Exam #2
EEL 3657 (Spring 2004)

Name: Solutions
SS#: ______________

Please show all work for partial credit. (100 points total)

Problem 1. (50 points) Consider the unity feedback system shown below (H(s)=1):

\[
\begin{align*}
R & \rightarrow \Sigma \rightarrow G_c \rightarrow G_p \rightarrow C \\
- & \downarrow H \uparrow
\end{align*}
\]

where \( G_p(s) = \frac{5}{s(s+1)} \), \( G_c(s) = \frac{K(s+4)}{s+b} \) is to be designed for a peak time \( t_p \approx 0.785\ s \) and damping ratio \( \xi = 0.8 \).

1) Find the location of the dominate closed-loop poles.
2) Find the location of the compensator pole (i.e., the value of b).
3) Find the gain K.
4) What is the steady-state error for a unit step input?
Solution 7.9.3.3

For the system of Figure 1 we have

\[ G_p = \frac{5}{s(s+1)}, \quad \text{and} \quad G_c = \frac{K_c(s+4)}{s+b}. \]

To achieve \( t_p = 0.785 \text{ s} \), we solve

\[ t_p = \frac{\pi}{\omega_d} \]

for

\[ \omega_d = \frac{\pi}{t_p} = \frac{3.4142}{0.785} = 4 \text{ rad/s}. \]

Since \( \omega_d = \omega_n \sqrt{1 - \zeta^2} \), and \( \zeta = 0.8 \), we have

\[ \omega_n = \frac{\omega_d}{\sqrt{1 - \zeta^2}} = \frac{4.0}{\sqrt{0.36}} = 6.667 \]

Then the real part of the complex roots is

\[ -\zeta \omega_n = -0.8 \times 6.667 = -5.34. \]

Figure 2 shows the vector evaluation of \( G_c G_p \) at \( s = -5.34 + j4 \). To get the root locus to pass through this point we must have

\[ \angle G_c G_p(s) \bigg|_{s=-5.34+j4} = -180^\circ. \]

Each of the vectors in Figure 2 is the polar representation of one of the factors in \( G_c G_p \) evaluated at \( s = -5.34 + j4 \). That is the vector \( V_1 \) is the polar representation of the factor \( s \) in the denominator of \( G_c G_p \), the vector \( V_2 \) the polar representation of the factor \( s+1 \), the vector \( V_3 \) the polar representation of the factor \( s+4 \), and the vector \( V_4 \) is the polar representation of the factor \( s+b \). Thus
Figure 2: Satisfaction of Angle Condition at \( s = -5.55 + j4.0 \)

\[ G_cG_p(s) \bigg|_{s=-5.34+j4} = \frac{5K_c|V_4|L\alpha}{(|V_1|L\theta_1)(|V_2|L\theta_2)(|V_3|L\beta)} = \frac{5K_c|V_4|}{|V_1||V_2||V_3|} \cdot L(\alpha - \beta - \theta_1 - \theta_2) \]

The evaluation of \( G_cG_p \) at \( s = -5.34 + j4 \) has now been broken down into a composite magnitude and a composite angle. For the root locus to pass through \( s = -5.34 + j4 \) we must have

\[ \alpha - \beta - \theta_1 - \theta_2 = -180^\circ. \]

The gain that places a closed loop pole at \( s = -5.34 + j4 \), and another at \( s = -5.34 - j4 \) is obtained by solving

\[ \frac{5K_c|V_4|}{|V_1||V_2||V_3|} = 1, \quad (1) \]

or

\[ K_c = \frac{|V_1||V_2||V_3|}{5|V_4|}. \quad (2) \]

It should be clear that it is the angle condition that drives this whole business. The angle condition will be used to find \( b \). Once \( b \) is determined then \( K_c \) can easily be calculated. All of the angles in equation (1) are known except \( \beta \). So we can write

\[ \beta = \alpha - \theta_1 - \theta_2 + 180^\circ \]

\[ = 180^\circ + [180^\circ - \tan^{-1}(4/1.33)] - [180^\circ - \tan^{-1}(4/5.34)] - [180^\circ - \tan^{-1}(4/4.33)] \]

\[ = 180^\circ + 108.46^\circ - 143.13^\circ - 137.29^\circ \]

\[ = 8.04^\circ \]
We now use simple trigonometry to find

\[ b = 5.34 + \frac{4}{\tan(8.04^\circ)} = 5.34 + \frac{4}{0.1412} = 5.34 + 28.35 = 33.66 \]

We can now find the gain using equation (2).

\[ K_c = \frac{|V_1||V_2||V_3|}{5|V_5|} = \frac{6.67 \times 5.9 \times 28.61}{5 \times 4.219} = 53.38 \]

Thus the complete compensator is

\[ G_c = \frac{53.38(s + 4)}{s + 33.66} \]

(4).

\[ K_p = \lim_{s \to 0} G_c G_p(s) = \lim_{s \to 0} \frac{53.38(s + 4) \times 5}{s(s + 1)(s + 33.66)} = \frac{53.38 \times 5}{0} = \infty \]

\[ \varepsilon_{ss} = \frac{1}{1 + kp} = 0 \]
Problem 2. (25 points) For the transfer function \( G(s) = \frac{100(s + 10)}{s(s + 2)(s + 50)} \), sketch asymptotic (straight-line) Bode magnitude plot. Mark the slopes in your diagram.

**Solution:**

The first step is to put the transfer function in time constant form. So we have

\[
G(s) = \frac{100(s + 10)}{s(s + 2)(s + 50)} = \frac{1000(1 + s/10)}{(2)(50)s(1 + s/2)(1 + s/50)} = \frac{10(1 + s/10)}{s(1 + s/2)(1 + s/50)}.
\]

Then the terms to be plotted are

\[
10, \quad \frac{1}{s}, \quad \frac{1}{1 + s/2}, \quad \frac{1}{1 + s/50}, \quad \text{and} \quad 1 + s/10
\]

\(2.0\log_{10}(10) = 20\ \text{db}\)

At low frequencies the only terms that contribute are the gain and \(1/s\). The term \(1/s\), which is a straight line crossing the 0-dB line at \(\omega = 1\), will, when the gain is added in, cross at 20 dB. The other terms are straight lines at 0 dB out to the break frequencies, where they then break upwards or downwards, at 20 dB/decade, depending on whether they are zeros or poles. The pieces of the asymptotic plot, and the composite asymptotic plot are shown in Figure 1.
**Problem 3.** (25 points) Given the control system shown below, find the value of $K$ so that there is a 10% steady state error for a unit ramp input.

\[
\begin{array}{c}
\text{Solution:} \\
K_u = \lim_{s \to 0} sG(s) = \lim_{s \to 0} \frac{K(s+5)}{s(s+6)(s+7)(s+8)} \\
= \frac{5K}{6 \times 7 \times 8} = \frac{5K}{336} \\

E_{ss} = \frac{1}{K_u} = \frac{336}{5K} = 10\% \\
\Rightarrow K = 672.
\end{array}
\]