1 Introduction

Illumination in an environment is, in general, a complex function over space and directions. In an ILLumination computation method one attempts to evaluate this function as accurately as possible. Many of these illumination computation methods reconstruct the complex illumination function by computing a large number of piece-wise constant functions over the environment. The choice of piece-wise constant function comes from the fact that the solution methods using them are relatively simpler. Much recently we come across methods which attempt to reconstruct the illumination function by computing higher order piece-wise polynomials [1, 2, 3, 4, 5] over the environment. Computation of higher order functions are preferable because they make good approximations with lesser discretisation. However, the methods using them are highly complex for an implementation. Here, in this paper, we provide a much simpler algorithm for computation of higher order functions. The method is based on Monte Carlo/Random walk and may be seen as an extension to the existing particle tracing method [6, 7].

Monte Carlo based methods can be classified into a ray-tracing class and into a particle tracing class. Ray-tracing class of methods compute view dependent illumination by tracing rays from the eye point towards the environment. These methods are basically the extensions of distribution ray tracing [8] and/or path tracing [9]. The particle tracing class of methods compute view-independent illumination of the environment by tracing rays from the light source. The main implementations so far compute this view-independent illumination by estimating piecewise constant functions over meshed surfaces and reconstruct the illumination function of the environment from them.

Among the view-independent illumination computation methods, particle tracing is very simple both by concept and by implementation. This method may be seen as the simulation of the particle model of light. Light particles are emitted from the emitter surface and propagated in the environment. The illumination of any surface in the environment is estimated as the total number of particles leaving that surface as a result of a large number of particle propagations.

In spite of the inherent simplicity this method is not very widely used. It is because of the statistical noise arising in the estimates. At times the noise is so much that it made the images generated from the estimated illumination completely unacceptable. The early particle tracing method did not provide any mechanism to rectify this problem. Furthermore, as the method restricted itself to evaluate constant functions, to capture any non-constant illumination function one was forced to use very fine discretisation. The fineness of the discretisation gave rise to very fine surface patches and amplified the problem much more because of the increased uncertainty of any particle hitting these fine patches. Thus in order to make the particle tracing more acceptable one has to provide the capability of estimating higher order functions and some mechanism to reduce the noise.

In this paper we propose an improved particle tracing method which follows a two step approach :

1. **Global Illumination Step** which carries out particle tracing and directly estimates piecewise higher order llumination function over the surfaces.

2. **Post-Processing Step** which tries to reduce the statistical noise from the estimated result, so as to be able to reconstruct visually acceptable illumination functions over the surface.

In addition we propose an improved random walk estimator to improve the statistical performance of the general particle tracing method.
The most interesting part of most of the improvements is that they come with very little modification to the original particle tracing algorithm.

The organisation of this article is as follows. First we discuss the function approximation problem in general, then that of illumination function in particular. Then we introduce the potential equation which is an adjoint formulation of radiance equation. Because of the adjointness, it is possible to compute anything related to illumination either solving an equation involving radiance equation or involving potential equation. Using this potential equation we formulate a dual to the illumination function approximation. Then we formulate its Monte Carlo/Random Walk solution and draw its connection to the existing particle tracing method. Then we propose the use of *next event estimator* which improves the efficiency of the random walk estimator. We show some of the results obtained by the modified particle tracing algorithm. Finally we discuss a reprojection technique to reduce the statistical noise in the estimates.

2 Illumination Function Approximation

Any continuous function g can be approximated by using a set of trial functions $\{\mathcal{N}_m;\ m = 1, 2, \ldots\}$ as follows [10] :

$$g(x) \approx \hat{g}(x) = \sum_{m=1}^{M} a_m \mathcal{N}_m(x)$$

where a_m are *coefficients of approximation* which are computed in a manner such that the approximation is accurate. The trial function is also known as *shape* function (in CAD) or *basis* function (in functional analysis). The basic requirement of choosing the trial function set is that the approximation must improve as the number M of the trial functions increases. At the limit when $M \rightarrow \infty$, the approximation is an equality for any g.

A standard method of computing the approximation coefficients is the weighted minimisation method. This method generates a set of M linear equations with unknowns, a_m, by the following minimisation technique.

$$\int \mathcal{W}_l(x)\,(g(x) - \hat{g}(x))\,dx = 0 \qquad \text{where } l = 1 \ldots M \text{ and } \mathcal{W}_l(x) \text{ is some weight function.}$$

The resulting linear equation set is $\mathbf{Ka} = \mathbf{f}$, where

$$\mathbf{a} = (a_1, a_2, \ldots, a_M)^T, \qquad K_{l,m} = \int \mathcal{W}_l(x)\mathcal{N}_m(x)dx, \qquad f_l \;=\; \int \mathcal{W}_l(x)g(x)dx$$

Thus the computation of the coefficients require the evaluations of K's and f's and solving the linear system. The evaluation of the coefficients $K_{l,m}$ is independent of the function to be approximated. So these coefficients can be precomputed. The only quantities that are needed to be evaluated are the coefficients f_l's.

Among the various weight functions used, the choice of the trial function itself as the weight function (*i.e.* $\mathcal{W} = \mathcal{N}$), is most popular and is widely known as *Galerkin* method. Particularly, if the trial function are orthonormal or we use the biorthogonal complement, $\hat{\mathcal{N}}_l(x)$, of the trial function as the weight function then we have a very simple expression for the coefficients as:

$$a_l = f_l = \int \mathcal{W}_l(x)g(x)dx = \int \hat{\mathcal{N}}_l(x)g(x)dx.$$

In this latter case the need to solve the linear equation set to compute the coefficients is avoided. The various deterministic methods used for illumination function evaluation make use of Galerkin methods [2, 3, 4]. However, for the purpose of discussion in this article, we shall keep the generality and shall not make any assumption about the use of either Galerkin method or normalised basis function set and concentrate on the evaluation of f_l values.

We can extend the function approximation discussion given above to higher dimensional problems, particularly, to a function of positions $\bar{x}$ defined over a surface as:

$$g(\bar{x}) \approx \sum_{m=1}^{M} a_m \mathcal{N}_m(\bar{x})$$

For the evaluation of the coefficients a_m we will need K_{lm} and f_m which are now defined as:

$$K_{l,m} \;=\; \int_{\text{surface}} \mathcal{W}_l(\bar{x})\mathcal{N}_m(\bar{x})dA_{\bar{x}} \quad \text{and} \quad f_m = \int_{\text{surface}} g(\bar{x})\mathcal{W}_m(\bar{x})dA_{\bar{x}}$$

where $dA_{\bar{x}}$ is the differential area around the surface point $\bar{x}$.
Similarly we can write an approximation to a function of directions, Θ, defined over a hemisphere as

$$g(\Theta) \approx \sum_{m=1}^{M} a_m \mathcal{N}_m(\Theta) \quad \text{where } K_{l,m} = \int_\Omega \mathcal{W}_l(\Theta)\mathcal{N}_m(\Theta)d\omega \text{ and } f_m = \int_\Omega g(\Theta)\mathcal{W}_m(\Theta)d\omega$$

The domain of integration, Ω, in this latter case represents the hemisphere.
Applying these formulations to the fundamental illumination function, L, defined over the surfaces of the environment, we can write the approximation of the illumination function over a surface k as follows:

$$L^k(\bar{x}, \Theta_{\bar{x}}) \approx \sum_{m=1}^{M} a_m^k \mathcal{N}_m(\bar{x}, \Theta_{\bar{x}}) \tag{1}$$

where $\Theta_{\bar{x}}$ is the outgoing direction over the point $\bar{x}$ on the surface k, and

$$\begin{aligned} K_{l,m}^k &= \int_{\text{surface}_k} \int_{\Omega_{\bar{x}}} \mathcal{W}_l(\bar{x}, \Theta_{\bar{x}})\mathcal{N}_m(\bar{x}, \Theta_{\bar{x}}) d\omega dA_{\bar{x}} \\ \text{and} \quad f_m^k &= \int_{\text{surface}_k} \int_{\Omega_{\bar{x}}} L^k(\bar{x}, \Theta_{\bar{x}})\mathcal{W}_m(\bar{x}, \Theta_{\bar{x}}) d\omega dA_{\bar{x}} \end{aligned} \tag{2}$$

As said earlier, $K_{l,m}^k$ is independent of the illumination function and hence its evaluation does not pose a problem. The main problem now remains in the evaluation of f_m^k.

