Adjoint Equations and Random Walks for Illumination Computation

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Abstract

In this paper we introduce the potential equation which along with the rendering equation forms an adjoint system of equations and provides the mathematical framework for all known approaches to illumination computation. The potential equation is more natural for illumination computations which simulate light propagation starting from the light sources, such as, progressive radiosity and particle tracing. Using the mathematical handles provided by the adjoint system of equations and the random walk model, we present a number of biasing schemes for improving the computation of flux estimation. Of particular significance is the scheme to use an approximate potential value as the biasing function for directing a majority of the random walks through regions of importance in the environment thus reducing the variance in the estimates of flux in these regions. Finally results from a simple implementation of this scheme is presented.

1 Introduction

There are basically two approaches used for the computation of global illumination. In one the luminous flux reaching a surface or eye point is estimated by sampling its surroundings. Ray tracing, path tracing and full matrix radiosity solution are extensively researched methods[1, 2, 3, 4] based on this approach. The other approach is based on simulating the propagation of light starting from the light sources of which progressive radiosity and particle tracing[5, 6, 7] are representative methods. So far Kajiya's rendering equation[3] has provided the mathematical basis for these computations. The essence of the rendering equation is the balancing of point to point light transport through the use of a *Transport Intensity* function which is closely related to the *Luminance* function. In this paper we introduce an adjoint formulation which we shall call as the *Potential* function. This function captures the potential that any point of the environment has towards the illumination of another point. The luminous flux at any point of a surface in the environment can be expressed using either the *luminance* (*transport intensity*) or the *potential* function. Together they form an adjoint system of equations which provides the mathematical basis for any illumination computation. While illumination computation methods for computing the flux reaching the surface or the eye are primarily based on the rendering equation, we believe that the potential equation is more natural for describing methods which simulate light propagation starting from the light sources. Just as Kajiya has shown[3] full radiosity to be an approximate analytical solution of the rendering equation, analogously we have also shown progressive radiosity[5] to be an approximate analytical solution of the potential equation. The particle tracing[7] (or light ray tracing method[6]) mathematically speaking is in fact a random walk based solution of the potential equation. Similarly the biasing scheme of absorption suppression[8, 7] found useful in particle tracing automatically falls within the mathematical framework of the random walk.

In this paper we introduce the notion of the potential function intuitively with the help of a simple hypothetical experiment placing light detectors in an environment and then proceed with its mathematical formulation. We then discuss the adjoint system of equations for illumination computation and their corresponding approximate analytic solutions. Monte Carlo quadrature and random walk techniques are then proposed for obtaining solutions to the general equations. Using the mathematical handles provided by the adjoint system of equations and the random walk model, we present a number of biasing schemes for improving the computation of flux estimation. Of particular significance is the scheme to use an approximate potential value as the biasing function for directing a majority of the random walks through regions of importance in the environment thus reducing the variance in the estimates of flux in these regions. Finally performance improvements resulting from a simple straight forward implementation of this biasing scheme are presented.

2 Adjoint System of Illumination Equations

The illumination of any point of a surface in a complex 3D environment is due to the emission of light from that point (if any) and/or due to the reflection from that point of the light received from incoming hemispherical directions around that point. This fundamental concept forms the basis for the derivation of the adjoint system of illumination equations. To simplify our discussion, we have restricted our attention to environments containing only opaque solid objects. However, in no way should this assumption be considered as a limitation of the discussed framework. Illumination of environments containing transmitting surfaces can also be easily explained within the given framework.

2.1 Luminance Equation

The luminance equation is basically Kajiya's rendering equation. We have preferred to use its original form with integration over a hemisphere mainly to retain its similarity with the potential equation which we are introducing immediately after this. From the definition of surface bidirectional reflectance function [9, page 64], the outgoing luminance (L) at any point x of a surface in the environment in any direction Θ_{out}, due to the luminance incident at x from direction Θ_{in} can be given by

$$L_{out}(x, \Theta_{out}) = \rho_x(\Theta_{out}, \Theta_{in}) L_{in}(x, \Theta_{in}) cos\theta_{in} d\omega_{in}$$

where θ_{in} and $d\omega_{in}$ are as shown in the Fig 1. Taking into account incoming luminance from all the directions in the incoming hemisphere around the point x, the outgoing luminance can be expressed as

$$L_{out}(x, \Theta_{out}) = \int_{\Omega_x} \rho_x(\Theta_{out}, \Theta_{in}) L_{in}(x, \Theta_{in}) cos\theta_{in} d\omega_{in}$$

where the integration range Ω_x represents the hemisphere around x. If we include emitting surfaces also in the general expression for the outgoing luminance then it takes the form:

$$L_{out}(x, \Theta_{out}) = \epsilon_{out}(x, \Theta_{out}) + \int_{\Omega_x} \rho_x(\Theta_{out}, \Theta_{in}) L_{in}(x, \Theta_{in}) cos\theta_{in} d\omega_{in}$$

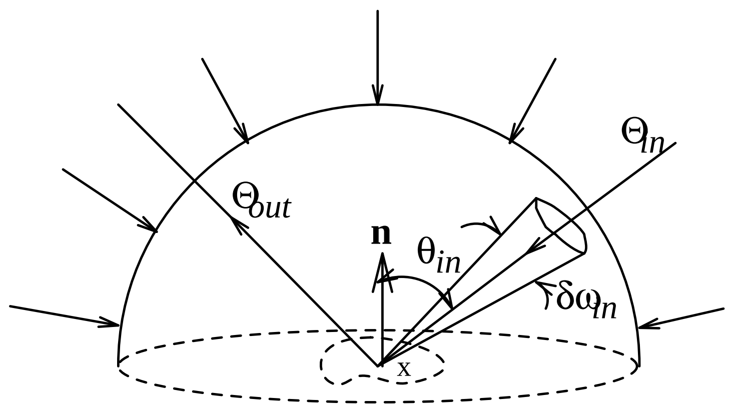

Figure 1: Hemispherical Directions for the Incoming illumination.

For the kind of environment under discussion the luminance in any incoming direction at x must be due to the outgoing luminance from some surface point y in an outgoing direction Θ_y where Θ_y is defined by the vector joining the point x to y. If we now wish to rewrite the luminance equation in terms of outgoing luminance and outgoing directions only, then by representing the outgoing directions at x and y as Θ_x and Θ_y we get:

$$L(x, \Theta_x) = \epsilon(x, \Theta_x) + \int_{\Omega_x} \rho_x(\Theta_x, \Theta_y) L(y, \Theta_y) cos\theta_x d\omega_x \tag{1}$$

Note that there is an implicit assumption in this equation that y represents a surface point visible to x and Θ_y is the direction from y to x. For ease of understanding, Kajiya introduced the concept of two point transport intensity, $I(x, x')$, which is the luminous flux density coming from a differential area dx' around x' and received by the differential area dx around x and reformulated equation 1 as the rendering equation shown below: [3]

$$I(x, x') = v(x, x') \left[\epsilon(x, x') + \int_A \rho(x, x', x'') I(x', x'') dx'' \right]$$

where the integration range is defined as all the surfaces (A) in the environment. In this equation all the illumination terms are from one point to another point of the environment. He introduced the additional term $v(x, x')$ to denote visibility which was not so explicit in the earlier equation. In his paper he has also clearly stated the relationship of each of the multipoint terms with the standard photometric terms used in the equation 1. As long as the similarity between these two forms of the illumination equations is understood, either form of the equation can form the mathematical basis for the illumination computation.

