Solution 5.8.1.1

For the system of Figure 1 let we have

\[ GH = \frac{K(s+2)}{s(s+1)} \]

The first step is to plot the poles and zeros of \( GH \). The poles of \( GH \) are not the closed loop pole locations, but they can be used to find the closed loop poles. The closed loop zeros can be found immediately: they are the zeros of \( G \) and the poles of \( H \). The zeros of \( GH \) also help in finding the poles of the closed loop system.

The root locus is shown in Figure 2. Note that there is root locus between the poles at \( s = 0 \) and \( s = -1 \) and to the left of the zero at \( s = -2 \). All locations between the poles and to the left of the zero are closed loop poles for some gain. As can be seen, root locus also appears off the real axis for the following reasons. We know that there is a closed loop pole associated with each of the poles of \( GH \). Heuristically, as the the gain is increased from zero through small values one can think of a pole emanating from each of the poles of \( GH \). As the gain is increased, these two poles migrate towards each other. For a certain gain, the largest gain that can be found along the real axis between \( s = 0 \) and \( s = -1 \), they 'collide,' that is for some gain there is a double real pole in the closed loop system.

As the gain is increased further, the two poles break out of the real axis. They migrate to the left because we know from the 'rules' that one of these poles must eventually go to the finite zero at \( s = -2 \) and the other to the zero at \( s = -\infty \). Since there is root locus everywhere to the left of the zero at \( s = -2 \) the logical conclusion is that the two poles break in somewhere to the left of \( s = -2 \). That is, for some gain the closed loop system has a double pole to the left of the zero at \( s = -2 \). For higher gains one pole
migrates right towards \( s = -2 \) and the other left towards \( s = -\infty \). Thus, insofar as the real axis is concerned, the value of gain at the break-in point represents the minimum gain. That is, if we compute the gain required to place closed loop poles anywhere to the left of \( s = -2 \) on the real axis, the minimum value will occur at the break-in point.

To find the break out and break in points along the real axis, we can do one of two things: build a table of specific values of \( K \) versus \( s \) along the real axis, or find the critical points of \( K \) as a function of \( s \). The second method has analytical appeal, and in this case is not difficult to use:

\[
\frac{d}{ds} K(s) = \frac{d}{ds} \frac{-s(s + 1)}{s + 2} = \frac{-(s^2 + 4s + 2)}{(s + 2)^2}
\]

The critical points (maximums and minimums) occur where the derivative is equal to zero or where

\[-(s^2 + 4s + 2) = 0,
\]

namely

\[s = -2 \pm \sqrt{2}\]

Note that the break in at \( s = -2 - \sqrt{2} \) and the break-out at \( s = -2 + \sqrt{2} \) are equal distance from the zero of \( GH \). Actually, the root locus off the real axis is a circle centered at the zero at \( s = -2 \). This is always the case if \( GH \) has exactly one pole and two zeros.
Figure 3: Gain Calculation to Find Break-in/Break-out Points

\[
\text{Gain at point } s = \frac{V_1 \times V_2}{V_3}
\]

<table>
<thead>
<tr>
<th>( s )</th>
<th>-0.8</th>
<th>-0.7</th>
<th>-0.6</th>
<th>-0.5</th>
<th>-0.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K )</td>
<td>0.1333</td>
<td>0.1615</td>
<td>0.1714</td>
<td>0.1667</td>
<td>0.15</td>
</tr>
</tbody>
</table>

Table 1: \( K \) versus \( s \) for \( s \) Between Poles of \( GH \)

We could also find the approximate break-in and break-out points by computing the gain at specific points along the real axis. The calculation is based upon Figure 3. We know that

\[
GH = \frac{K(s + 2)}{s(s + 1)}
\]

and that for any \( s \) on the root locus, including those along the real axis,

\[
K = \frac{|s||s + 1|}{|s + 2|}
\]

and that the magnitudes in this equation are simply the lengths of the vectors drawn from the poles and zero of \( GH \) to the chosen \( s \). These magnitudes are shown in Figure 3. Tables 1 and 2 can be used to find the approximate break out and break in points.