In the discussion above, all the functions and domain of surface integration were over the surface k. We can however generalise and extend the domain of integration to the whole environment simply by assuming that the weight functions and the basis functions are sets of global functions having compact support and the extents of these functions are at best limited to the surface k. This modification makes the basis functions and weight functions different for different surfaces, and to distinguish them we add the superscript k to each of them *i.e.* $\mathcal{W}_m^k$ and $\mathcal{N}_m^k$. The expression of f_m^k now takes the following form.

$$f_m^k = \int_{\text{Env}} \int_{\Omega_{\bar{x}}} L(\bar{x}, \Theta_{\bar{x}})\mathcal{W}_m^k(\bar{x}, \Theta_{\bar{x}}) d\omega dA_{\bar{x}} \tag{3}$$

Note that L is no more restricted to the surface k, and this has been indicated in this latter formulation by removing the superscript k from L^k.

It is possible to develop path tracing like Monte Carlo solution for evaluating f_m^k. However, it may not be advisable to do that because, in general there are large number of surfaces in the environment and each surface is likely to have a large number of basis function. So the large number of such individual f_m^k evaluations will make the process formidable. For this reason, we look for a dual formulation of equation 3 which can lead to a particle tracing like efficient Monte Carlo method [11]. Before we give the dual formulation, we discuss briefly about the adjointness which is the necessary mathematical tool for such formulation.

3 Adjoint Illumination Equation and Dual Formulation

3.1 Adjointness

In the above formulations we are dealing with radiance function. Radiance function in a non-participating environment is given by the following equation:

$$L(\bar{x}, \Theta_{\bar{x}}) = L^e(\bar{x}, \Theta_{\bar{x}}) + \int_{\Omega_{\bar{x},in}} f_r(\bar{x}, \Theta_{\bar{y}}, \Theta_{\bar{x}}) \cos\theta_{\bar{x},in} L(\bar{y}, \Theta_{\bar{y}}) d\omega_{in} \tag{4}$$

where $L(\bar{x}, \Theta_{\bar{x}})$ and $L^e(\bar{x}, \Theta_{\bar{x}})$, are respectively the total outgoing radiance and the emittance at x along the direction $\Theta_{\bar{x}}$. $\bar{y}$ is a point visible to $\bar{x}$ in its incoming hemisphere $\Omega_{\bar{x},in}$, $\Theta_{\bar{y}}$ is the outgoing direction at $\bar{y}$

Figure 1: Consequence of Adjointness.

leading to point $\bar{x}$, $f_r(\bar{x}, \Theta_{\bar{y}}, \Theta_{\bar{x}})$ is the bidirectional reflectance function (*brdf*) of the surface at point $\bar{x}$ and $\theta_{\bar{x},in}$ is the angle that the line joining $\bar{y}$ and $\bar{x}$ makes with the surface normal at $\bar{x}$.
This above equation can be abstracted into the form of integral equation of the second kind, *i.e.*

$$f(x) \quad = \quad a(x) + \int_a^b K(x,y) f(y) dy = a(x) + \kappa(f)(x) \tag{5}$$

where $\kappa(f)$ is the integral operator used as a compact notation of $\int_a^b K(x,y)f(y)dy$.
For every such operator κ there exists another operator κ^*, such that they are adjoint to each other. By adjoint one means that the operators, κ and κ^*, satisfy the condition $\langle \kappa(f), f^* \rangle \; = \; \langle \kappa^*(f^*), f \rangle$. Here the symbol $\langle a, b \rangle$ implies the inner product $\int a(x)b(x)dx$. One can write an integral equation corresponding to the operator κ^* as

$$f^*(x) = b(x) + \kappa^*(f^*)(x) = b(x) + \int_a^b K^*(x,y) f^*(y) dy \tag{6}$$

The equations 5 and 6 are said to be an adjoint pair of equations.

An important consequence to the adjointness [12] as shown in figure (1) is :

$$\int_a^b f(x)b(x)dx = \int_a^b f^*(x)a(x)dx = \mathcal{F} \tag{7}$$

This consequence tells us that $\mathcal{F}$ in equation 7 can be computed either by evaluating $\int_a^b f(x)b(x)dx$ or by evaluating $\int_a^b f^*(x)a(x)dx$. The possibility of having two different ways of solving a single problem involving integral equation makes the concept of adjointness popular in many complex transport processes.

It must be emphasised here that the adjointness is in the operator. Keeping the operator same we can write many different integral equations. So strictly speaking, the adjoint pair of equations is not unique. If one of the integral equations is kept constant, then we can find an infinity number of integral equations which can form adjoint pair with it, each one differing from the other in the shape of the function $b(x)$.

3.2 Potential Equations

For the radiance equation one can derive an adjoint equation. Potential equation is one such adjoint formulation. It is given as follows:

$$\mathcal{P}(\bar{x}, \Theta_{\bar{x}}) \quad = \quad g(\bar{x}, \Theta_{\bar{x}}) + \cos\theta_{\bar{x}} \int_{\Omega_{\bar{y}}} f_r(\bar{y}, \Theta_{\bar{x}}, \Theta_{\bar{y}}) \mathcal{P}(\bar{y}, \Theta_{\bar{y}}) d\omega \tag{8}$$

where $\mathcal{P}$ is the potential function. One must note the difference between the equations 4 and 8. Here the $\bar{y}$ is the point visible to $\bar{x}$ along an outgoing direction from $\bar{x}$ and the integration is over the outgoing hemisphere around $\bar{x}$. Whereas in the radiance equation (equation 4) $\bar{y}$ is the point visible to $\bar{x}$ along an incoming direction at $\bar{x}$, and the integration is over the incoming hemisphere around $\bar{x}$.
For the derivation of this equation, see appendix A and also [7, 11, 13]. Potential is a general concept and was introduced in [7, 11] to denote the potential characteristic of any $(\bar{x}, \Theta_{\bar{x}})$ towards the illumination of some other point or some region. The function $g(\bar{x}, \Theta_{\bar{x}})$ provides the exact meaning to the potential.
For example, when $g(\bar{x}, \Theta_{\bar{x}})$ is a delta function and given as

$$g(\bar{x}, \Theta_{\bar{x}}) = \begin{cases} \frac{1}{dA_{\bar{x}} d\omega_{\bar{x}}} & \text{iff } (\bar{x}, \Theta_{\bar{x}}) = (\bar{z}, \Theta_{\bar{z}}), \\ 0 & \text{otherwise,} \end{cases}$$

then $\mathcal{P}(\bar{x},\Theta_{\bar{x}})$ defines the potential capacity of $(\bar{x},\Theta_{\bar{x}})$ to contribute towards the radiance at $(\bar{z},\Theta_{\bar{z}})$. This can be verified using the consequence of adjointness (equation 7) :

$$L(\bar{z},\Theta_{\bar{z}}) = \int_{\text{Env}}\int_{\Omega_{\bar{x}}} L(\bar{x},\Theta_{\bar{x}})g(\bar{x},\Theta_{\bar{x}})d\omega dA_{\bar{x}} = \int_{\text{Env}}\int_{\Omega_{\bar{x}}} \mathcal{P}(\bar{x},\Theta_{\bar{x}})L^e(\bar{x},\Theta_{\bar{x}})d\omega dA_{\bar{x}} \tag{9}$$