2.2 Potential Equation

Because of the optical properties of surfaces, in our case primarily reflection, the light emitted from any surface in any direction can illuminate many other surfaces of an environment. Alternatively we can say that a surface can be illuminated by lights placed anywhere in the environment. The placement of the lights will of course determine how brightly or how dimly lit that surface is. This phenomenon can be elegantly captured by the notion of a potential associated with every point and direction in the environment. We shall describe a simple experiment to make the concept of potential easier to understand.

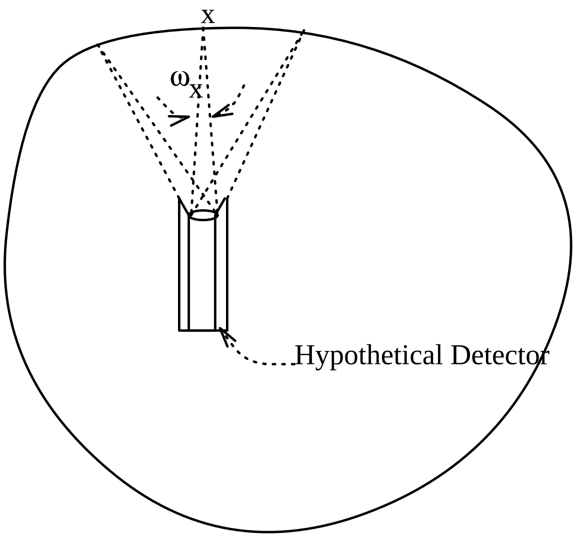

Figure 2: Hypothetical Detector Focused on the Surface Points of the Environment.

For the purpose of illumination computation an environment is generally described in terms of the geometry of its surfaces and their optical properties such as reflection, transmission and emission. To start with, consider an environment completely specified except that its emission characteristics are omitted. Position some hypothetical light detectors in this environment such that outgoing illumination from any surface point and direction gets registered in one and only one detector. In other words each detector exclusively sees some directional emission of some surface region (Fig.2). The detectors are hypothetical and in no way affect the flow of light. Next take a hypothetical point source with highly directional emission, emitting unit amount of luminous flux in any particular direction. If we position this light source at a surface point in some orientation, it is clear that some or all of the hypothetical detectors will register some amount of luminous flux passing through them. Let us concentrate only on one of these, say the k-th detector and note the flux received by that detector because of the placement of the hypothetical emitter. Carry out this exercise for all possible orientations of the hypothetical emitter at that point and at all other surface points of the environments. In the process we will have collected data as a function of all the points and directions of the environment. We will call this as the illumination potential function as this function captures the potential capability of every point and every direction around that point, in illuminating the region on which the k-th detector is focused. Let us denote this function as W_k. Other detectors would similarly define potential functions, say W_i.

Next we shall derive an expression for such a function. Let H_k denote the set of all points x over which the k-th detector is focused. Similarly let $\mathcal{D}_k$ denote the set of all directions made by these point with the aperture of the k-th detector. Then we define a function g_k as follows:

$$g_k(x, \Theta_x) = \begin{cases} 1 & \text{iff } (x \in H_k \text{ and } \Theta_x \in \mathcal{D}_k) \\ 0 & \text{otherwise.} \end{cases}$$

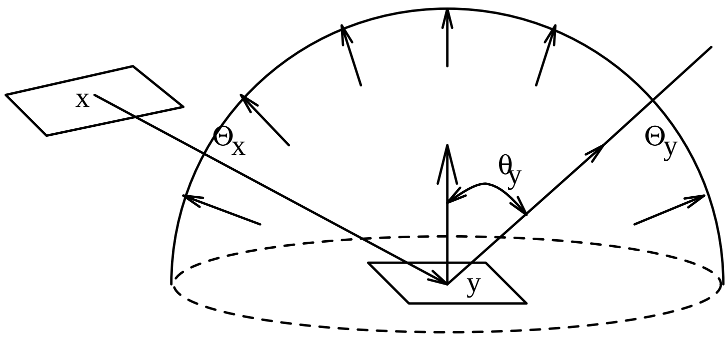

Figure 3: Hemispherical Directions for the Outgoing illumination.

Recall that the potential function W_k is the value of light detected by placing hypothetical unit light sources at every surface point and direction in the environment. Then the immediate contribution of the unit light source placed at (x, Θ_x) in the environment is captured by the function $g_k(x, \Theta_x)$ as the detector would register an immediate unit amount of emission flux only from those emitter positions and orientations, (x, Θ_x), such that $x \in H_k$ and $\Theta_x \in \mathcal{D}_k$ and would register an immediate zero emission flux from any other emitter position and orientation. We also have to account for an indirect contribution which is the flux received by the detector due to any number of reflections of the light emitted form this unit light source. For this component we will provide a recursive expression. The emission from the hypothetical emitter at x along direction Θ_x will reach the nearest surface point y and then possibly reflected. If we take the probability of the whole amount of flux getting reflected in any one of the hemispherical directions Θ_y around y as $\rho_y(\Theta_y, \Theta_x) cos\theta_y d\omega_y$, where the symbols used are as in Fig 3, then its contribution to the indirect component will be this probability times the potential of the point y along Θ_y, i.e. $\rho_y(\Theta_y, \Theta_x) cos\theta_y d\omega_y W_k(y, \Theta_y)$. The indirect component is then the cumulative result of this expression obtained over the outgoing hemisphere around y, i.e. $\int_{\Omega_y} \rho_y(\Theta_y, \Theta_x) W_k(y, \Theta_y) cos\theta_y d\omega_y$. The complete expression for the potential function is therefore given by:

$$W_k(x, \Theta_x) = g_k(x, \Theta_x) + \int_{\Omega_y} \rho_y(\Theta_y, \Theta_x) W_k(y, \Theta_y) cos\theta_y d\omega_y \tag{2}$$

If we look back at the equation 1 for the luminance equation, we find a striking similarity in the form of this equation with that of the other. However it must be noted that in equation 1 the integration is over the incoming hemisphere around x whereas in equation 2 the integration is over the outgoing hemisphere around y, where y is the surface point visible to x in the direction Θ_x. We now proceed to find the relationship between these two equations.

2.3 Duality

Here we show that the equations 1 and 2 are duals of each other for the purpose of computation of flux. Duality means that either equation may be used.

In most of the illumination computations one is interested in computing flux from a small region in a small spread of directions. For example: in image rendering the color of a pixel is assigned by computing the luminance from all the surface points visible to the eye through that pixel and in a spread of directions made by each such point with the aperture of the eye. Similarly in the computation of view independent global illumination of a diffuse environment by radiosity based methods one is computing flux from a small surface patch in the hemispherical direction.