One might ask why an approximate calculation would be preferred over an exact calculation. An answer to that is the following. In this problem, \( GH \) has two poles and the equation we needed to solve to find the critical points was second order. If \( GH \) had four poles the equation for the critical

<table>
<thead>
<tr>
<th>( s )</th>
<th>-4</th>
<th>-3.5</th>
<th>-3.4</th>
<th>-3.2</th>
<th>-3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K )</td>
<td>6.0</td>
<td>5.883</td>
<td>0.5826</td>
<td>5.8667</td>
<td>6.0</td>
</tr>
</tbody>
</table>

Table 2: \( K \) versus \( s \) for \( s \) to the Left of \( s = -2 \)

3
points of $K(s)$ would have been fourth order. With a modern calculator, and supposing we made no error in calculating $dK/ds$, this is not a big problem. However, finding $dK/ds$ if $K(s)$ is fourth order is tedious and one can easily make a mistake, a mistake that can be hard to spot. On the other hand, the tables can be generated with a simple calculator and they are self correcting. That is, if we make a mistake in computing a particular value of gain, it should stick out like a sore thumb. Further, we generally are not interested in finding break-in and break-out points exactly, we just want to know their approximate locations.
Solution 5.8.1.15

The first step is to plot the poles and zeros of $GH$ in the $s$-plane and then find the root locus on the real axis. The shaded regions of the real axis in Figure 1 show where the root locus occurs on the real axis. The rule is that root locus occurs on the real axis to the left of an odd count of poles and zeros. That is, if you stand on the real axis and look to your right you must count an odd number of poles and zeros. The two poles off the root locus have no effect, because at any point along real axis their angle contributions sum to zero.

The next step is to compute the asymptotes.

\[ p_{ex} = 2 - 1 = 2. \]

\[ \theta_0 = \left( \frac{1+2 \times 0}{p_{ex}} \right) \times 180^\circ = \left( \frac{1}{2} \right) \times 180^\circ = 90^\circ \]

\[ \theta_1 = \left( \frac{1+2 \times 1}{p_{ex}} \right) \times 180^\circ = \left( \frac{3}{2} \right) \times 180^\circ = 270^\circ. \]

\[ \sigma_i = \frac{\text{Sum of poles of } GH - \text{Sum of zeros of } GH}{p_{ex}} = \frac{[-40 - 2] - [-5]}{2} = -18.5 \]

One finite zero, three poles so two poles migrate to ‘zeros’ infinitely far away at ends of asymptotes (called zeros at infinity). Because pole at $s = -40$ is distant from other poles and zeros will be a break-in point near $s = -11$ and a break-out near $s = -20$. Table below shows gain for some points along real axis.

<table>
<thead>
<tr>
<th>$s$</th>
<th>$-30$</th>
<th>$-25$</th>
<th>$-20$</th>
<th>$-17$</th>
<th>$-16.5$</th>
<th>$-16$</th>
<th>$-12$</th>
<th>$-11$</th>
<th>$-10.5$</th>
<th>$-6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$</td>
<td>336</td>
<td>431.3</td>
<td>480</td>
<td>488.8</td>
<td>488.9</td>
<td>488.7</td>
<td>480</td>
<td>478.5</td>
<td>478.7</td>
<td>816</td>
</tr>
</tbody>
</table>

Break-out is closer to $s = -16.5$ The root locus is shown in Figure 2.
Solution 5.8.1.25

\[ \begin{align*}
R & \rightarrow G \\
G & \rightarrow C \\
H & \rightarrow R
\end{align*} \]

Figure 1:

For the system shown above

\[ GH(s) = \frac{K(s + 2)}{s^2(s + 8)} \]

The first step is to plot the poles and zeros of \( GH \) in the \( s \)-plane and then find the root locus on the real axis. The shaded regions of the real axis in Figure 2 show where the root locus occurs on the real axis. The rule is that root locus occurs on the real axis to the left of an odd count of poles and zeros. That is, if you stand on the real axis and look to your right you must count an odd number of poles and zeros.