In another example: when $g(.,.)$ defined as follows:

$$g(\bar{x},\Theta_{\bar{x}}) = \begin{cases} \cos\theta_{\bar{x}} & \text{iff } \bar{x}\in \text{Surface}_k, \\ 0 & \text{otherwise,} \end{cases}$$

then $\mathcal{P}(\bar{x},\Theta_{\bar{x}})$ defines the potential capacity of $(\bar{x},\Theta_{\bar{x}})$ to contribute towards the total outgoing flux of the surface k in the environment. This can be verified from the equation below.

$$\int_{\text{Env}}\int_{\Omega_{\bar{x}}} L(\bar{x},\Theta_{\bar{x}})g(\bar{x},\Theta_{\bar{x}})d\omega dA_{\bar{x}} = \Phi_k = \int_{\text{Env}}\int_{\Omega_{\bar{x}}} \mathcal{P}(\bar{x},\Theta_{\bar{x}})L^e(\bar{x},\Theta_{\bar{x}})d\omega dA_{\bar{x}}$$

Note that this expression above is exactly same as that in equation 9. Only the integration result is different. It is because, the function $g(.,.)$ is different in both cases. That changes the meaning of $\mathcal{P}(.,.)$ and consequently the integration result changes.
We can sum up the discussion in this section by saying that :
• one can express any illumination related quantity using not only the radiance equation but also using potential equation which is adjoint to the radiance equation;
• though the concept *potential* is not unique, it is still given by the unique expression, equation 8, with the appropriate definition of $g(,)$.

3.3 Dual Formulation

We shall make use of the concept developed above (equation 7 and figure 1), to the adjoint equation pair equations 4 and 8 and give a dual representation of the expression of f^k_m (given in equation 3) as

$$\begin{aligned} f^k_m &= \int_{\text{Env}}\int_{\Omega_{\bar{x}}} L(\bar{x},\Theta_{\bar{x}})\mathcal{W}^k_m(\bar{x},\Theta_{\bar{x}})d\omega dA_{\bar{x}} \\ &= \int_{\text{Env}}\int_{\Omega_{\bar{x}}} \mathcal{P}^{(k,m)}(\bar{x},\Theta_{\bar{x}})L^e(\bar{x},\Theta_{\bar{x}})d\omega dA_{\bar{x}} \end{aligned} \tag{10}$$

where

$$\mathcal{P}^{(k,m)}(\bar{x},\Theta_{\bar{x}}) = \mathcal{W}^k_m(\bar{x},\Theta_{\bar{x}}) + \cos\theta_{\bar{x}}\int_{\Omega_{\bar{y}}} f_r(\bar{y},\Theta_{\bar{x}},\Theta_{\bar{y}})\mathcal{P}^{(k,m)}(\bar{y},\Theta_{\bar{y}})d\omega \tag{11}$$

is the necessary integral equation which is adjoint to the radiance equation given in equation 4, *i.e.*

$$L(\bar{x},\Theta_{\bar{x}}) = L^e(\bar{x},\Theta_{\bar{x}}) + \int_{\Omega_{\bar{x},in}} f_r(\bar{x},\Theta_{\bar{y}},\Theta_{\bar{x}})\cos\theta_{\bar{x},in}L(\bar{y},\Theta_{\bar{y}})d\omega_{in}$$

and $\mathcal{W}^k_m(\bar{x},\Theta_{\bar{x}})$ is the weight function used for the error minimisation in the radiance function approximation. It must be noted here that the potential function $\mathcal{P}^{(k,m)}$ is neither local to surface k nor has any compact support. The superscript (k,m) in this function indicates that it is a potential of any $(\bar{x},\Theta_{\bar{x}})$ towards the m-th basis function of the k-th surface.
Now we can estimate f^k_m by solving the equation pair (10,11) and then solve the linear equation set $\mathbf{K}^k\mathbf{a}^k = \mathbf{f}^k$ to derive the approximation to the illumination function given in equation 1.

The complexity of the equations involved indicate that one must resort to some numerical solution method. As said earlier, we will resort to a Monte Carlo method for the solution. First we shall briefly introduce the general Monte Carlo/Random walk methods for solving the type of equations discussed above. then we shall proceed to discuss the particular method of estimating f^k_m.

4 Monte Carlo Quadrature and Random Walk

The Monte Carlo quadrature for estimating

$$G = \int_a^b p(s)g(s)ds \qquad \text{where } p(s) \text{ is a } pdf \ i.e. \int_a^b p(s)ds = 1 \tag{12}$$

in its simplest form is : Generate n samples according to the pdf $p()$ and for each sample s_i evaluate $g(s_i)$, then estimate G as

$$G \approx \frac{1}{n}\sum_{i=1}^{n} g(s_i)$$

Here we assume that $g(s)$ is a known function and can be evaluated for a given sample point s_i. However $g(s)$ could be an unknown function. In particular $g(s)$ could be the solution of an integral equation like :

$$g(s) = h(s) + \int_a^b T(s,t)g(t)dt \ \text{ where } \int_a^b T(s,t)dt < 1. \tag{13}$$

Then also we can use the above estimate, with the modification that instead of the exact value of $g(s_i)$ we shall use its estimated value. One of the methods of getting the estimated value of $g(s_i)$ is the random walk method [14], using which the estimating equation is

$$g(s_i) = h(s_i) + \sum_{j=1}^{m_i} h(s_{i,j})$$

where for each value of s_i we generate a m_i number of samples $s_{i,j}$ by sampling from the distribution function $T(s_{i,j-1}, t)$, the very first one of which, *i.e.* $s_{i,1}$, is drawn by sampling the distribution function $T(s,t)$. Note that the function $T(s_{i,j-1}, t)$ is not a strict probability distribution function, because $\int_a^b T(s_{i,j-1}, t)dt < 1$. The sampling from such a distribution function is guaranteed to terminate after a finite number of draws of samples. It is carried out in two steps. First a binary choice between *continuation* or *termination* is made by sampling a binary probability distribution $[\rho, 1-\rho]$ where $\rho = \int_a^b T(s_{i,j-1}, t)dt$. If the sampling indicates *continuation* then only, one sample, $s_{i,j}$, is generated from the actual probability distribution function $T(s_{i,j-1}, t)/\rho$.

Thus to sum up : the estimate of an integration (equation 12) containing a function defined by an integral equation (equation 13) is

$$G \approx \frac{1}{n}\sum_{i=1}^{n}\left[\sum_{j=0}^{m_i} h(s_{i,j})\right]. \tag{14}$$

This estimation is known as *random walk estimation* and $h(s_{i,j})$'s are known as random walk estimators.

