Solution 4.6.3.1

The characteristic equation is

\[ 1 + \frac{K(s + 1)}{(s - 2)(s + 2)} = 0, \]

which can be rewritten as

\[ \frac{s^2 - 4 + Ks + K}{(s - 2)(s + 2)} = 0, \]

or

\[ s^2 + Ks + (K - 4) = 0. \]

The initial Routh table is

| \( s^2 \) | 1 | K-4 | 0 |
| \( s^1 \) | K | 0 | 0 |
| \( s^0 \) | \( b_1 \) | \( b_2 \) | 0 |

Then

\[ b_1 = \frac{-\text{Det} \begin{bmatrix} 1 & K-4 \\ K & 0 \end{bmatrix}}{K} = \frac{-(0 - K(K - 4))}{K} = K - 4, \]

\[ b_2 = \frac{-\text{Det} \begin{bmatrix} 1 & 0 \\ K & 0 \end{bmatrix}}{K} = \frac{-(0 - 0)}{K} = 0 \]

\[ b_3 = \frac{-\text{Det} \begin{bmatrix} 1 & 0 \\ K & 0 \end{bmatrix}}{K} = \frac{-(0 - 0)}{K} = 0 \]

The completed Routh table is

| \( s^2 \) | 1 | K-4 | 0 |
| \( s^1 \) | K | 0 | 0 |
| \( s^0 \) | \( K - 4 \) | 0 | 0 |

For all the terms in the first column to be positive we must have

\[ K > 0 \quad \text{and} \quad (K - 4) > 0, \]

or, equivalently

\[ K > 4. \]
Solution 4.6.3.4

The characteristic equation is

\[ 1 + \frac{K}{s(s + 10)(s + 50)} = 0, \]

which can be rewritten as

\[ \frac{s^3 + 60s^2 + 500s + K}{s(s + 10)(s + 50)} = 0, \]

or

\[ s^3 + 60s^2 + 500s + K = 0. \]

The initial Routh table is

<table>
<thead>
<tr>
<th>s^3</th>
<th>1</th>
<th>500</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>s^2</td>
<td>60</td>
<td>K</td>
<td>0</td>
</tr>
<tr>
<td>s^1</td>
<td>b_1</td>
<td>b_2</td>
<td>0</td>
</tr>
<tr>
<td>s^0</td>
<td>c_1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Then

\[ b_1 = \frac{-\text{Det} \begin{bmatrix} 1 & 500 \\ 60 & K \end{bmatrix}}{60} = \frac{-(K - 30,000)}{60} \]

\[ b_2 = \frac{-\text{Det} \begin{bmatrix} 1 & 0 \\ 60 & 0 \end{bmatrix}}{60} = \frac{-(0 - 0)}{60} = 0 \]

The partially completed Routh table is

<table>
<thead>
<tr>
<th>s^3</th>
<th>1</th>
<th>500</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>s^2</td>
<td>60</td>
<td>K</td>
<td>0</td>
</tr>
<tr>
<td>s^1</td>
<td>-(K - 30,000)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>s^0</td>
<td>c_1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Then

\[ c_1 = \frac{-\text{Det} \begin{bmatrix} 60 & K \\ -(K - 30,000) & 0 \end{bmatrix}}{-(K - 30,000)} = K \]

The completed Routh table is

1
<table>
<thead>
<tr>
<th>$s^3$</th>
<th>1</th>
<th>500</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^2$</td>
<td>60</td>
<td>$K$</td>
<td>0</td>
</tr>
<tr>
<td>$s^1$</td>
<td>$(K-30,000)$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$s^0$</td>
<td>$K$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

For all the terms in the first column to be positive we must have

$$\frac{-(K - 30,000)}{60} > 0 \quad \text{and} \quad K > 0,$$

or, equivalently

$$0 < K < 30,000.$$
Solution 4.6.3.11

The characteristic equation is

\[ 1 + \frac{K}{s^2(s + 10)} = 0, \]

which can be rewritten as

\[ \frac{s^3 + 10s^2 + 0s + K}{s^3 + 10s^2} = 0, \]

or

\[ s^3 + 10s^2 + 0s + K = 0. \]