The next step is to compute the asymptotes.

\[ p_{ex} = 3 - 1 = 2. \]

\[ \theta_0 = \left( \frac{1 + 2 \times 0}{p_{ex}} \right) \times 180^\circ = \left( \frac{1}{2} \right) \times 180^\circ = 90^\circ \]

Figure 2:
Figure 3: Completed Root Locus

\[ \theta_1 = \left( \frac{1 + 2 \times 1}{p_{ex}} \right) \times 180^\circ = \left( \frac{2}{2} \right) \times 180^\circ = 270^\circ \]

\[ \sigma_i = \frac{\text{Sum of poles of } GH - \text{Sum of zeros of } GH}{p_{ex}} = \frac{[-8] - [(-2)]}{2} = -3 \]

Two poles at origin break-out at ±90°. One finite zero three poles so two poles migrate to ‘zeros’ (called zeros at infinity), infinitely far away at ends of asymptotes. The root locus is shown in Figure 3
Solution 5.8.1.33

![Feedback configuration diagram]

Figure 1: Feedback configuration

For the system shown above

\[ GH(s) = \frac{K(s + 2)}{(s + 1 + j)(s + 1 - j)(s + 20)(s + 40)} \]

To sketch the root locus, we compile the data shown in Table 1, which determine if there are break-in or break-out points. From Table 1 we see

<table>
<thead>
<tr>
<th>s</th>
<th>-19</th>
<th>-17</th>
<th>-13</th>
<th>-10</th>
<th>-8.5</th>
<th>-8</th>
<th>-7.5</th>
<th>-4.5</th>
<th>-4</th>
<th>-3.5</th>
<th>-3</th>
</tr>
</thead>
<tbody>
<tr>
<td>K</td>
<td>401</td>
<td>1182</td>
<td>2491</td>
<td>3075</td>
<td>3191</td>
<td>3200</td>
<td>3194</td>
<td>2916</td>
<td>2880</td>
<td>2911</td>
<td>3145</td>
</tr>
</tbody>
</table>

Table 1: Gain along real axis

that there is a break-out point near \( s = -8 \). That means there must be a break-in point to the right of \( s = -8 \). There are four poles and one zero so we have three asymptotes at 60°, 140° and 300°. The asymptotes intersect the negative real axis at

\[ s = \frac{-1 - 1 - 20 - 40 + 2}{3} = -20. \]

The completed root locus is shown in Figure 2. The root locus shown in Figure 3 was generated by the MATLAB dialogue:

```
EDU> K = linspace(0,2000,200);
EDU> gh = zpk([-2],[0 -1 -20 -40],10)
```
Figure 2: Sketch of root locus
Figure 3: MATLAB generated root locus

Zero/pole/gain:
10 (s+2)
------------------------
s (s+1) (s+20) (s+40)

EDU>rlocus(gh)
EDU>print -deps rl58133.eps
EDU>
5.8.2.5

For the system of Figure 1 let

\[
G(s)H(s) = \frac{K(s + 1)}{s(s - 2)(s + 40)}.
\]

The root locus is shown in Figure 2. The pole zero excess (pze) is two, meaning two asymptotes at ±90°. The gain to place a poles the imaginary axis can be found in a number of ways. One is to search along the imaginary axis until we find where the angles contributions of the pole and zeros of GH sum to −180°. Another is to solve for the for break-out point, either analytically. In either case the basic formula is

\[
|GH(s)| = 1,
\]

which in the present case yields

\[
\frac{K|s + 1|}{|s||s - 2||s + 40|} = 1,
\]

or

\[
K = \frac{|s||s - 2||s + 40|}{|s + 1|}.
\]

The vector interpretation of equation 1 is shown in Figure 3.

We simply pick points along the positive real axis to the left of the pole of GH at s = 2. We know that all these points are closed loop poles for some value of K. Table 1 summarizes the search using the equation

\[
K = \frac{|s||s - 2|}{|s + 1|}
\]
Figure 2: Root locus

Figure 3: Gain calculation along real axis

<table>
<thead>
<tr>
<th>s</th>
<th>0.72</th>
<th>0.73</th>
<th>0.74</th>
<th>0.735</th>
<th>0.741</th>
<th>0.739</th>
</tr>
</thead>
</table>

Table 1: Gain values along positive real axis
The break-out point is near $s = 0.741$.

To find where the root locus crosses the imaginary axis we solve the angle equation

$$\angle(GH) = -180^\circ,$$

as shown in Figure 4. We thus have

$$\angle(s + 1) - \angle(s + 40) - \angle(s - 2) = -180^\circ.$$

Table 2 shows the search.