5 Illumination Function Estimation

We shall now proceed to estimate f_m^k by applying the technique described above to equation pair 10 and 11. Before doing that we shall rewrite these equations in such a way that the sampling functions satisfy the requirements dictated by the equations 12 and 13. First we look at the equation 11. Here $f_r()$, the *brdf*, is a known function and can be used for sampling directions. From the definition of *brdf* it is known that

$$\int_\Omega f_r(y,.,.)\cos\theta d\omega = \rho_y < 1 \qquad \text{where } \rho_y \text{ is the reflection coefficient at } y.$$

As we need a cosine term with f_r, we can rewrite the equation 11 to bring in the cosine term along with $f_r()$ inside the integral as follows:

$$\begin{aligned}
\frac{\mathcal{P}^{(k,m)}(\bar{x},\Theta_{\bar{x}})}{\cos\theta_{\bar{x}}} &= \frac{\mathcal{W}_m^k(\bar{x},\Theta_{\bar{x}})}{\cos\theta_{\bar{x}}} + \int_{\Omega_{\bar{y}}} f_r(\bar{y},\Theta_{\bar{x}},\Theta_{\bar{y}})\cos\theta_{\bar{y}}\frac{\mathcal{P}^{(k,m)}(\bar{y},\Theta_{\bar{y}})}{\cos\theta_{\bar{y}}}d\omega \qquad \text{or} \\
\mathcal{P}'^{(k,m)}(\bar{x},\Theta_{\bar{x}}) &= \frac{\mathcal{W}_m^k(\bar{x},\Theta_{\bar{x}})}{\cos\theta_{\bar{x}}} + \int_{\Omega_{\bar{y}}} f_r(\bar{y},\Theta_{\bar{x}},\Theta_{\bar{y}})\cos\theta_{\bar{y}}\mathcal{P}'^{(k,m)}(\bar{y},\Theta_{\bar{y}})d\omega \qquad (15) \\
& \text{where } \mathcal{P}'^{(k,m)}(\bar{x},\Theta_{\bar{x}}) = \frac{\mathcal{P}^{(k,m)}(\bar{x},\Theta_{\bar{x}})}{\cos\theta_{\bar{x}}}.
\end{aligned}$$

Now we modify[1] equation 10 to accommodate $\mathcal{P}'^{(k,m)}(x,\Theta_x)$ as:

$$f_m^k = \int_{\text{Env}}\int_{\Omega_{\bar{x}}} \mathcal{P}'^{(k,m)}(\bar{x},\Theta_{\bar{x}})\cos\theta_{\bar{x}}L^e(\bar{x},\Theta_{\bar{x}})d\omega dA_{\bar{x}} \qquad (16)$$

In this equation, $L^e(\bar{x},\Theta_{\bar{x}})\cos\theta_{\bar{x}}$ is also a known function and can be normalised and used for sampling the position and direction to start each random walk.

The emission function, L^e, is non-zero only over the emitting surfaces and in general the emitting surfaces make a very small fraction of the total surfaces. So we shall write equation 16 as a sum of integrals as follows:

$$\begin{aligned}
f_m^k &= \sum_{l=1}^{N_{src}} \int_{\text{emitter}_l}\int_{\Omega_{\bar{x}}} \mathcal{P}'^{(k,m)}(\bar{x},\Theta_{\bar{x}})\cos\theta_{\bar{x}}L_l^e(\bar{x},\Theta_{\bar{x}})d\omega dA_{\bar{x}} \\
&= \sum_{l=1}^{N_{src}} \Phi_l \int_{\text{emitter}_l}\int_{\Omega_{\bar{x}}} \mathcal{P}'^{(k,m)}(\bar{x},\Theta_{\bar{x}})\frac{L_l^e(\bar{x},\Theta_{\bar{x}})\cos\theta_{\bar{x}}}{\Phi_l}d\omega dA_{\bar{x}} \\
&\qquad \text{where } \Phi_l = \int_{\text{emitter}_l}\int_{\Omega_{\bar{x}}} L_l^e(\bar{x},\Theta_{\bar{x}})\cos\theta_{\bar{x}}d\omega dA_{\bar{x}} \\
&= \Phi_{\text{Env}}\sum_{l=1}^{N_{src}} \frac{\Phi_l}{\Phi_{\text{Env}}} \int_{\text{emitter}_l}\int_{\Omega_{\bar{x}}} \mathcal{P}'^{(k,m)}(\bar{x},\Theta_{\bar{x}})\frac{L_l^e(\bar{x},\Theta_{\bar{x}})\cos\theta_{\bar{x}}}{\Phi_l}d\omega dA_{\bar{x}} \qquad (17)
\end{aligned}$$

N_{src}, in the above equations is the number of light sources, L_l^e is the radiance function due to the self emission of the source l, Φ_l is the total emission power of source l and $\Phi_{\text{Env}} = \sum_l \Phi_l$ is the total emission power of the whole environment.

Applying random walk estimation technique as given in equation 14, to the equation pair 17 and 15 we can estimate f_m^k as

$$f_m^k \approx \frac{\Phi_{\text{Env}}}{n}\sum_{i=1}^{n}\left[\sum_{j=0}^{m_i}\frac{\mathcal{W}_m^k(\bar{x}_{i,j},\Theta_{\bar{x}_{i,j}})}{\cos\theta_{\bar{x}_{i,j}}}\right] \qquad (18)$$

where the random samples $(\bar{x}_{i,0},\Theta_{\bar{x}_{i,0}})$ are generated by : first randomly deciding on a light source l by sampling discrete probability distribution

$$\left\{\frac{\Phi_1}{\Phi_{\text{Env}}},\ldots,\frac{\Phi_{N_{src}}}{\Phi_{\text{Env}}}\right\} \qquad (19)$$

and then by drawing samples $(\bar{x}_{i,0},\Theta_{\bar{x}_{i,0}})$ from the selected light source l by sampling the continuous distribution function $\frac{L_l^e(\bar{x},\Theta_{\bar{x}})\cos\theta_{\bar{x}}}{\Phi_l}$. For each i, sample $(\bar{x}_{i,j},\Theta_{\bar{x}_{i,j}})$ is generated from the previous sample

[1] Note that [7, 11] used the equation 15 as the expression for potential equation.

$(\bar{x}_{i,j-1}, \Theta_{\bar{x}_{i,j-1}})$ by equating the nearest surface point $\bar{y}$, from $\bar{x}_{i,j-1}$ along the direction $\Theta_{\bar{x}_{i,j-1}}$, to $\bar{x}_{i,j}$ and sampling $f_r(\bar{y}, \Theta_{\bar{x}_{i,j-1}}, \Theta_{\bar{y}})$ for getting the direction $\Theta_{\bar{x}_{i,j}}$.

Thus we have derived a particle tracing algorithm for estimating f_m^k of a illumination function belonging to a surface k. We initialise f_m^k to 0, initiate a random walk from a random position over an emitting surface, along a random direction and continue the walk by drawing random directions from the surfaces along the path. If the surface encountered along the path is k then increment its f_m^k estimate by $\frac{\mathcal{N}_m^k(\bar{x}_{i,j}, \Theta_{\bar{x}_{i,j}})}{\cos\theta_{\bar{x}_{i,j}}}$. If we carry out a large number of such random walks we will arrive at a good estimate for f_m^k.
The important aspect of this technique is that at each time a random walk originates by sampling the emission function and continues its walk by sampling the reflection function. Thus the walk at no stage has any preference towards any surface. So **the same random walk can be used for the estimation of all f_m^k's corresponding to the illumination functions over all the surfaces, and thus for the estimation of the approximation of the illumination function over all surfaces**. Figure 2 shows the resulting algorithm.

At this point it may be worthwhile to recall the *particle tracing* technique described in [6, 7]. It followed an almost similar procedure and estimated the total flux over every surface. The difference is only in the final estimation strategy, *i.e.* the action taken as each particle left a surface. In the earlier method the flux estimate associated with surface was incremented, whereas in the new method the estimates of f_m^k's associated with the surface are incremented and the increment requires the evaluation of a function at the point from where the particle leaves the surface. Thus we can look at the current method as an *improved particle tracing* technique. We have actually generalised the particle tracing method so as to be able to choose any basis function and weight function to better approximate the illumination function. The full mechanism of the existing particle tracing algorithm remains unchanged. We reemphasise here that, unlike the early particle tracing algorithm, in the modified particle tracing algorithm, the increment is not 1, but is related to evaluated value of the weight function. Also the number of increments per each particle leaving a surface is upto M where M is the number of basis function chosen for approximation of the illumination function (equation 1). This increase in computation time must be taken into account while choosing the number of basis functions.