The initial Routh table is

\[
\begin{array}{ccc}
 s^3 & 1 & 0 & 0 \\
 s^2 & 10 & K & 0 \\
 s^1 & b_1 & b_2 & 0 \\
 s^0 & c_1 & 0 & 0 \\
\end{array}
\]

Then

\[ b_1 = \frac{-\text{Det} \begin{bmatrix} 1 & 0 \\ 10 & K \end{bmatrix}}{10} \]

\[ = \frac{-K}{10} \]

\[ b_2 = \frac{-\text{Det} \begin{bmatrix} 1 & 0 \\ 10 & 0 \end{bmatrix}}{52} \]

\[ = \frac{-(0 - 0)}{10} \]

\[ = 0 \]

The partially completed Routh table is

\[
\begin{array}{ccc}
 s^3 & 1 & 0 & 0 \\
 s^2 & 10 & K & 0 \\
 s^1 & -\frac{K}{10} & 0 & 0 \\
 s^0 & c_1 & 0 & 0 \\
\end{array}
\]
Then

\[ c_1 = \frac{-\text{Det} \begin{bmatrix} 10 & K \\ \frac{-K}{10} & 0 \end{bmatrix}}{\frac{-K}{10}} = \frac{-(0 + K^2)}{\frac{-K}{10}} = 10K. \]

The completed Routh table is

<table>
<thead>
<tr>
<th>( s^3 )</th>
<th>1</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s^2 )</td>
<td>10</td>
<td>( -K )</td>
<td>0</td>
</tr>
<tr>
<td>( s^1 )</td>
<td>( \frac{-K}{10} )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( s^0 )</td>
<td>10K</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

For all the terms in the first column to be positive we must have

\[ \frac{-K}{10} > 0 \quad \text{and} \quad 10K > 0. \]

Since \( K \) cannot be both greater than zero and less than zero, this system is unstable for all values of \( K \). We will see why in Chapter 5.
Solution 4.6.4.1

The closed loop transfer function is

\[
\frac{K(s+1)}{s^2} = \frac{K(s+1)}{s^2 + Ks + K}.
\]

The characteristic equation is

\[
s^2 + Ks + K = 0.
\]

The MATLAB program

```matlab
K = 2
p=[1 K K]
roots(p)
K = 4
p=[1 K K]
roots(p)
K = 20
p=[1 K K]
roots(p)
K=[2 40 20]
gh = zpk([],[-1 -10],1)
[R,K] = rlocus(gh,K)
plot(R,'kd')
print -deps rl4641.eps
```

generates the following output

EDU>sm4641

\[
K =
\[
2
\]

\[
p =
\[
1 \quad 2 \quad 2
\]
ans =

-1.0000+ 1.0000i
-1.0000- 1.0000i

K =

4

p =

1  4  4

ans =

-2
-2

K =

20

p =

1  20  20

ans =

-18.9443
-1.0557

K =
Zero/pole/gain:
\[ \begin{array}{ccc}
1 \\
(s+1) (s+10)
\end{array} \]

\[ R = \begin{array}{ccc}
-9.7720 & -5.5000+ 4.4411i & -6.0000 \\
-1.2280 & -5.5000- 4.4411i & -5.0000
\end{array} \]

\[ K = \begin{array}{c}
2 \\
40 \\
20
\end{array} \]

EDU>
EDU>

The plot of the points is shown in Figure 1.
For \( K = 2 \), the poles of the closed loop system are at \( s = -1 \pm j \). Thus, the closed loop transfer function is
\[ T_c(s) = \frac{2(s + 1)}{(s + 1 - j)(s + 1 + j)}. \]

Then the step response is
\[ C(s) = \frac{2(s + 1)}{s(s + 1 - j)(s + 1 + j)} = \frac{A}{s} + \frac{M}{s + 1 - j} + \frac{M^*}{s + 1 + j}. \]