Expression of this flux using the Luminance equation will therefore be an integral of the form:

$$\Phi_k = \int_{position\ spread} \int_{direction\ Spread} L(x, \Theta_x) cos\theta_x d\omega_x dx$$

If we assume that this flux represent the flux received by the k-th hypothetical detector then we can use the earlier defined function $g_k(x, \Theta_x)$ which evaluates to 1 in the limits of the integration and 0 everywhere else, and rewrite the above equation as follows:

$$\Phi_k = \int_A \int_{\Omega_x} L(x, \Theta_x) g_k(x, \Theta_x) cos\theta_x d\omega_x dx \tag{3}$$

To get an expression for the above discussed flux using the potential function we will remove the hypothetical source and complete the environment description by adding the actual emissive characteristics to some of the surface points thus defining a function ϵ which is zero everywhere except at the positions belonging to emissive surfaces. By introducing ϵ we get the emission luminance at (x, Θ_x) to be $\epsilon(x, \Theta_x)$ and hence the emission flux leaving (x, Θ_x) is $\epsilon(x, \Theta_x) cos\theta_x d\omega_x dx$. The potential of (x, Θ_x) towards the k-th detector is $W_k(x, \Theta_x)$. Then the flux received by the k-th detector due to the actual emission at (x, Θ_x) will be $W_k(x, \Theta_x) \times \epsilon(x, \Theta_x) cos\theta_x d\omega_x dx$. Thus the expression for the total flux received by the k-th detector will be

$$\Phi_k = \int_A \int_{\Omega_x} W_k(x, \Theta_x) \epsilon(x, \Theta_x) cos\theta_x d\omega_x dx \tag{4}$$

To sum up

- We have given two different equations, equations 3 and 4, to express the same quantity Φ_k using two different functions L and W_k.
- The equations 3 and 4 are similar in form and so also are the equation 1 and 2 for L and W_k.
- Equations 1, 2, 3 and 4 together form a closed system.

We will write again all these four equations together to highlight the above mentioned points.

$$\begin{aligned}
\int_A \int_{\Omega_x} L(x, \Theta_x) g_k(x, \Theta_x) cos\theta_x d\omega_x dx = \quad \Phi_k \quad &= \int_A \int_{\Omega_x} W_k(x, \Theta_x) \epsilon(x, \Theta_x) cos\theta_x d\omega_x dx \\
L(x, \Theta_x) \quad &= \quad \epsilon(x, \Theta_x) + \int_{\Omega_x} \rho_x(\Theta_x, \Theta_y) L(y, \Theta_y) cos\theta_x d\omega_x \\
W_k(x, \Theta_x) \quad &= \quad g_k(x, \Theta_x) + \int_{\Omega_y} \rho_y(\Theta_y, \Theta_x) W_k(y, \Theta_y) cos\theta_y d\omega_y
\end{aligned}$$

The equations satisfying above mentioned properties are said to form an adjoint system.

One may wish to solve equation 3 or 4 to compute Φ_k. In the subsequent chapters we will discuss solution methods for computing this flux using either of the equations.

3 Analytical Solution for a Diffuse Environment

Because of their inherently complex nature it is not possible to analytically solve equations 3 and 4. However, simplified forms of these may be amenable to analytical solutions. We will derive the simplified equations by making the following assumptions[1]:

1. The environment is a collection of a finite number, say N, of small uniformly diffuse patches.
2. As the luminance from any point of any such uniformly diffuse patch is $1/\pi$ times the flux per unit area we shall compute this total flux from any patch leaving that patch in all the hemispherical direction.
3. The solution is carried out in an enclosure, i.e. the hemispherical direction around any point in the environment is assumed to be covered by one or more of the patches of that environment and every patch, j, may be assumed to occupy a solid angle, ω_j (which may be zero) in the hemisphere over any surface point.

Because of the diffuse nature of the surfaces the luminance functions, $\epsilon(x, \Theta_x)$ and $L(x, \Theta_x)$ become independent of Θ_x and because of the uniformity of the patch they are also independent of the position x on any patch. Thus they may be denoted as ϵ_i and L_i respectively for all the x belonging to the patch i. Similarly $\rho_x(\Theta_x, \Theta_y)$ is independent of directions Θ_x and Θ_y and is independent of the position x on any patch i and hence may be denoted as ρ_i.

Under these assumptions equation 1 for the luminance function will simplify to:

$$L_i = \epsilon_i + \rho_i \int_{\Omega_i} L_j cos\theta_i d\omega_i = \epsilon_i + \rho_i \sum_{j=1}^{N} L_j \int_{\omega_{ij}} cos\theta_i d\omega_i = \epsilon_i + \rho_i \sum_{j=1}^{N} L_j F_{ij} \tag{5}$$

where L_j is the luminance of any point y belonging to the j-th patch; ω_{ij} is the solid angle occupied by the j-th patch in the visible hemisphere around the point x of the small patch i, and F_{ij} is the geometric factor between patch i and j. It may be recalled that F_{ij} is similar to the formfactor used in the radiosity computation[4] with the only difference that the summation of this factor over all the patches, i.e. $\sum_{j=1}^{N} F_{ij}$, is equal to π.

As we are computing the flux leaving a surface patch in the hemispherical directions and the surface patch has uniform illumination properties, again by the above assumptions, $g_k(x, \Theta_x)$ is independent both of the direction Θ_x and x on any patch i and hence may also be denoted as $g_{k,i}$. Further by the original definition of the g function, $g_{k,i}$ would evaluate to 1 for $i = k$ and 0 otherwise. Using these relations we derive the new expression for the flux leaving the k-th patch or equivalently the flux reaching the k-th detector which sees the total outgoing flux from the k-th patch, as follows:

$$\begin{aligned}
\Phi_k &= \int_A \int_{\Omega_x} L(x, \Theta_x) g_k(x, \Theta_x) cos\theta_x d\omega_x dx = \sum_{i=1}^{N} \int_{A_i} \int_{\Omega_x} L(x, \Theta_x) g_k(x, \Theta_x) cos\theta_x d\omega_x dx \\
&= \sum_{i=1}^{N} L_i g_{k,i} \int_{A_i} \int_{\Omega_x} cos\theta_x d\omega_x dx = \sum_{i=1}^{N} L_i g_{k,i} \pi A_i = L_k A_k \pi
\end{aligned}$$

[1]These assumptions are typical of radiosity based solutions[4].

$$
\begin{aligned}
&= A_k \pi \left(\epsilon_k + \rho_k \sum_{j=1}^{N} L_j F_{kj} \right) = E_k + A_k \pi \rho_k \sum_{j=1}^{N} L_j F_{kj} \\
&= E_k + A_k \rho_k \sum_{j=1}^{N} \frac{\Phi_j}{A_j} F_{kj} \qquad (6)
\end{aligned}
$$

where E_k is the total emission flux leaving the k-th patch. We can see that this equation forms the basis of well established radiosity method[4].