6 Improved Random Walk Estimation

A general guideline for a random-walk based methods is: the more a walk contributes towards the estimated result better is the performance. Some of the proven methods are : *next event estimation*, *absorption supression* and *importance sampling*. In this section we discuss only the next event estimation. More detail discussion on the other two can be found in [11]. Next event estimation changes the estimator without causing any change to the random walk.

In equation 14 of section 4 we describe the simplest of the random walk estimator. Section 5 used this estimator to estimate the coefficients f_m^k. We shall propose here an improved estimator [11] known as *next event estimator*, which uses $h'(s_{i,j})$ in the place of $h(s_{i,j})$ of equation 14. $h'(s_{i,j})$ in this case being

$$h'(s_{i,j}) = \int_a^b T(s_{i,j}, t)h(t)dt.$$

This estimator can be derived easily by an one level expansion of the integral equation 13 as follows:

$$\begin{aligned} g(s) &= h(s) + \int_a^b T(s,t)g(t)dt = h(s) + \int_a^b T(s,t)\left[h(t) + \int_a^b T(t,u)g(u)du\right]dt \\ &= h(s) + \int_a^b T(s,t)h(t)dt + \int_a^b T(s,t)\int_a^b T(t,u)g(u)dudt \\ &= h(s) + h'(s) + \int_a^b T(s,t)g'(t)dt \end{aligned}$$

main
begin
- Initialise;
- Normal Particle Trace;
- Compute Coefficients;

end main.

Initialise
begin
- For each surface k initialise its f_m^k's to zero.
- Compute the discrete power distribution (equation 19).

end Initialise;

Normal Particle Trace
begin
- do
  - /*...Emit a particle ...*/
  - Choose an emitter by sampling the discrete power distribution.
  - Choose the position and direction $(\bar{x}_{i,0}, \Theta_{\bar{x}_{i,0}})$ for a particle on the emitter surface by sampling its emitted power distribution function.
  - Increment all f_m^k's corresponding to the emitting surface k by $\frac{\mathcal{W}_m^k(\bar{x}_{i,0},\Theta_{\bar{x}_{i,0}})}{\cos\theta_{\bar{x}_{i,0}}}$.
  - /*...Track the particle until the particle is absorbed.../*.
  - do
    - Find the nearest surface k and the surface point y along the particle path.
    - Choose the type of interaction i.e. *absorption* or *reflection*, by sampling the discrete interaction distribution function $\{\rho_{\bar{y}}, 1-\rho_{\bar{y}}\}$ where $\rho_{\bar{y}}$ is the reflection coefficient at $\bar{y}$.
    - If the interaction is *reflection*
      - Set $\bar{x}_{i,j} = y$.
      - Assign a reflected direction, $\Theta_{\bar{x}_{i,j}}$ to the particle by sampling $\frac{f_r \cos\theta_{\bar{y}}}{\rho_{\bar{y}}}$.
      - Increment all f_m^k terms corresponding to the reflecting surface k by $\frac{\mathcal{W}_m^k(\bar{x}_{i,j},\Theta_{\bar{x}_{i,j}})}{\cos\theta_{\bar{x}_{i,j}}}$.
  - while(interaction is not *absorption*);
- until(a large number of particles are emitted);

end Normal Particle Trace;

Compute Coefficients
begin
- For each surface k do
  - Evaluate all $K_{l,m}^k$'s and compute $\left(\mathbf{K}^k\right)^{-1}$.
  - Evaluate $\mathbf{a}^k$, the coefficients of illumination function from computed $K_{l,m}^k$'s and estimated f_m^k's as $\mathbf{a}^k = \frac{\Phi_{\mathrm{Env}}}{n}\left(\mathbf{K}^k\right)^{-1}\mathbf{f}^k$, where n is the total number of emitted particles.
- end for;

end Compute Coefficients;

Figure 2: Normal Particle Tracing Algorithm with higher order function estimation.

$$\text{where } g'(t) = \int_a^b T(t,u)g(u)du \text{ and } h'(s) = \int_a^b T(s,t)h(t)dt$$

$$\begin{aligned} &= h(s) + g'(s) \\ g'(s) &= h'(s) + \int_a^b T(s,t)g'(t)dt \end{aligned}$$

To this modified version of the integral equation, we can apply random walk estimator (equation 14) to arrive at the new estimate of G as

$$G \approx \frac{1}{n}\sum_{i=1}^{n}\left[h(s_{i,0}) + \sum_{j=0}^{m_i} h'(s_{i,j})\right]$$

This is an improved estimator, because at each step of the walk it contributes towards the estimate an average quantity that would result due to all possible next steps from the current step. Using such estimator, the estimate of f_m^k will be :

$$f_m^k \approx \frac{\Phi_{\text{Env}}}{n}\sum_{i=1}^{n}\left[\frac{\mathcal{W}_m^k(\bar{x}_{i,0}, \Theta_{\bar{x}_{i,0}})}{\cos\theta_{\bar{x}_{i,0}}} + \sum_{j=0}^{m_i}\int_{\Omega_{\bar{y}}} f_r(\bar{y}, \Theta_{\bar{x}_{i,j}}, \Theta_{\bar{y}})\cos\theta_{\bar{y}}\frac{\mathcal{W}_m^k(\bar{y}, \Theta_{\bar{y}})}{\cos\theta_{\bar{y}}}d\omega\right] \tag{20}$$

The physical interpretation of this estimation strategy is that now for each incident particle there is a contribution towards the estimated value. In the earlier strategy the particle contributed to the estimated value only if underwent a non-absorption event, whereas in the current technique the contribution is independent of the absorption. Accommodating this technique requires a little modification to the above algorithm and the modified algorithm is as follows: This algorithm differs from that shown in figure 2 on when the update takes place. In earlier case the update was taking place only when the particle is reflected, whereas in the later case update takes place independent of the absorption or reflection of the particle. Also instead of a simple increment, now we have an integration result as the increment. All the functions in the integration are known. So, in principle one can compute this increment. Also, note that this increment is being carried out for every particle incident, independent of whether it is absorbed or reflected. Thus every particle incident on the surface makes a contribution and hence is preferable. In general this improvement comes at an additional cost of evaluation of an integral. However, we shall see in the next section that, in the special case, where one is estimating the surface radiosity function, the integral computation is simplified to an evaluation of a function and hence does not involve any extra cost.

7 Implementation

In the above sections we have given algorithms for general illumination function estimation. We can adapt it to any particular case. In this section we shall adapt it to the evaluation of the radiosity function. We shall first compute the radiance functions by approximating them with direction independent basis functions. Then we shall derive the radiosity function by using the simple relationship between radiosity(B) and radiance(L) for diffuse surfaces, $B = \pi L$.

7.1 Basis Functions

So far in the discussions we have made no restrictions on the basis function and the weight function used. Because the algorithm evaluates the weight function for each particle hit, it is evident that we cannot use a delta function for the weight function as done in point collocation methods. Apart from this any combination of weight and basis function known can be used. We have carried out implementation of Galerkin error minimisation technique with various basis functions. Here we show the results of the implementation using

Next Event Particle Trace

begin

- do

  /*... Emit a particle ... */

  - Choose an emitter by sampling the discrete power distribution.
  - Choose the position and direction $(\bar{x}_{i,0}, \Theta_{\bar{x}_{i,0}})$ for a particle on the emitter surface by sampling its emitted power distribution function.
  - Increment all f_m^k's corresponding to the emitting surface k by $\frac{\mathcal{W}_m^k(\bar{x}_{i,0}, \Theta_{\bar{x}_{i,0}})}{\cos\theta_{\bar{x}_{i,0}}}$.