We see from the table that the root locus crosses the imaginary axis at

$$\omega = 1.47 \text{ rad}.$$

The gain to place poles on the imaginary axis is then

$$K = \frac{|s||s - 2||s + 40|}{|s + 1|} |_{s = j1.47} = 82.15.$$

Thus, the system is stable for $0 < K < 82.15$. 

3
Solution 5.8.2.6

For the system of Figure 1 let

\[
\begin{align*}
G(s)H(s) &= \frac{K(s + 1)}{s^2(s + 2)(s + 4)}
\end{align*}
\]

Figure 1: Standard Closed Loop Configuration

The root locus is shown above. The pole zero/excess(pze) is three, meaning three asymptotes at

\[
\theta_1 = 60^\circ, \quad \theta_2 = 180^\circ, \quad \theta_3 = 300^\circ.
\]

The asymptotes intersect at

\[
\begin{align*}
\sigma_i &= \sum_{\text{pole locations (of } GH \text{) } - \sum_{\text{zero locations (of } GH \text{) }}} \text{pole/zero excess} \\
&= \frac{(0 - 0 - 2 - 4) - (-1)}{3} \\
&= -1.67
\end{align*}
\]

The pole locations on the \( j\omega \)-axis can be found by satisfying the angle condition, namely

\[
\angle GH(j\omega) = -180^\circ
\]

for some \( \omega \). We find \( \omega \) by searching along the \( j\omega \)-axis until we satisfy 1. The calculation can be set up using Figure 2 That is

\[
\angle GH(j\omega) = \tan^{-1}(\omega/1) - 90^\circ - \tan^{-1}(\omega/2) - \tan^{-1}(\omega/4)
\]

Table 1 summarizes the search for the \( \omega \) that satisfies this equation.

From the table we see that the root locus crosses at approximately \( s = \pm j1.415 \). Students resist this approach because it does not lead to
Figure 2: Finding crossing point on imaginary axis

<table>
<thead>
<tr>
<th>$s$</th>
<th>$j1$</th>
<th>$j1.5$</th>
<th>$j1.4$</th>
<th>$j1.42$</th>
<th>$1.415$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\angle GH(s)$</td>
<td>$-175.6^\circ$</td>
<td>$-181.12^\circ$</td>
<td>$-179.8^\circ$</td>
<td>$-180.1^\circ$</td>
<td>$-180.01^\circ$</td>
</tr>
</tbody>
</table>

Table 1: Search for crossing of imaginary axis
\begin{tabular}{cccccccc}
\hline
s & -1.5 & -1.3 & -1.2 & -1.25 & -1.27 & -1.26 & -1.268 \\
K & 5.63 & 10.64 & 16.1 & 12.9 & 11.9 & 12.3 & 11.99 \\
\hline
\end{tabular}

Table 2: Search for closed loop pole to the right of \( s = -2 \)

\begin{tabular}{cccccccc}
\hline
s & -5 & -4.5 & -4.7 & -4.8 & -4.75 & -4.73 & 4.732 \\
K & 18.75 & 7.23 & 11.28 & 13.58 & 12.4 & 11.95 & 12.0 \\
\hline
\end{tabular}

Table 3: Search for closed loop pole to left of \( s = -4 \)

an analytical solution. However, as can be seen, we find the crossing points relatively quickly. In addition, the process is self correcting. If we make a mistake in any particular calculation, it will generally correct itself if we do the next one right.

The gain to place poles at \( s = \pm j1.415 \) can be calculated from the basic formula:

\[ |GH(s)| = 1, \quad (3) \]

which in the present case yields

\[
K = \left. \frac{(s^2|s + 2||s + 4|)}{|s + 1|} \right|_{s = j1.415} \\
= \frac{1.415^2 \times |2 + j1.415| \times |4 + j1.415|}{|1 + j1.415|} \\
= \frac{1.415^2 \times \sqrt{2^2 + 1.415^2} \times \sqrt{4^2 + 1.415^2}}{\sqrt{1 + 1.415^2}} \\
= 12.01
\]

For this system we can find the other two closed loop poles by simply picking points along the negative real axis to the right of the pole of \( GH \) at \( s = -2 \) and to the left of the pole at \( s = -4 \) and compute the gain. We know that all these points are closed loop poles for some value of \( K \). What we want is the particular values of \( s \) that correspond to a gain \( K = 12.01 \). Table 2 summarizes the search for the closed loop pole to the right of the pole of \( GH \) at \( s = -2 \).

