tr>
<th>$s$</th>
<th>-0.12</th>
<th>-0.122</th>
<th>-0.121</th>
<th>-0.1207</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$</td>
<td>1297</td>
<td>1409</td>
<td>11351</td>
<td>1334.3</td>
</tr>
</tbody>
</table>

Table 1: Search for $K = 1334.5$ Near Zero at $s = -0.15$

To find the other two poles of the closed loop system, we simply look along the appropriate parts of the real axis for the same gain that we used to place the dominant complex poles. The calculation for the closed loop pole between $s = -0.17$ and $s = 0$ is shown in Figure 4. As always the vectors representing each term in $G_cG_p$ are drawn to the point where $G_cG_p$ is to be evaluated. Then

$$K = \frac{||V_1||||V_2||||V_3||V_4||}{5||V_5||}$$

Note that the plant gain of 5 has to be divided out. Table 1 shows the search for the gain of 1335. Note that we do not find a gain of 1334.3 exactly. The gain changes very rapidly in this region because $K$ goes from zero when $s = 0$ to $\infty$ as $s$ approaches -0.15. So we settle for a closed loop pole location *approximately* at $s = -0.1207$.

The search for the closed loop pole to the left of $s = -186$ proceeds in the same way. Figure 5 shows the vectors used in the calculation. The calculation is the same, but the location of $s$ has changed. In this case we use the following MATLAB dialogue to find the other closed loop pole

`EDU> g = zpk([-0.15],[0 -1 -10 -186],6672.4)`

Zero/pole/gain:

6672.4 (s+0.15)
Figure 5: Calculation of $K$ to Left of $s = -186$

\[
s (s+1) (s+10) (s+186)
\]

EDU>h = 1

h = 1

EDU>tc = feedback(g,h)

Zero/pole/gain:

\[
\frac{6672.4 (s+0.15)}{(s+186.2) (s+0.1208) (s^2 + 10.67s + 44.6)}
\]

EDU>

In this case the closed loop pole is very close to $s = -186$. There is a reason for this. For $K$ small the closed loop pole is near the pole of $G_cG_p$ at $s = -186$. As $K \to \infty$ the closed loop pole must travel all the way to $s = -\infty$. Thus the closed loop pole has plenty of room to move in, so a gain of 1334.3 doesn’t move it very far. It is just the opposite of the situation near the zero at $s = -0.15$.

Solution

Part e.

We can now write down the closed loop transfer function by inspection.

\[
T_c(s) = \frac{6672.4(s + 0.15)}{(s + 5.33 - j4)(s + 5.33 + j4)(s + 0.1207)(s + 186.2)}
\]
The output $C(s)$ for a step is

$$C(s) = \frac{1}{s} T_c(s)$$

$$= \frac{6672.4(s + 0.15)}{s(s + 5.33 - j4)(s + 5.33 + j4)(s + 0.1207)(s + 186.2)}$$

$$= \frac{B}{s + 0.1207} + \frac{D}{s + 186.2} + \frac{M}{s + 5.33 - j4} + \frac{M^*}{s + 5.33 + j4}$$

At this point we know the general form of $c(t)$. That is,

$$c(t) = [1 + B e^{-0.1207t} + D e^{-186.2t} + 2M e^{-5.33t} \cos(4t + \phi)] 1(t),$$

where $\phi = \angle M$. So we know quite a bit. What remains to be done is to find the constants $B$, $D$ and $M$. 