  /*... Track the particle until the particle is absorbed... /*.

  - do
    - Find the nearest surface k and the surface point $\bar{x}_{i,j}$ along the particle path.
    - Increment all f_l^k terms corresponding to the surface k by $\int_{\Omega_{\bar{y}}} f_r(\bar{y}, \Theta_{\bar{x}_{i,(j-1)}}, \Theta_{\bar{y}})\mathcal{W}_m^k(\bar{y}, \Theta_{\bar{y}})d\omega$ where y is the nearest surface from $\bar{x}_{i,j}$ along $\Theta_{\bar{y}}$.
    - Choose the type of interaction i.e. *absorption* or *reflection*, by sampling the discrete interaction distribution function $\{\rho_{\bar{y}}, 1-\rho_{\bar{y}}\}$ where $\rho_{\bar{y}}$ is the reflection coefficient at $\bar{y}$.
    - If the interaction is *reflection*

      Assign a reflected direction, $\Theta_{\bar{x}_{i,j}}$ to the particle by sampling $\frac{f_r()\cos\theta_{\bar{y}}}{\rho_{\bar{y}}}$.

  while (interaction is not *absorption*);

  until (a large number of particles are emitted);

end Normal Particle Trace;

Figure 3: Algorithm with Next Event Random Walk Estimator

orthogonal scaling functions of the multi-wavelet basis. We have used the simplest construction for generating direction independent basis functions for the radiance function defined over a biparametric surface from 1D multi-wavelet basis functions as follows:

$$\mathcal{W}(\bar{x},\Theta_{\bar{x}}) = \mathcal{N}(\bar{x},\Theta_{\bar{x}}) = \mathcal{N}(\bar{x}) = \mathcal{S}(u)\mathcal{R}(v)$$

where u, v are the parameters corresponding to the surface point $\bar{x}$, $\mathcal{S}$ and $\mathcal{R}$ are respectively the 1D multiwavelet functions of u and v.
Using such functions, the increments for the algorithm in figure 2 becomes:

$$\frac{\mathcal{N}(\bar{x},\Theta_{\bar{x}})}{\cos\theta_{\bar{x}}} = \frac{\mathcal{S}(u)\mathcal{R}(v)}{\cos\theta_{u,v}}$$

and using the fact that for diffuse surfaces $f_r(\bar{y},\Theta_{\bar{x}},\Theta_{\bar{y}}) = f_r(\bar{y}) = \rho(\bar{y})/\pi$ where ρ is the surface reflectivity, the increments for the algorithm in figure 3 becomes

$$\begin{aligned}\int_{\Omega_{\bar{y}}} f_r(\bar{y},\Theta_{\bar{x}},\Theta_{\bar{y}})\mathcal{N}_m^k(\bar{y},\Theta_{\bar{y}})d\omega &= \frac{\rho(s,t)}{\pi}\mathcal{S}(s)\mathcal{R}(t)\int_{\Omega_{s,t}} d\omega \\ &= \frac{\rho(s,t)}{\pi}\mathcal{S}(s)\mathcal{R}(t)2\pi = 2\rho(s,t)\mathcal{S}(s)\mathcal{R}(t)\end{aligned}$$

where s, t are the surface parameters corresponding to the point y.
We see from the above that the next event estimation algorithm for radiosity function estimation is quite simple. For every particle hit at a surface k, the estimates of f_m^k of the surface are incremented by twice the reflectivity times the basis function evaluated at that point.

7.2 Evaluation of $K_{l,m}^k$

Because of the orthogonality of the multi-wavelet basis functions, $K_{l,m}^k$'s vanish for $l \neq m$. It leaves us to compute only $K_{m,m}^k$ values[2].

$$\begin{aligned}K_{m,m}^k &= \int_{\text{surface}_k}\int_{\Omega_x} N_m^k(\bar{x},\Theta_{\bar{x}})N_m^k(\bar{x},\Theta_{\bar{x}})d\omega dA_{\bar{x}} \\ &= \int_{\text{surface}_k} N_m^k(\bar{x},\Theta_{\bar{x}})N_m^k(\bar{x},\Theta_{\bar{x}})\left[\int_{\Omega_x} d\omega\right] dA_{\bar{x}} \\ &= 2\pi\int_{\text{surface}_k} N_m^k(\bar{x},\Theta_{\bar{x}})N_m^k(\bar{x},\Theta_{\bar{x}})dA_{\bar{x}} \\ &= 2\pi\int_0^1\int_0^1 \mathcal{S}_m(u)\mathcal{R}_m(v)\mathcal{S}_m(u)\mathcal{R}_m(v)\left\|\frac{\delta\bar{x}}{\delta u}\times\frac{\delta\bar{x}}{\delta v}\right\| du\, dv\end{aligned}$$

Because of the polynomial nature of the multi-wavelet basis functions it is possible to find a closed form evaluation of this integral when the parametric area is defined as a simple polynomial function *s.a.* spline. Otherwise some numerical quadrature technique must be used for the evaluation. However, these values can always be computed offline, so, are not too much of a botheration. It only suffices here to say that one has to make the appropriate variable transformation in the equation before applying the quadrature.

In figures 4 and 5 we show the application results in a simple room. Figure 4 shows the side view of a room. The illumination function of all the surfaces of this room has been captured using lowest resolution MultiWavlet basis functions with 4 vanishing moments. Figure 5 shows the same room with a few furnitures at the centre. The resolution of the basis functions for the floor has been increased by 1. The shadow below the furnitures have been very crudely captured.

[2]It must be noted that in Multi-wavelet basis each of these coefficients $K_{m,m}^k$ represents a matrix of $M^2 \times M^2$ terms, where M is the number of vanishing moments of the basis function. So in fact, one would be computing $M^2 \times M^2$ terms of the type $K_{m,m}^k[i,j] = \int_{\text{surface}_k}\int_{\Omega_x} N_m^k[i](\bar{x},\Theta_{\bar{x}})N_m^k[j](\bar{x},\Theta_{\bar{x}})d\omega dA_{\bar{x}}$.

8 Function Refinement

In particle tracing, we are using a nondeterministic method for estimating values related to the illumination function. Any estimation based on a nondeterministic method is subject to a random noise. The only remedy to this problem is to make sure that the estimate be derived from a large number of samples. In our case, it amounts to saying that the estimates of f_m^k be based on a large number of particle incidences. The problem we address here is, if the estimates are based on a smaller number of particle incidences, other than saying that they are not acceptable, what better can be done. This particular problem comes, when one has carried out a particle tracing operation within a prespecified time constraint. At the end of such operation we will certainly find surfaces whose f_m^k estimates are based on only smaller number of particle incidences (If time were not a constraint one could continue tracing more particles till have any such offending surface.). The illumination function derived using these noisy estimates are very likely to be incorrect. The end goal of most of the illumination computation method is image creation. Thus the images created using these incorrect illumination functions are bound to be unacceptable. If we have to derive visually acceptable images with the estimates at hand, we must carry out some post-processing operation. We propose below some such technique based on reprojection. Before we proceed we must say that the concept of largeness is not predefined, *i.e.* there does not exist a unique number which can be said to be large. It depends on what one is estimating and how much variation one can tolerate. In the particular case of our particle tracing operation, the estimates of f_m^k's are based on the multiwavelet basis functions. Multi-wavelet is a class of basis functions which differ from each other depending on the number of their vanishing moment. If we are using a set with M as the vanishing moment that means we are approximating our illumination function with piece-wise polynomials of degree upto $M-1$. In the context of statistical error one must understand that a sample number which may be considered large the use of multi-wavelet with $M=1$ is very likely to be not large for the use of multi-wavelet with $M=4$.