To derive the analytical approximation of Φ_x using the potential function we introduce a hemispherical potential function over any point of patch i, $\mathcal{W}_k$, as the average potential of the surface points in any hemispherical direction. If the patches are sufficiently small this hemispherical potential function may be assumed to be independent of the position on each patch. The expression for this hemispherical potential function will be:

$$
\begin{aligned}
\mathcal{W}_{k,i} &= \frac{1}{\pi} \int_{\Omega_{x_i}} W_k(x_i, \Theta_{x_i}) cos\theta_{x_i} d\omega_{x_i} \\
&= \frac{1}{\pi} \int_{\Omega_{x_i}} \left[g_k(x, \Theta_x) + \int_{\Omega_y} \rho_y(\Theta_y, \Theta_x) W_y(y, \Theta_y) cos\theta_y d\omega_y \right] cos\theta_{x_i} d\omega_{x_i} \\
&= \frac{1}{\pi} \int_{\Omega_{x_i}} \left[g_{k,i} + \pi \rho_j \mathcal{W}_{k,j} \right] cos\theta_{x_i} d\omega_{x_i} = \frac{1}{\pi} g_{k,i} \int_{\Omega_{x_i}} cos\theta_{x_i} d\omega_{x_i} + \int_{\Omega_{x_i}} \rho_j \mathcal{W}_{k,j} cos\theta_{x_i} d\omega_{x_i} \\
&= g_{k,i} + \sum_{j=1}^{N} \rho_j \int_{\omega_{ij}} \mathcal{W}_{k,j} cos\theta_{x_i} d\omega_{x_i} = g_{k,i} + \sum_{j=1}^{N} \rho_j \mathcal{W}_{k,j} \int_{\omega_{ij}} cos\theta_{x_i} d\omega_{x_i} \\
&= g_{k,i} + \sum_{j=1}^{N} \rho_j \mathcal{W}_{k,j} F_{ij} = g_{k,i} + \sum_{j=1}^{N} \rho_j F_{ij} \left[g_{k,j} + \sum_{l=1}^{N} \rho_l \mathcal{W}_{k,l} F_{jl} \right] \\
&= g_{k,i} + \sum_{j=1}^{N} \rho_j F_{ij} g_{k,j} + \sum_{j=1}^{N} \rho_j F_{ij} \sum_{l=1}^{N} \rho_l \mathcal{W}_{k,l} F_{jl} \\
&= g_{k,i} + \rho_k F_{ik} + \sum_{j=1}^{N} \rho_j F_{ij} \rho_k F_{jk} + \ldots \qquad (7)
\end{aligned}
$$

Using this hemispherical potential function we can derive the simplified expression for the flux over the k-th patch as follows:

$$
\begin{aligned}
\Phi_k &= \int_A \int_{\Omega_x} W_k(x, \Theta_x) \epsilon(x, \Theta_x) cos\theta_x d\omega_x dx = \sum_{i=1}^{ns} \int_{A_i} \int_{\Omega_x} W_k(x, \Theta_x) \epsilon(x, \Theta_x) cos\theta_x d\omega_x dx \\
&= \pi \sum_{i=1}^{ns} \epsilon_i \mathcal{W}_{k,i} \int_{A_i} dx = \pi \sum_{i=1}^{ns} \epsilon_i A_i \mathcal{W}_{k,i} \\
&= \pi \sum_{i=1}^{ns} \epsilon_i A_i \left[g_{k,i} + \rho_k F_{ik} + \sum_{j=1}^{N} \rho_j F_{ij} \rho_k F_{jk} + \ldots \right] \qquad (8)
\end{aligned}
$$

where ns is the total number of source patches in the environment. And this is how the computation proceeds in the progressive refinement approach for the radiosity computation[5]. Thus just as the full matrix radiosity solution is an approximate solution to the rendering equation, progressive radiosity solution is analogously an approximate solution to the potential equation.

4 Monte Carlo and Random Walk for General Solution

We now attempt to provide a general solution method for computing flux using equation 3 and 4. Given equation 3 or 4, basically we have to carry out a multidimensional integration. The integration is further complicated by the fact that a component of integral in turn has a form of an integral equation of the second kind.

It is by now well known that solution of multidimensional integrations are best carried out by Monte Carlo quadrature techniques. The main principle behind a Monte Carlo quadrature technique for computing the integral $\int F(x)dx$ is as follows[10]:

1. Write *F(x)* as a product $f_1(x)f_2(x)$ such that $\int f_1(x) = 1$, *i.e.* $f_1(x)$ is a *pdf*.
2. Sample f_1 for a x_i.
3. For each such sample x_i evaluate $f_2(x_i)$.
4. Carry out the steps (2) and (3) for some n times. The average, $\frac{1}{n}\sum_{i=1}^{n} f_2(x_i)$, is the estimate of the integral. Larger the n better is the estimation.

We will follow similar steps to evaluate equations 3 and 4. We have to find a *pdf* for the purpose. In both of the equations we have a predefined known function each, ϵ and g_k respectively. We can convert these functions to constant times a normalised function. That means emission function $\epsilon(x, \Theta_x)$ may be converted to $\mathcal{E} \times S(x, \Theta_x)$ where $\mathcal{E} = \int_A \int_{\Omega_x} \epsilon(x, \Theta_x) cos\theta_x d\omega_x dx$ and $g_k(x, \Theta_x)$ may be converted into $\mathcal{G}_k \times G_k(x, \Theta_x)$ where $\mathcal{G}_k = \int_A \int_{\Omega_x} g_k(x, \Theta_x) cos\theta_x d\omega_x dx$. Then the quadrature process will start by sampling S and G_k using any standard sampling technique. For each such sample the next task will be to evaluate W_k and L. As said earlier the L and W_k are integral equations of the second kind. Random walk, a versatile mathematical method, is known to be useful in solving such integral equations[11]. We shall discuss below in detail its use for the evaluation of luminance(L) and potential(W_k) values.

A random walk or a Markov chain is basically a sequence of states. Its formulation requires the definition of all possible states (discrete or continuous) of the system, a starting state and the transition probability function (T) for transition from one state (s) to another (s') such that $\int T(s \to s')ds' \leq 1$. From a current state the next state is chosen by sampling this transition probability function. The transition kernel is said to be normalised if $\int T(s \to s')ds' = 1$, whereas it is said to be subcritical when $\int T(s \to s')ds' < 1$. In a subcritical situation, the probability of $(1 - \int T(s \to s')ds')$ is taken as the probability of no transition (absorption) from in any state s. Hence random walk with a subcritical transition kernel is bound to terminate in finite number of steps on undergoing absorption at some state, whereas any random walk with a normalised kernel can go on for ever. So in the latter cases, the walk has to be terminated by some external criterion. Any interaction of the light with the medium or the surface is always associated with some absorption. Thus the environment for illumination computation is always subcritical with $\rho_y(\Theta_y, \Theta_x) cos\theta_y$ as the transition kernel for potential equation solution and $\rho_x(\Theta_x, \Theta_y) cos\theta_x$ as the transition kernel for luminance equation solution. Thus straight forward evaluation of L or W_k using a random walk leads to paths consisting of finite number of steps and hence are computable. The states in our environment are the continuum of surface positions and hemispherical directions around each such surface position. The starting states are sampled from the respective *pdf*s i.e. $S(x, \Theta_x)$ or $G_k(x, \Theta_x)$.

