Problem 1. (25 points) Find the transfer function, \( G(s) = \frac{C(s)}{R(s)} \), corresponding to the differential equation

\[
\frac{d^3 c}{dt^3} + 3 \frac{d^2 c}{dt^2} + 7 \frac{dc}{dt} + 5c = \frac{d^2 r}{dt^2} + 4 \frac{dr}{dt} + 3r
\]

Solution:

Taking the Laplace transform of the differential equation assuming zero initial conditions yields:

\[ s^3 C(s) + 3s^2 C(s) + 7sC(s) + 5C(s) = s^2 R(s) + 4sR(s) + 3R(s) \]

Collecting terms,

\[ (s^3 + 3s^2 + 7s + 5)C(s) = (s^2 + 4s + 3)R(s) \]

Thus,

\[
\frac{C(s)}{R(s)} = \frac{s^2 + 4s + 3}{s^3 + 3s^2 + 7s + 5}
\]
Problem 2. (25 points) Find the equivalent transfer function, $T(s) = C(s)/R(s)$, for the system shown below:

![System Diagram]

**Solution:**

**Forward paths:**
1. $s \cdot s \cdot \frac{1}{s} = s$,
2. $\frac{1}{s} \cdot \frac{1}{s} = \frac{1}{s^2}$

**Loops:**
1. $-s \cdot s \cdot \frac{1}{s} \cdot s = -s^2$,
2. $-s \cdot s \cdot 1 = -s^2$,
3. $-\frac{1}{s} \cdot \frac{1}{s} \cdot s = -\frac{1}{s}$,
4. $-\frac{1}{s} \cdot 1 = -\frac{1}{s}$

**Use the simplified version of Mason’s rule:**

$$G(s) = \frac{C(s)}{R(s)} = \frac{s + \frac{1}{s^2}}{1 + s^2 + s^2 + \frac{1}{s} + \frac{1}{s}} = \frac{s^3 + 1}{2s^4 + s^2 + 2s}.$$
Method 2:

Combine the parallel blocks in the forward path. Then, push $\frac{1}{s}$ to the left past the pickoff point.

Combine the parallel feedback paths and get $2s$. Then, apply the feedback formula, simplify, and get, $T(s) = \frac{s^3 + 1}{2s^4 + s^2 + 2s}$. 
Problem 3. (50 points) For a unity feedback system shown below,

\[ G(s) = \frac{K(s^2 - 2s + 2)}{(s + 1)(s + 2)} \]

where \( G(s) \), sketch the root locus and find:
1. break-away/break-in points;
2. \( \omega_n \) axis crossings;
3. \( K \) value at the \( \omega_n \) axis crossings;
4. angle of arrival at the zeros;
5. the value of \( K \) at the point \( s = -0.75 \pm j1.199 \).
6. The range of \( K \) within which the closed-loop system is stable.

Solution:

\[ a(s) b(s) = a(s) b'(s) \]

\[ (2s - 2) (s^2 + 3s + 2) = (2s + 3) (s^2 - 2s + 2) \]

\[ 5s^2 - 10 = 0 \implies s = \pm 1.41 \]

Since \( s = 1.41 \) is not on the root locus, the real break-away point is \( -1.41 \).
2). Characteristic equation:

\[ 1 + G(s) = 0 \]

\[ \Rightarrow (s^2 + 3s + 2) + k(s^2 - 2s + 2) = 0 \]

\[ \Rightarrow (1+k)s^2 + (3-2k)s + (2+2k) = 0 \]

Let \( s = j \omega \),

\[ \left[ -\omega^2 (1+k) + (2+2k) \right] + j (3-2k) \omega = 0 \]

\[ \Rightarrow \omega^2 (1+k) + (2+2k) = 0 \]

\[ (3-2k) \omega = 0 \]

\[ \Rightarrow \omega = \pm \sqrt{2} = \pm 1.41 \]

\[ k = \frac{3}{2} = 1.5 \]

3). From step 2), we get \( k = 1.5 \).

4). \( 90^\circ + \theta_3 - \theta_1 - \theta_2 = 180^\circ \)

\[ \theta_1 = \tan^{-1} \frac{1}{3}, \quad \theta_2 = \tan^{-1} \frac{1}{2} \]

\[ \Rightarrow \theta_3 = 135^\circ \]
5). \[
\left| G(s) \right|_{s = -0.75 \pm j1.199} = 1
\]

\[
\Rightarrow K = \left| \frac{s^2 - 2s + 2}{(s+1)(s+2)} \right|_{s = -0.75 \pm j1.199} = 1
\]

\[
\Rightarrow K = \left| \frac{|s+1| \cdot |s+2|}{|s^2 - 2s + 2|} \right|_{s = -0.75 \pm j1.199} = 0.429
\]

6). From the root locus and step 3),

\[ 0 < K < 1.5 \]