8.1 Reprojection

Reprojection is a general technique, which involves deriving an approximation to a function using a given set of basis functions from the approximation already obtained using another set of basis functions. This is easier to achieve, because we already have one approximation of the function in hand. All that remains to it is choosing the new basis function set. Using scaling functions of multi-wavelet basis of vanishing moment M we have two choices in hand for carrying out reprojection.

1. Generate the estimates with lesser M from the already obtained estimates. This is the simplest of all. Carrying out this reprojection amounts to saying that at a lesser M, the number of samples on which the estimate is based could be considered large. With this operation we can go upto the limit $M=1$. Even if at this vanishing moment the number of samples are not significant then we have not achieved much. In fact we are back to the same problem as we had with the early particle tracing technique.

2. Carry out a method similar to that used in Mallat's hierarchical signal decomposition [15]. Hierarchical signal decomposition technique carries out reprojections of the signal to lower resolution from the approximation of the signal at a higher resolution. However, the reprojection is localised to the region of low frequency, leaving the high frequency regions unaltered. This operation is carried out recursively till no more such reprojection is possible. Using multi-wavelet basis at sufficiently high resolution we have f_m^kestimates over a surface k which can be seen as a signal at higher resolution. Carrying out similar hierarchical operation means we reproject the regions with statistical noise so that the noise is smoothed out and at the same time the region with noise-free information and even discontinuities remain unaltered. This technique applies as well to the early particle tracing algorithm.

To demonstrate the usefulness of the proposed technique, we took the scene of the figure 5 and attempted to capture the shadow properly. The illumination function was estimated using the multi-wavelet coefficients with vanishing moment 4 at a very fine resolution (2^6). Figure 6 shows how well the illumination complexity of the floor has been captured. At the same time it shows the extreme statistical noise in the estimated results. Figure 7 shows the result after a combination of above reprojection techniques. The resulting illumination function is much better. However, still there are some ugly artifacts remaining. These are due to the lack of continuity along the border of the multi-wavelet function. We are trying to see if we can further reproject using a set of basis functions which have continuity along the boundary, for example splines which will remove this problem.

9 Importance Driven Particle Tracing

In an accompanied paper [16], we have provided a further development, the importance driven particle tracing algorithm, which provides an efficient method of view dependent illumination computation. The efficiency is derived by bringing changes to the particle emission pattern.

In the algorithms discussed so far, we have generated particles by sampling the light source according to their emitting strength. Thus, if we have a number of light sources with equal emission power, then statistically the same number of particles will be emitted from each of the light sources. Generating particles in this fashion is essential so as to estimate the illumination of the whole environment. However, when we are interested in computing illumination for a subset of the regions of the total environment, we can do something better than the above. One of the possibility is: we can find out the importance $\mathcal{I}$ associated with each light source towards the region of interest and use this $\mathcal{I}$ to modify the emission probability of the light sources. The change in the natural emission probability must be compensated in order to have a proper illumination estimation. Also to make the algorithm attractive one must find a very fast method of estimating $\mathcal{I}$. In [16] we have provided a very efficient algorithm to carry out all this.

10 Discussion

Having provided a number of improvements to the particle tracing technique to make it more attractive we shall now discuss some of the problems still remaining. The one problem is that the stochastic uncertainty still remains with surface elements which do not even have enough number of outgoing particles for reasonable estimation of the lowest order illumination function (*i.e.* constant illumination over the whole surface). For such surfaces one can carry out a one level deterministic/stochastic gathering from the bright and already estimated surfaces. This one level gathering can be justified by saying that the surfaces are either small or dark enough to have any important illumination contribution over the environment and to have a large variation in their illumination distribution.

Further more, though we are able to capture the discontinuity in the illumination function to some extent, it may not be always satisfactory. The captured discontinuity will depend very much on the resolution of the basis used. The higher resolution results in a memory overhead. Also the final function is likely to be somewhat blurred because of the multi-resolution decomposition. So carrying out discontinuity meshing before the particle tracing will certainly give much better results.

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A Potential Equation and its derivation

For computing illumination radiance equation is fundamental. It expresses the radiance leaving a point $\bar{x}$ along a direction $\Theta_{\bar{x}}$ and is given in equation 4 as:

$$L(\bar{x}, \Theta_{\bar{x}}) \;=\; L^e(\bar{x}, \Theta_{\bar{x}}) + \int_{\Omega_{\bar{x},in}} f_r(\bar{x}, \Theta_{\bar{y}}, \Theta_{\bar{x}}) \cos\theta_{\bar{x},in} L(\bar{y}, \Theta_{\bar{y}}) d\omega_{in} \tag{21}$$

Any illumination in an environment is due to the emission from the emitting surface(s). However, the expression above does not directly relate the expression of the radiance to the emission of the emitters. Sometimes it may be beneficial to relate them. One can always expand the above equation to get an expression relating to emitters but with infinite terms.

$$\begin{aligned} L(\bar{x}, \Theta_{\bar{x}}) \;=\; & L^e(\bar{x}, \Theta_{\bar{x}}) + \int_{\Omega_{\bar{x},in}} f_r(\bar{x}, \Theta_{\bar{y}}, \Theta_{\bar{x}}) \cos\theta_{\bar{x},in} L^e(\bar{y}, \Theta_{\bar{y}}) d\omega_{in} \\ & + \int_{\Omega_{\bar{x},in}} \mathbf{f_r}(\bar{x}, \Theta_{\bar{y}}, \Theta_{\bar{x}}) \cos\theta_{\bar{x},in} \int_{\Omega_{\bar{y},in}} f_r(\bar{y}, \Theta_{\bar{z}}, \Theta_{\bar{y}}) \cos\theta_{\bar{y},in} L^e(\bar{z}, \Theta_{\bar{z}}) d\omega_{in} \\ & + \ldots \end{aligned} \tag{22}$$

The same relationship can be expressed in a compact fashion by using the concept of *potential*. This concept was introduced in [11, 7]. It may be worth while to briefly reintroduce this concept. It is as follows:
Let $\bar{y}$ be a point on the light source and $\Theta_{\bar{y}}$ a direction at $\bar{y}$. Then we define a quantity, $\mathcal{P}'(\bar{y}, \Theta_{\bar{y}})$, to be the differential radiance at $\bar{x}$ along $\Theta_{\bar{x}}$ due to a *unit* amount (1 watt) of light power emitted from the differential area $dA_{\bar{y}}$ around $\bar{y}$, along a differential direction $d\omega_{\bar{y}}$ around $\Theta_{\bar{y}}$.
The actual emission power leaving $\bar{y}$ along $\Theta_{\bar{y}}$ is $L^e(\bar{y}, \Theta_{\bar{y}}) \cos\theta_{\bar{y}} d\omega_{\bar{y}} dA_{\bar{y}}$. So

$$d^2 L(\bar{x}, \Theta_{\bar{x}}) = L^e(\bar{y}, \Theta_{\bar{y}}) \cos\theta_{\bar{y}} d\omega_{\bar{y}} dA_{\bar{y}} \mathcal{P}'(\bar{y}, \Theta_{\bar{y}})$$

The cumulative radiance due to all the points of the light source and all the directions around them will be

$$L(\bar{x}, \Theta_{\bar{x}}) \;=\; \int_{\bar{y} \in Source} \int_{\Omega_{\bar{y}}} L^e(\bar{y}, \Theta_{\bar{y}}) \cos\theta_{\bar{y}} \mathcal{P}'(\bar{y}, \Theta_{\bar{y}}) d\omega_{\bar{y}} dA_{\bar{y}}$$