We see there is a closed loop pole very close to \( s = -1.268 \) Table 3 summarizes the search for the closed loop pole to the left of \( s = -4 \).
Figure 3: Accurate root locus

Thus we know that for $K = 12$

$$C(s) = \frac{12\alpha(s)}{s - j1.415)(s + j1.415)(s + 1.268)(s + 4.732)}$$

and the system is stable for $0 < K < 12$. The root locus is shown in Figure 3.
Solution 5.8.3.1

For the system of Figure 1 let

\[ GH = \frac{K(s + 2)}{(s + 1 - j)(s + 1 + j)(s + 20)(s + 40)} \]

Figure 1: Standard Closed Loop Configuration

The first step in drawing the root locus is to plot the poles and zeros of \( GH \). The poles of \( GH \) are not the closed loop pole locations, but they can be used to find the closed loop poles. The closed loop zeros can be found immediately: they are the zeros of \( G \) and the poles of \( H \). The poles and zeros of \( GH \) serves as landmarks that help in finding the polés of the closed loop system.

The portion of the root locus on the real axis is the shaded regions shown in Figure 2. These regions are determined by invoking the rule that states that root locus on the real axis is found to the left of an odd count of poles and zeros of \( GH \).

The root locus has four poles and one finite zero. One of the limbs of the root locus will end at the finite zero for \( K = \infty \). The other three limbs will end at so-called ‘zeros at infinity.’ These zeros are at the ‘end’ of each of the three asymptotes. The asymptotes are at \( \theta = 60^\circ, 180^\circ, \) and \( 300^\circ \).

Figure 2: Root Locus on Real Axis
The asymptotes intersect at

\[ \sigma_i = \frac{[0 + (-1) + (-1) + (-20) + (-40)] - [-2]}{3} \]
\[ = -20. \]

The two potential root loci are shown in Figure 3. If a break-in and break-out point exist between \( s = -2 \) and \( s = -20 \) then the root locus will look like part (a) of the figure. If there are no break-in and break-out points the root locus will look like part (b) of the figure.

To check for the possibility of break-in and break-out points the gain can be calculated at some representative values along the real axis. The plot will look like part (a) of Figure 4 if there are break-in and break-out points, and
like part(b) if there are no break-in or break-out points. Figure 4.

The gain can be plotted using Figure 5. A representative point on the line segment is shown in the figure. For this choice of s we know that

\[
|GH(s)| = \frac{K|s + 2|}{|s + 1 - j||s + 1 + j||s + 20||s + 40|} = 1
\]

Thus

\[
K = \frac{|s + 1 - j||s + 1 + j||s + 20||s + 40|}{|s + 2|}
\]

\[\alpha - \theta_1 - \theta_2 - \theta_3 - \theta_4 = -180^\circ.\]

As the circle shrinks in radius all the angles except \(\theta_1\) can be computed. That is,

\[
\theta_1 = \tan^{-1}(1/1) + 180^\circ - \tan^{-1}(1/19) - \tan^{-1}(1/39) - 90^\circ
\]

\[= 45^\circ + 180^\circ - 90^\circ - 3.01^\circ - 1.47^\circ\]

\[= 130.52^\circ\]

The angle of departure is really a neutral indicator in this case. The MATLAB dialogue

EDU>gh = zpk([-2],[-1+j*1 -1-j*1 -20 -40],100)

Zero/pole/gain:
100 (s+2)

-------------------------
(s+20) (s+40) (s^2 + 2s + 2)

EDU>rlocus(gh)
EDU>print -deps rl5831f.epw
EDU>

draws the root locus shown in Figure 7.
Figure 4: Gain Versus Position for $s \in [-20, -2]$
Figure 5: Computation of Gain Along Real Axis

Figure 6: Calculation of angle of Departure
Figure 7: MATLAB generated root locus