The evaluation of equation 3 may be carried out by drawing n samples from the *pdf*, $G_k(x, \Theta_x)$, and evaluating L by the random walk for each sample (x_i, Θ_{x_i}). If the i-th random walk starting from the state $(x_{i_0}, \Theta_{x_{i_0}})$ covers m_i steps, $(x_{i_1}, \Theta_{x_{i_1}})$,... $(x_{i_{m_i}}, \Theta_{x_{i_{m_i}}})$, then the luminance estimate from this walk will be given by

$$< L(x_i, \Theta_i) >= \epsilon(x_{i_0}, \Theta_{x_{i_0}}) + \sum_{k=1}^{m_i} \epsilon(x_{i_k}, \Theta_{x_{i_k}}) \tag{9}$$

From this, the estimate of Φ_k will be arrived by averaging over n such random walks.

$$\begin{aligned}
\Phi_k &= \int_A \int_{\Omega_x} L(x, \Theta_x) g_k(x, \Theta_x) cos\theta_x d\omega_x dx \\
&= \mathcal{G} \int_A \int_{\Omega_x} L(x, \Theta_x) G_k(x, \Theta_x) cos\theta_x d\omega_x dx \\
&= \mathcal{G} \times \frac{1}{n} \sum_{i=1}^{n} \left[\epsilon(x_{i_0}, \Theta_{x_{i_0}}) + \sum_{k=1}^{m_i} \epsilon(x_{i_k}, \Theta_{x_{i_k}}) \right] \\
&= \mathcal{G} \times \frac{1}{n} \sum_{i=1}^{n} \sum_{k=0}^{m_i} \epsilon(x_{i_k}, \Theta_{x_{i_k}})
\end{aligned} \tag{10}$$

This method of evaluating Φ_k by first sampling the G_k function is the essence of Kajiya's *path tracing* method.

Similarly the evaluation of equation 4 may be carried out by drawing samples $(x_{i_0}, \Theta_{x_{i_0}})$ from the source function, $S(x, \Theta_x)$ and carrying out the random walk. A random walk may terminate at the state $(x_{i_k}, \Theta_{x_{i_k}})$ with probability

$$\sigma_{(x_{i_k}, \Theta_{x_{i_k}})} = 1 - \int_{\Omega_y} \rho_y(\Theta_y, \Theta_{x_{i_k}}) cos\theta_y d\omega_y \tag{11}$$

or proceed to the next state $(x_{i_{k+1}}, \Theta_{x_{i_{k+1}}})$ chosen with probability $\rho_{x_{i_{k+1}}}(\Theta_{x_{i_{k+1}}}, \Theta_{x_{i_k}}) cos\theta_{x_{i_{k+1}}}$ and so on. For each such sample W_k can be evaluated by carrying out the random walk. The potential estimate from this walk will be given by

$$< W_k(x_i, \Theta_i) >= g_k(x_{i_0}, \Theta_{x_{i_0}}) + \sum_{k=1}^{m_i} g_k(x_{i_k}, \Theta_{x_{i_k}}) \tag{12}$$

Thus Φ_k will be estimated from n such walks as:

$$\begin{aligned}
\Phi_k &= \int_A \int_{\Omega_x} W_k(x, \Theta_x) \epsilon(x, \Theta_x) cos\theta_x d\omega_x dx \\
&= \mathcal{E} \times \int_A \int_{\Omega_x} W_k(x, \Theta_x) S(x, \Theta_x) cos\theta_x d\omega_x dx \\
&= \mathcal{E} \times \frac{1}{n} \sum_{i=1}^{n} \left[g_k(x_{i_0}, \Theta_{x_{i_0}}) + \sum_{k=1}^{m_i} g_k(x_{i_k}, \Theta_{x_{i_k}}) \right] \\
&= \mathcal{E} \times \frac{1}{n} \sum_{i=1}^{n} \sum_{k=0}^{m_i} g_k(x_{i_k}, \Theta_{x_{i_k}})
\end{aligned} \tag{13}$$

This method of evaluating Φ_k by first sampling the source function is the essence of the *particle tracing* or light ray tracing.

Of the two solution methods, *particle tracing* is highly intuitive as it resembles the physical illumination process [7]. Sampling of the source for a start state may be thought of as the emission of a photon from the source and the transition for simulation of random walks may be thought of as the wandering of the photon in the environment as it gets reflected at the surface boundaries until it is absorbed. *Path tracing* though not directly related to the physical process is by now well known to the *computer graphics* community. The eye point, p_{eye}, and a random point on the pixel, p_{pixel}, define the direction, $\Theta_{out} = p_{eye} - p_{pixel}$. This direction along with the nearest surface position along $-\Theta_{out}$ define the starting state for the random walk. At that nearest surface the ray is absorbed and the walk terminates or is reflected along one of the incoming hemispherical directions, Θ_{in}, by sampling the *brdf* and the walk continues.

What is more important in the discussion so far is that both the random walk processes attempt to solve the same problem and are subjected to similar statistical errors which in montecarlo studies is known as *variance*. But one thing which makes the *particle tracing* more attractive is that the simulation proceeds by sampling the source function. If we partition the space into a finite number of subregions $sr_1, sr_2, sr_3, \ldots$, then we can locate detectors focused over each of these i.e. formulate equal number of g functions $g_1, g_2, g_3, \ldots$, such that g_i is nonzero in the respective subregion sr_i and zero otherwise. Then each random walk originating from the source contributes towards the estimation of the Φ_i for each of the subregions. At the end of the simulation we have the estimates for Φ_i for all the subregions. Whereas in *path tracing* the random walk starts by sampling a particular g_i, for example: directions through a particular pixel. So each random walk contributes towards the estimation of only the Φ_i for that region for which g_i is defined to be nonzero. This is not meant to be understood as saying that the computational efforts required to compute the brightness of a pixel by path tracing and to compute the illumination of all the subregions visible through a pixel by particle tracing are of equal magnitude. One may arrive at a low variance in the brightness estimate of the pixel by tracing a small number of paths whereas it is possible that even after a large number of particle tracings the brightness estimates of a few of the subregions continue to show high variance. However, the difference is worth repeating:

> In particle tracing a single random walk contributes towards the estimation of many Φ_is as against many random walks contributing to the single Φ_i in path tracing.

There have been a number of efforts to combine these approaches and thus derive benefits of both. These have typically come to be known as two pass methods or more generally multi pass methods[12, 13, 14, 7]. In the initial passes simulation proceeds starting from the light sources and estimates are obtained for the flux in different subregions. For example, radiosity[13, 14] or particle tracing[7] is used in the first pass to estimate the flux over diffuse surfaces. Chen et al[14] have an additional pass in which rays are traced from the light sources through non-diffuse surfaces to estimate caustics. In the case of multiple initial passes, care is taken to ensure that the flux computations are non-intrusive. The final rendering pass is always from the eye which is based on the random walk solution for equation 1 with slight difference from path tracing in that the walks are absorption suppressed[2] and the walk terminates at a diffuse surface whose illumination computation has already been carried out in the earlier passes.

[2] See section 5.2.1 for Absorption Suppression.

Knowing the basic solution processes now we shall discuss some strategies for reduction in variance. Most of our discussions will be based on the particle tracing method. However, it must be emphasised that both the solution methods shall be equally benefited by these strategies.

5 Improved Estimation Strategies

We discuss a few methods based on the observations that each random walk contributes either zero or nonzero values to the estimation of a Φ_i. In most of the situations of interest, more specifically in the problem of illumination computation of a reasonably complex environment the fraction of random walks contributing nonzero values towards the estimate of any single Φ_i is small[3]. The basic principle of any computation based on random sampling is that *larger the number of samples better is the confidence on the estimated result.* A similar principle applied to particle tracing would mean that larger the number of random walk visits to any subregion better could be the confidence in the estimated flux. A simple minded approach of improving the estimated result will be to increase the number of random walks. Each random walk requires some amount of computational effort for – sampling the initial state, sampling the transition probability function for moving to the next state and computing the nearest surface along a given direction. So any increase in the number of random walks involves proportionate increase in computation and must be contained. It can be seen that many random walks may in fact never visit the subregion(s) of interest or may visit subregions in which there have already been an adequate number of visits and hence not contributing further to the flux estimates of those subregions. So an ideal strategy would be either to transform the basic underlying random walk process or to change the estimator or to do both such that each random walk almost always contributes towards any subregion of our interest.