The $\mathcal{P}'(\bar{y}, \Theta_{\bar{y}})$ term defined here can be said to be a potential function, because it is the potential capacity of the of the point y and direction Θ_y to contribute towards the radiance at $\bar{x}$ along $\Theta_{\bar{x}}$.
There is nothing special about the points on the light source. Any point in the environment and any direction around that point will have its own potential. So we can as well write above equation as

$$L(\bar{x}, \Theta_{\bar{x}}) \;=\; \int_{\bar{y} \in Env} \int_{\Omega_{\bar{y}}} L^e(\bar{y}, \Theta_{\bar{y}}) \cos\theta_{\bar{y}} \mathcal{P}'(\bar{y}, \Theta_{\bar{y}}) d\omega_{\bar{y}} dA_{\bar{y}} \tag{23}$$

Any light leaving $(\bar{y}, \Theta_{\bar{y}})$ can come out at $(\bar{x}, \Theta_{\bar{x}})$ by zero or more interreflection. The possibility of zero reflection comes from the fact that one cannot rule out the equality $(\bar{y}, \Theta_{\bar{y}}) = (\bar{x}, \Theta_{\bar{x}})$. Thus like the radiance equation the expression for the potential function will have a simple direct term and an integral indirect term. It is as follows :

$$\mathcal{P}'(\bar{y}, \Theta_{\bar{y}}) \;=\; g(\bar{y}, \Theta_{\bar{y}}) + \int_{\Omega_{\bar{z}}} f_r(\bar{z}, \Theta_{\bar{y}}, \Theta_{\bar{z}}) \cos\theta_{\bar{z}} \mathcal{P}'(\bar{z}, \Theta_{\bar{z}}) d\omega_{\bar{z}} \tag{24}$$

where $g(.,.)$ is a delta like function and is defined as [3]

$$g(\bar{y}, \Theta_{\bar{y}}) = \begin{cases} \frac{1}{dA_{\bar{y}} \cos\theta_{\bar{y}} d\omega_{\bar{y}}} & \text{iff } (\bar{y}, \Theta_{\bar{y}}) = (\bar{x}, \Theta_{\bar{x}}), \\ 0 & \text{otherwise.} \end{cases}$$

[3]The term $\frac{1}{dA_{\bar{x}} \cos\theta_{\bar{x}} d\omega_{\bar{x}}}$ arises in the definition of g because this is the radiance that you will get out of 1 watt of light power leaving differential area $dA_{\bar{x}}$ in a differential direction $d\omega_{\bar{x}}$.

$\bar{z}$ is the nearest surface point visible to $\bar{y}$ along $\Theta_{\bar{y}}$ direction and the $f_r(\bar{z}, \Theta_{\bar{y}}, \Theta_{\bar{z}}) \cos\theta_{\bar{z}}$ term gives the reflected power along $\Theta_{\bar{z}}$ direction due to the incidence of 1 watt of light from $\Theta_{\bar{y}}$ direction.

Through the equations 23 and 24 we have arrived at a compact representation for the relationship of the radiance to the emittance. The correctness of the relationship can be checked by expanding the equations 23 and 24 and equating each term of the expansion to each term of equation 22.

Adjoint Illumination Equations

We have so far derived two integral equations. They are

$$\begin{aligned} L(\bar{y}, \Theta_{\bar{y}}) &= L^e(\bar{y}, \Theta_{\bar{y}}) + \int_{\Omega_{\bar{y},in}} f_r(\bar{y}, \Theta_{\bar{z}}, \Theta_{\bar{y}}) \cos\theta_{\bar{y},in} L(\bar{z}, \Theta_{\bar{z}}) d\omega_{in} \\ \mathcal{P}'(\bar{y}, \Theta_{\bar{y}}) &= g(\bar{y}, \Theta_{\bar{y}}) + \int_{\Omega_{\bar{z}}} f_r(\bar{z}, \Theta_{\bar{y}}, \Theta_{\bar{z}}) \cos\theta_{\bar{z}} \mathcal{P}'(\bar{z}, \Theta_{\bar{z}}) d\omega \end{aligned}$$

We have also shown that

$$L(\bar{x}, \Theta_{\bar{x}}) = \int_{\bar{y} \in Env} \int_{\Omega_{\bar{y}}} \mathcal{P}'(\bar{y}, \Theta_{\bar{y}}) L^e(\bar{y}, \Theta_{\bar{y}}) \cos\theta_{\bar{y}} d\omega dA_{\bar{y}}$$

As $g(\bar{y}, \Theta_{\bar{y}})$ is a delta like function we can write another expression for $L(\bar{x}, \Theta_{\bar{x}})$ involving double integrations

$$L(\bar{x}, \Theta_{\bar{x}}) = \int_{\bar{y} \in Env} \int_{\Omega_{\bar{y}}} L(\bar{y}, \Theta_{\bar{y}}) g(\bar{y}, \Theta_{\bar{y}}) \cos\theta_{\bar{y}} d\omega dA_{\bar{y}}$$

This last formulation[4] is deliberate as it gives expression similar to the equation 7, *i.e.*

$$\int_{\bar{y} \in Env} \int_{\Omega_{\bar{y}}} L(\bar{y}, \Theta_{\bar{y}}) g(\bar{y}, \Theta_{\bar{y}}) \cos\theta_{\bar{y}} d\omega dA_{\bar{y}} = L(\bar{x}, \Theta_{\bar{x}}) = \int_{\bar{y} \in Env} \int_{\Omega_{\bar{y}}} \mathcal{P}'(\bar{y}, \Theta_{\bar{y}}) \cos\theta_{\bar{y}} L^e(\bar{y}, \Theta_{\bar{y}}) d\omega dA_{\bar{y}} \quad (25)$$

And from the analogy given in the previous section, on the consequence of adjointness (see figure 1), we can say that the equation pairs

$$\begin{aligned} L(\bar{y}, \Theta_{\bar{y}}) &= L^e(\bar{y}, \Theta_{\bar{y}}) + \int_{\Omega_{\bar{y},in}} f_r(\bar{y}, \Theta_{\bar{z}}, \Theta_{\bar{y}}) \cos\theta_{\bar{y},in} L(\bar{z}, \Theta_{\bar{z}}) d\omega_{in} \\ \mathcal{P}'(\bar{y}, \Theta_{\bar{y}}) \cos\theta_{\bar{y}} &= g(\bar{y}, \Theta_{\bar{y}}) \cos\theta_{\bar{y}} + \cos\theta_{\bar{y}} \int_{\Omega_{\bar{z}}} f_r(\bar{z}, \Theta_{\bar{y}}, \Theta_{\bar{z}}) \cos\theta_{\bar{z}} \mathcal{P}'(\bar{z}, \Theta_{\bar{z}}) d\omega \end{aligned}$$

form an adjoint pair of equations.
If we replace $\mathcal{P}'(\bar{y}, \Theta_{\bar{y}}) \cos\theta_{\bar{y}}$ by $\mathcal{P}(\bar{y}, \Theta_{\bar{y}})$ then we have an adjoint equation to the radiance equation as

$$\mathcal{P}(\bar{y}, \Theta_{\bar{y}}) = g(\bar{y}, \Theta_{\bar{y}}) + \cos\theta_{\bar{y}} \int_{\Omega_{\bar{z}}} f_r(\bar{z}, \Theta_{\bar{y}}, \Theta_{\bar{z}}) \mathcal{P}(\bar{z}, \Theta_{\bar{z}}) d\omega \quad (26)$$

The above equation gives the actual adjoint operator to the integral operator corresponding to the radiance equation.

[4]The multiplication of $\cos\theta_{\bar{y}}$ is to cancel out the corresponding term appearing in the denominator in the definition of g.

Figure 4: Simple room, Side view.

Figure 5: Room+Furniture.

Figure 6: Discontinuity capture with high resolution Multiwavelets

Figure 7: Improvement after the post-processing.