The residues \( A \) and \( M \) are determined by the following MATLAB program, which also plots the step response, shown in Figure 2.

\[ K = 2; \]
\[ p0 = [1 0]; \]
\[ p1 = [1 1+j*1]; \]
Figure 1: Plot of solutions

p2 = [1 1-j*1];
B = 2*[1 1]
A = conv(p1,p2);
A = conv(A,p0)
[R,P,K] = residue(B,A)
M = R(1)
absm = abs(M)
abs2m = 2*abs(M)
angm = angle(M)
test = 1 + 2*abs(M)*cos(angle(M))
t = 0
dt = 0.1
kount = 1
while t < 3
    c(kount) = 1+0.02728*exp(-1.1436*t) + 2*absm*exp(-2.928*t)*cos(3*t + angm);
time(kount) = t;
t = t + dt;
kount = kount + 1;
end
plot(time,c)
print -deps sr4641a.eps

For $K = 4$, there are two closed loop poles are at $s = -2$. For $K = 2$, the poles of the closed loop system are at $s = -1 \pm j$. Thus, the closed loop transfer function is

$$T_c(s) = \frac{4(s + 1)}{(s - 2)^2}.$$

Then the step response is

$$C(s) = \frac{4(s + 1)}{s(s + 2)^2} = \frac{A}{s} + \frac{B}{s + 2} + \frac{C}{(s + 2)^2}.$$

Then

$$A = \left[ \frac{4(s + 1)}{(s + 2)^2} \right]_{s = 0} = 1$$
\[ B = \left\{ \frac{d}{ds} \left[ \frac{4(s+1)}{s} \right] \right\}_{s=-2} = \left\{ \frac{d}{ds} \left[ \frac{-4(s+1)}{s^2} + \frac{4}{s} \right] \right\}_{s=-2} = \left[ \frac{-4s - 4 + 4s}{(s+2)^2} \right]_{s=-2} = -1 \]

\[ C = \left[ \frac{4(s+1)}{s} \right]_{s=-2} = \frac{2}{2} = 1 \]

The residues, A, B, and C are determined by the MATLAB program:

```matlab
K = 4;
p0 = [1 0];
p1 = [1 2];
p2 = [1 2];
B = K*[1 1];
A = conv(p1,p2);
A = conv(A,p0);[R,P,K] = residue(B,A);A = R(3);
B = R(1);
C = R(2);
t = 0;
dt = 0.1;
kount = 1;while t < 3
        c(kount) = 1*B*exp(-2*t) + C*t*exp(-2*t);
time(kount) = t;
t = t + dt;
kount = kount + 1;
end
plot(time,c)
print -deps sr4641b.eps
```

which also plots the step response, shown in Figure 3. For \( K = 20 \), there
are two closed loop poles are at \( s = -1.0557 \) and \( s = -18.9443 \) Thus, the closed loop transfer function is

\[
T_c(s) = \frac{20(s+1)}{(s+1.0557)(s+18.9443)}.
\]

Then the step response is

\[
C(s) = \frac{20(s+1)}{s(s+1.0557)(s+18.9443)} = \frac{A}{s} + \frac{B}{s+1.0557} + \frac{C}{s+18.9443}.
\]

The residues \( A, B, \) and \( C \) are determined by the MATLAB program

\[
\begin{align*}
K &= 20; \\
p0 &= [1 0] \\
p1 &= [1 1.0557] \\
p2 &= [1 18.9443] \\
B &= K*[1 1] \\
A &= \text{conv}(p1,p2) \\
A &= \text{conv}(A,p0)
\end{align*}
\]
Figure 4: Step Response for $K = 4$

$[R,P,K] = \text{residue}(B,A)$
$A = R(3)$
$B = R(2)$
$C = R(1)$
$t = \text{linspace}(0,3,100);$  
$A = \text{ones}(100,1)$
$c = A + (B*\exp(-1.0557*t))' + (C*\exp(-18.9443*t))'$
plot(t,c)
print -deps sr4641c.eps

which also plots the step response, shown in Figure 4.