5.1 Next Event Estimation

This technique[15] leaves the stochastic process under study unaltered but modifies the form of the estimator. The modification involves the use of

$$g_k(x_{i_0}, \Theta_{x_{i_0}}) + \sum_{k=0}^{m_i} W_k^1(x_{i_k}, \Theta_{x_{i_k}})$$

as the estimator of $W_k(x_i, \Theta_{x_i})$ and thus uses

$$\Phi_k = \mathcal{E} \times \frac{1}{n} \sum_{i=1}^{n} \left[g_k(x_{i_0}, \Theta_{x_{i_0}}) + \sum_{k=0}^{m_i} W_k^1(x_{i_k}, \Theta_{x_{i_k}}) \right] \tag{14}$$

instead of the eqn.13 for the estimation of Φ_k using *particle tracing* where

$$W_k^1(x, \Theta_x) = \int_{\Omega_y} \rho_y(\Theta_y, \Theta_x) g_k(y, \Theta_y) cos\theta_y d\omega_y$$

and the use of

$$\epsilon(x_{i_0}, \Theta_{x_{i_0}}) + \sum_{k=0}^{m_i} L^1(x_{i_k}, \Theta_{x_{i_k}})$$

[3]In *particle tracing* it rarely happens that every subregion of the space is visited in a single walk. Similarly in the *path tracing* it is also equally rare that every random walk starting from the eye will at all visit the light source during its walk.

as the estimator of $L(x_i, \Theta_{x_i})$ and thus uses

$$\Phi_k = \mathcal{G} \times \frac{1}{n} \sum_{i=1}^{n} \left[\epsilon(x_{i_0}, \Theta_{x_{i_0}}) + \sum_{k=0}^{m_i} L^1(x_{i_k}, \Theta_{x_{i_k}}) \right] \tag{15}$$

instead of the eqn.10 for the estimation using *path tracing* where

$$L^1(x, \Theta_x) = \int_{\Omega_x} \rho_x(\Theta_x, \Theta_y) \epsilon(y, \Theta_y) cos\theta_x d\omega_x$$

The choice of this estimator is based on the intuition that replacing the contribution $g_k(x, \Theta_x)$ of an event by $W_k^1(x, \Theta_x)$, the expected direct potential contribution averaged over all possible next events, and replacing the contribution $\epsilon(x, \Theta_x)$ of an event by $L^1(x, \Theta_x)$, the expected direct source contribution averaged over all possible sampling over the source may lead to faster convergence to the actual result. The next event estimation in the *path tracing* would mean computing the local model at the points of ray-hit. Though not explicitly mentioned we believe that Kajiya used this estimator in path tracing as he writes in [3, page 146] "*Calculating emitted ... factors is simply a matter of consulting the ... light models*". We also would like to point out that Chen et al[14, page 167] used a variant of the next event estimation principle in computing the final luminance $I(x, \Theta_x)$ by computing $I_{l,s}(x, \Theta_x)$, a part of $I(x, \Theta_x)$, by Monte Carlo sampling only the source contribution at x.

5.2 Biasing

All the methods discussed under this section attempt to transform the mathematical description of the stochastic process such that the modified form also gives the estimation of Φ and is likely to converge faster than the the original process. The illumination process as described in section(2) is completely described by the source function and the surface *brdf*s. If we replace them by biased functions satisfying some requirement and if we still wish to use them to estimate Φ then we must make the compensation for the change. For such a biasing in particle tracing the compensation required for the estimation may be derived as follows:

$$\begin{aligned}
\Phi_k &= \mathcal{E} \times \int_A \int_{\Omega_x} S(x, \Theta_x) W_k(x, \Theta_x) cos\theta_x d\omega_x dx \\
&= \mathcal{E} \times \int_A \int_{\Omega_x} S'(x, \Theta_x) \left(\frac{S(x, \Theta_x)}{S'(x, \Theta_x)} \right) W_k(x, \Theta_x) cos\theta_x d\omega_x dx \\
W_k(x, \Theta_x) &= g_k(x, \Theta_x) + \int_{\Omega_y} \rho_y(\Theta_y, \Theta_x) W_k(y, \Theta_y) cos\theta_y d\omega_y \\
&= g_k(x, \Theta_x) + \int_{\Omega_y} T'(\Theta_x \to \Theta_y) \left(\frac{T(\Theta_x \to \Theta_y)}{T'(\Theta_x \to \Theta_y)} \right) W_k(y, \Theta_y) d\omega_y
\end{aligned}$$

where S' is the biased normalised source function, $T(\Theta_x \to \Theta_y)$ is the transition function introduced only for notational convenience and is nothing but $\rho_y(\Theta_y, \Theta_x) cos\theta_y$ and T' is the biased transition function. In order to make the equation more compact we will define a multiplication factor f such that

$$f(x, y) = \frac{T(\Theta_x \to \Theta_y)}{T'(\Theta_x \to \Theta_y)}$$

Then the transformed potential equation can simply be written as

$$W_k(x,\Theta_x) = g_k(x,\Theta_x) + \int_{\Omega_y} T'(\Theta_x \to \Theta_y) f(x,y) W_k(y,\Theta_y) d\omega_y \tag{16}$$

The estimation equation with compensation will be:

$$\Phi_k = \mathcal{E} \times \frac{1}{n}\sum_{i=1}^{n} \frac{S(x_{i_0},\Theta_{x_{i_0}})}{S'(x_{i_0},\Theta_{x_{i_0}})} \left[g_k(x_{i_0},\Theta_{x_{i_0}}) + \sum_{k=1}^{m_i} \left(\prod_{l=0}^{k-1} f(x_{i_l}, x_{i_{l+1}}) \right) g_k(x_{i_k},\Theta_{x_{i_k}}) \right] \tag{17}$$

We can see below two special cases of this general biasing mechanism.

5.2.1 Survival Biasing or Absorption Suppression

As the name implies, in this method the absorption probability at the transition points is reduced (may even be made zero) and as a consequence the random walk stretches to longer distances and the probability of nonzero contribution of each random walk to the estimation of Φ_is is increased. The absorption probability σ at any state is given by equation 11. Any reduction in this probability can be achieved by appropriate increase in the reflection probabilities. A very convenient method is to scale the reflection probabilities simply by the factor $\frac{1}{1-\sigma}$, consequently making the absorption probability at every state to zero. Thus the compensated estimate can be derived from eqn.17 to be

$$\Phi_k = \mathcal{E} \times \frac{1}{n}\sum_{i=1}^{n} \left[g_k(x_{i_0},\Theta_{x_{i_0}}) + \sum_{k=1}^{m_i} \left(\prod_{l=0}^{k-1} (1-\sigma_{x_{i_{l+1}}}) \right) g_k(x_{i_k},\Theta_{x_{i_k}}) \right] \tag{18}$$

A word of caution is needed here; if the transition probability is changed such that there is no absorption at any state then any single random walk will go on for ever without terminating. In practice the walk is terminated when the product term in the above equations falls below some minimum threshold. However this termination process introduces a bias into the estimation. Unbiased termination technique like Russian Roulette may be used to overcome this[8, 7].