$$B = \left( s + 0.1207 \right) \left. C(s) \right|_{s = -0.1207}$$

$$= \left( s + 0.1207 \right) \left. \frac{6672.4(s + 0.15)}{s(s + 5.33 - j4)(s + 5.33 + j4)(s + 0.1207)(s + 186.2)} \right|_{s = -0.1207}$$

$$= -0.2007 \rightarrow -0.2$$

$$D = \left( s + 186.2 \right) \left. T_c(s) \right|_{s = -186.2}$$

$$= \left( s + 186.2 \right) \left. \frac{6672.4(s + 0.15)}{s(s + 5.33 - j4)(s + 5.33 + j4)(s + 0.1207)(s + 186.2)} \right|_{s = -186.2}$$

$$= -0.0012$$

$$M = \left( s + 5.33 - j4 \right) \left. T_c(s) \right|_{s = -5.33 + j4}$$

$$= \left( s + 5.33 - j4 \right) \left. \frac{6672.4(s + 0.15)}{s(s + 5.33 - j4)(s + 5.33 + j4)(s + 0.1207)(s + 186.2)} \right|_{s = -5.33 + j4}$$

$$= 0.6883 \angle 2.1893$$

Thus, finally we have

$$c(t) = \left[ 1 - 0.2e^{-0.1207t} - 0.0012e^{-186.2t} + 1.378e^{-5.33t} \cos(4t + 2.1893) \right] 1(t)$$

The time response is shown in Figure 6. The time response of the pole at $s = -186.2$ is not shown because it is almost instantly zero. Note that the pole at $s = -0.1207$ causes the system to creep towards its final steady state value. This is because the choice of complex poles forced the zero location of the lead compensator in close to the origin where it attracts the pole at the origin.
Figure 6: Step Response of Compensated System
Figure 1: Cascade Compensation with Unity Feedback

Solution 7.9.5.1

For the system of Figure 1 we have

\[ G_p = \frac{1}{s(s + 0.5)} \quad \text{and} \quad G_c = \frac{K_c(s + 1.5)}{s + p_1}. \]

Part a.

The calculation of \( p_1 \) and \( K_c \) is based on Figure 2.

\[ \angle G_c G_p = -180^\circ. \]

Each of the vectors in Figure 2 is the polar representation of one of the factors in \( G_c G_p \) evaluated at \( s = -3 + j2 \). That is the vector \( V_1 \) is the polar representation of the factor \( s \) in the denominator of \( G_c G_p \), \( V_2 \) the polar representation of the factor \( s + 0.5 \), \( V_3 \) the polar representation of \( s + b \), and \( V_4 \) the polar representation of \( s + 1.5 \). Thus,

\[ G_c G_p(s) \mid_{s=-3+j2} = \frac{K_c |V_4| \angle \alpha}{(|V_1| |V_2| |V_3| |V_4| \angle \theta_1 \angle \theta_2 \angle \beta)} = \frac{K_c |V_4|}{|V_1||V_2||V_3|} \angle (\alpha - \beta - \theta_1 - \theta_2) \]

The evaluation of \( G_c G_p \) at \( s = -3 + j2 \) has now been broken down into a composite magnitude and a composite angle. For the root locus to pass through this point, we must have

\[ \alpha - \beta - \theta_1 - \theta_2 = -180^\circ. \]
Figure 2: Satisfying Angle Condition for Desired Closed Loop Poles

Since all the angles except $\beta$ are known, we can write

\[
\beta = \alpha + 180^\circ - \theta_1 - \theta_2
\]

\[
= \left[180^\circ - \tan^{-1}(2/1.5)\right] + 180^\circ - \left[180^\circ - \tan^{-1}(2/3)\right] \left[180^\circ - \tan^{-1}(2/2.5)\right]
\]

\[
= 126.87^\circ + 180^\circ - 146.31^\circ - 141.34^\circ
\]

\[
= 19.22^\circ
\]

Then

\[
b = 3 + \frac{2}{\tan(19.22^\circ)} = 3 + 5.74 = 8.74.
\]

The gain that places a closed loop pole at $s = -3 + j2$, and another at $s = -3 - j2$ is obtained by solving

\[
\frac{K_c|V_4|}{|V_1||V_2||V_3|} = 1, \tag{1}
\]

or

\[
K = \frac{|V_1||V_2||V_3|}{|V_4|}. \tag{2}
\]

The gain to place the poles at this location is then

\[
K_c = \frac{\sqrt{13} \times \sqrt{10.25} \times \sqrt{36.91}}{\sqrt{6.25}} = 28.1.
\]