5.2.2 Source Biasing

In the particle tracing process, emission function, $S(x,\Theta_x)$, plays an important role as every random walk originates at the light source. From equation 17 it can be shown that any biasing of this density, still keeping the normalisation condition satisfied and keeping the transition probability unaltered will change the form of flux estimation equation from eqn.13 to

$$\Phi_k = \mathcal{E} \times \frac{1}{n}\sum_{i=1}^{n} \frac{S(x_{i_0},\Theta_{x_{i_0}})}{S'(x_{i_0},\Theta_{x_{i_0}})} \left[g_k(x_{i_0},\Theta_{x_{i_0}}) + \sum_{k=1}^{m_i} g_k(x_{i_k},\Theta_{x_{i_k}}) \right] \tag{19}$$

Source biasing provides a convenient mechanism for variance reduction using particle tracing. In the next section we have provided the results from its implementation.

5.3 The Approximate Potential as A Biasing Function

Any biasing need be carried out only if it results in improvement in the estimation either with increase in computation speed or in variance reduction. In the case of *particle tracing* this means that the minimum criterion based on which the biasing may be carried out must be such that most of the emissions and transitions, lead the random walk directly or indirectly to the region of interest, the hypothetical detector(s) of section 2. Further the biasing computation must be straight forward. So what we need is a biasing function, which can be easily computed and which can simply be multiplied with the emission function and/or transition probability function to give us the appropriate biased functions. Suppose we want to bias our random work process to improve the estimate variance of a some specified region of interest in the environment. Such region of interest, hence forward will be termed as the *region of importance*. The potential function as defined in equation 2 for this region of importance may be seen as one such function. For the source biasing if we know the potential towards the region of importance of all the source points i.e. where $S(x,\Theta_x) > 0$ then we can bias the source function to $S'(x,\Theta_x)$ such that the $S'(x,\Theta_x) \gg S(x,\Theta_x)$ for those points whose potential is higher and $S'(x,\Theta_x) \ll S(x,\Theta_x)$ for those points for which the potential is lower. Similarly the transition probability i.e. in our case the reflection probability of the particle from a surface point in the hemispherical directions can be preferentially scaled up only for those directions the nearest surface points along which have larger potential towards the region of importance than those along the other directions. However, it is clear that if we can derive this function exactly then we can as well derive the solution for the problem at hand and hence we will not require the simulation. However, if we can derive an approximate value of this function easily, then also this approximate value can be used to bias the emission and transition probability functions. Obviously such biasing is not optimal, but in any case is better for the region of importance than no biasing at all. Such region of importance can be arrived at adaptively in a view dependent global illumination or can be predefined, for example: in rendering one or more views of a scene all those surface points visible to the eye-point in the view(s) would form the region of importance. Our observation is that often it is possible to derive an approximate potential function $W_{imp}(x,\Theta_x)$ with respect to such region of interest. This approximate function or one which can be easily derived from this will be termed *importance function*[15], $Imp(x,\Theta_x)$. We can see the usefulness of this importance function in biasing the mathematical description of the process.

$$\begin{aligned}
\Phi_k &= \mathcal{E} \times \int_A \int_{\Omega_x} S(x,\Theta_x) W_k(x,\Theta_x) cos\theta_x d\omega_x dx \\
&= \mathcal{E} \times \int_A \int_{\Omega_x} (S(x,\Theta_x) Imp(x,\Theta_x)) \times \frac{W_k(x,\Theta_x)}{Imp(x,\Theta_x)} cos\theta_x d\omega_x dx \\
\frac{W_k(x,\Theta_x)}{Imp(x,\Theta_x)} &= \frac{g_k(x,\Theta_x)}{Imp(x,\Theta_x)} + \frac{1}{Imp(x,\Theta_x)} \int_{\Omega_y} T(\Theta_x \to \Theta_y) W_k(y,\Theta_y) d\omega_y \\
&= \frac{g(x,\Theta_x)}{Imp(x,\Theta_x)} + \int_{\Omega} T(\Theta_x \to \Theta_y) \frac{Imp(y,\Theta_y)}{Imp(x,\Theta_x)} \frac{W_k(y,\Theta_y)}{Imp(y,\Theta_y)} d\omega_y
\end{aligned}$$

If we denote $S(x,\Theta_x)Imp(x,\Theta_x)$ as a biased function $S'(x,\Theta_x)$, $\frac{W_k(x,\Theta_x)}{Imp(x,\Theta_x)}$ as $W'_k(x,\Theta_x)$ and $T(\Theta_x \to \Theta_y)\frac{Imp(y,\Theta_y)}{Imp(x,\Theta_x)}$ as $T'(\Theta_x \to \Theta_y)$ then we have the following representation of the biased potential function and the flux using this biased function:

$$\begin{aligned}\Phi &= \mathcal{E} \times \int_A \int_{\Omega_x} S'(x,\Theta_x) W'_k(x,\Theta_x) cos\theta_x d\omega_x dx \\ W'_k(x,\Theta_x) &= \frac{g(x,\Theta_x)}{Imp(x,\Theta_x)} + \int_{\Omega_y} T'(\Theta_x \to \Theta_y) W'_k(y,\Theta_y) d\omega_y\end{aligned}$$

As earlier we can proceed to solve this quadrature by sampling S' followed by the simulation of random walks as discussed in section 4, provided the following conditions are satisfied.

1. $S'(x,\Theta_x)$ is normalised.
2. $\int_{\Omega_y} T'(\Theta_x \to \Theta_y) d\omega_y \leq 1$.

To satisfy the condition (1) we choose $\frac{W_{imp}(x,\Theta_x)}{\int_A \int_{\Omega_x} S(x,\Theta_x) W_{imp}(x,\Theta_x) cos\theta_x d\omega_x dx}$ for $Imp(x,\Theta_x)$ which trivially assures the normalisation of $S'(x,\Theta_x)$ as

$$\int_A \int_{\Omega_x} S'(x,\Theta_x) cos\theta_x d\omega_x dx = \int_A \int_{\Omega_x} S(x,\Theta_x) \frac{W_{imp}(x,\Theta_x)}{\int_A \int_{\Omega_x} S(x,\Theta_x) W_{imp}(x,\Theta_x) cos\theta_x d\omega_x dx} cos\theta_x d\omega_x dx = 1$$

With the above definition of $Imp(x,\Theta_x)$,

$$T'(\Theta_x \to \Theta_y) = T(\Theta_x \to \Theta_y) \frac{Imp(y,\Theta_y)}{Imp(x,\Theta_x)} = T(\Theta_x \to \Theta_y) \frac{W_{imp}(y,\Theta_y)}{W_{imp}(x,\Theta_x)}$$

Using eqn.2 we can write that

$$\begin{aligned}W_k(x,\Theta_x) &= g_k(x,\Theta_x) + \int_{\Omega_y} T(\Theta_x \to \Theta_y) W_k(y,\Theta_y) d\omega_y \\ W_k(x,\Theta_x) - g_k(x,\Theta_x) &= \int_{\Omega_y} T(\Theta_x \to \Theta_y) W_k(y,\Theta_y) d\omega_y\end{aligned}$$

Thus

$$\begin{aligned}\int_{\Omega_y} T'(\Theta_x \to \Theta_y) d\omega_y &= \int_{\Omega_y} T(\Theta_x \to \Theta_y) \frac{W_{imp}(y,\Theta_y)}{W_{imp}(x,\Theta_x)} d\omega_y \\ &= \frac{W_{imp}(x,\Theta_x) - g_{imp}(x,\Theta_x)}{W_{imp}(x,\Theta_x)} \\ &= 1 - \frac{g_{imp}(x,\Theta_x)}{W_{imp}(x,\Theta_x)} \leq 1\end{aligned}$$

Thus both the conditions are satisfied. Further it is established that the transformed transition probability function is subcritical and the absorption probability at any (x,Θ_x) is $\frac{g_{imp}(x,\Theta_x)}{W_{imp}(x,\Theta_x)}$. Thus the estimator with the biased random walk is:

$$\Phi_k = \mathcal{E} \times \frac{1}{n} \sum_{i=1}^{n} \sum_{k=0}^{m_i} \frac{g_k(x_{i_k},\Theta_{x_{i_k}})}{Imp(x_{i_k},\Theta_{x_{i_k}})} \tag{20}$$

Having shown the usefulness of an approximate potential function for the biasing of emission and transition function we proceed to show an implementation of this scheme.

6 Implementation and Results

Just for the sake of testing out the ideas of biased random walk solutions to the potential equations we have carried out a simple implementation supporting only source biasing by computing and using an approximate illumination potential as the biasing function. The results obtained are quite encouraging.

In the earlier section we have shown that the approximate illumination potential of every point in the environment towards any region of importance forms an useful biasing function for biasing the random walks to visit that region of interest more often than usual. Any computational method may be used to compute this approximate potential. In our implementation we have used unbiased particle tracing with a small number of samples to arrive at this approximate potential.

a) In one case we kept track of the total number of random walks leaving each patch (say $n1$) and the total number out of these (say $n2$) visiting the region of importance. Assuming uniform behaviour over the patch, the fraction $\frac{n2}{n1}$ actually gives an approximate potential for the points of the patch averaged over the hemisphere. Because of this averaging the directional nature is lost.

This approximate value could be used for source biasing. In source biasing the biased source is computed as $S' = S \times Imp$. The computation of Imp requires W_{imp} and another double integral term which we shall denote as $S0$. The evaluation of $S0$ will be carried out as follows:

$$\begin{aligned} S0 &= \int_A \int_{\Omega_x} S(x,\Theta) W_{imp}(x,\Theta) cos\theta_x d\omega_x dx \\ &= \sum_{i=1}^{Ns} \int_{A_i} S(x,\Theta_x) \int_{\Omega_x} W_{imp}(x,\Theta) cos\theta_x d\omega_x dx = \sum_{i=1}^{Ns} S_i \pi \mathcal{W}_{imp,i} A_i \\ &= \sum_{i=1}^{Ns} \pi S_i \mathcal{W}_{imp,i} A_i \end{aligned}$$

With this value of $S0$, strength of each of the patch is computed as: $S_i' = S_i \frac{\mathcal{W}_{imp,i}}{S0}$. This biased emission is again uniformly diffuse over the patch as no change in directional distribution of the emission strength has taken place in this biasing. However now the random walk originates more often than usual from those patches whose average potential is higher than that of others.

b) In a second case we captured the potential as a function of a range of directions. In this the direction range over each patch is uniformly discretised into a number of ranges (say Nd direction ranges). For convenience this discretisation has been carried out by uniform subdivision of the θ and ϕ space. The total number of walks leaving each patch in the j-th direction range (say $n1_j$) and the total number out of these (say $n2_j$) that visiting patches in the region of interest is recorded. The fraction $\frac{n2_j}{n1_j}$ gives an approximate average potential for the points of the patch in the j-th direction range where the averaging has taken place over the solid angle made by the j-th direction range with the patch. With this directional capture of potential, $S0$ may be evaluated as follows:

$$\begin{aligned} S0 &= \int_A \int_{\Omega_x} S(x,\Theta_x) W(x,\Theta_x) cos\theta_x d\omega_x dx \\ &= \sum_{i=1}^{Ns} \int_{A_i} \sum_{j=1}^{Nd} \int_{\omega_{ij}} S(x,\Theta_x) W(x,\Theta_x) cos\theta_x d\omega_x dx \end{aligned}$$

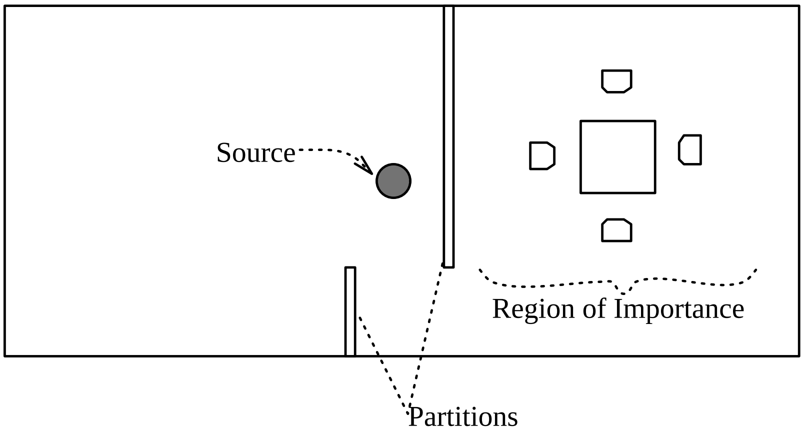

Figure 4: A Typical Scene for Testing the Biasing Scheme

$$
\begin{aligned}
&= \sum_{i=1}^{Ns} S_i \int_{A_i} \sum_{j=1}^{Nd} W_{ij} \int_{\omega_{ij}} cos\theta_x d\omega_x dx = \sum_{i=1}^{Ns} S_i \int_{A_i} \left(\sum_{j=1}^{Nd} W_{ij} D_{ij} \right) dx \\
&= \sum_{i=1}^{Ns} S_i \left(\sum_{j=1}^{Nd} W_{ij} D_{ij} \right) A_i
\end{aligned}
$$

where $D_{ij} = \int_{\omega_{ij}} cos\theta_x d\omega_x$ is independent of the patch and depends only on the θ and ϕ ranges and hence can be precomputed.

We carried out the biasing for a predefined region of importance in a typical environment where the region of importance has been chosen in such a way that no portion of the source is visible to this region and hence receives indirect illumination only (see figure 4). In order to compare the simulation results between biased and unbiased cases we have recorded the percentage of visits to the region of importance with respect to all walks originating from the source. Table 1 shows the results of the simulation.

Type of Biasing	% of Random Walk visits to the Region of Interest
Unbiased Simulation	18
Biased simulation: *Position Biasing*	30
Biased simulation: *Position* + *Direction* Biasing 8×8 hemisphere discretisation	45
Biased simulation: *Position* + *Direction* Biasing 8×32 hemisphere discretisation	50

7 Conclusion

Use of the approximate potential for biasing shows good promises in the computation of global illumination by random walks. Immediate extension of the current work will be to adaptively improve the particle tracing computation results of any environment by starting with the whole scene as the region of importance and then narrowing the region at intermediate stages to include only those surfaces whose results cannot be said to be free from error. The major problem in such a case will be finding a measure of the error in the estimation at each of the surface patches. Further use of better sampling techniques can make the method more efficient by directing more and more random walks to the regions of interest.

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