The Potential Equation and Importance in Illumination Computations

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Abstract

An equation adjoint to the luminance equation for describing the global illumination can be formulated using the notion of a surface potential to illuminate the region of interest. This adjoint equation which we shall call as the potential equation, is fundamental to the adjoint radiosity equation used to devise the importance driven radiosity algorithm. In this paper we first briefly derive the adjoint system of integral equations and then show that the adjoint linear equations used in the above algorithm are basically discrete formulations of the same. We also show that the importance entity of the linear equations is basically the potential function integrated over a patch. Further we prove that the linear operators in the two equations are indeed transposes of each other.

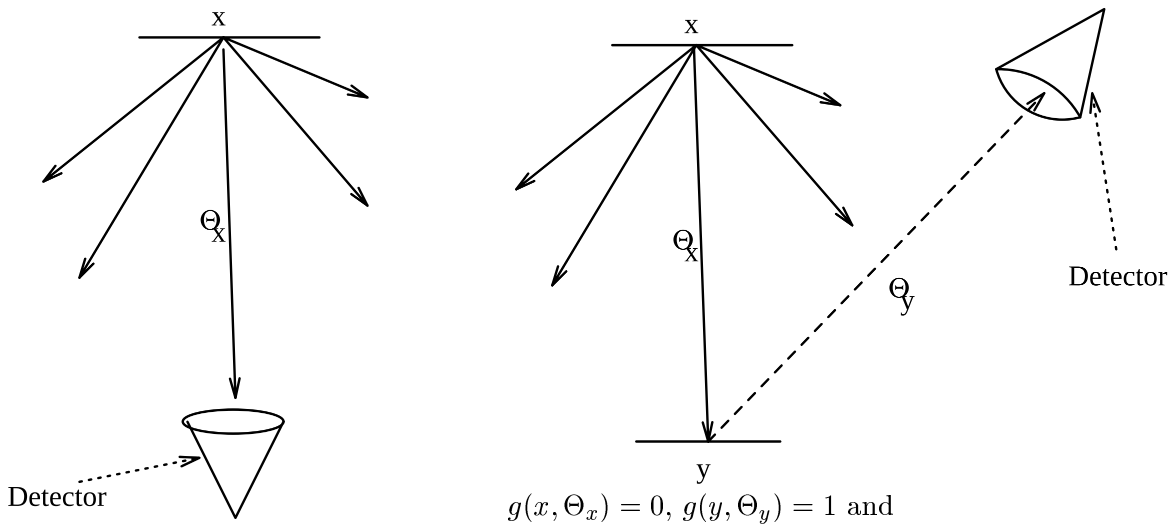
1. Introduction

In a recent paper[1] Smits et al present a new importance driven radiosity algorithm for efficiently computing global illumination solutions with respect to a constrained set of views. The basis of this new algorithm is a pair of adjoint light transport equations. One of these is the standard radiosity equation[2] which uses the form-factor matrix to relate the radiosity of any surface patch to the emitting sources in the environment. The other is the adjoint equation, which uses the transpose of the form-factor matrix to relate so called importance to the receivers. Importance and receivers are deemed as the duals of radiosity and emitting sources respectively. Smits et al visualise importance to be flowing out from the eye or camera and distributing itself amongst the patches of the environment, very much like light.

In our earlier work[3] we have similarly formulated a pair of integral equations as an adjoint set for describing the light transport. The first of these equations is basically Kajiya's rendering equation[4] defined in terms of luminance, emittance and *brdf*s, while the second is its adjoint defined in terms of potential, hypothetical detector(s) and *brdf*s. The potential equation is fundamental to the adjoint radiosity equation just as as much as the rendering equation is to the standard radiosity equation. In this paper we show how the adjoint radiosity equation can be derived from the potential equation. We show that importance is basically the potential integrated over a patch; we derive the detector values and prove that the linear operators in the two equations are indeed transposes of each other. In order to make the contents of this paper self-contained we briefly derive the adjoint equations below. For a detailed discussion on the subject the reader is referred to [3].

2. Potential Equation

Because of the optical properties of surfaces, such as reflection, transmission, etc the light emitted from any surface in any direction can illuminate many other surfaces of an environment. Alternatively we can say that a surface can be illuminated by lights placed anywhere in



$g(x, \Theta_x) = 1$ and $W(x, \Theta_x) = 1$.

$g(x, \Theta_x) = 0$, $g(y, \Theta_y) = 1$ and
$W(x, \Theta_x) = 0 + \int_{\Omega_y} \rho_y(\Theta_y, \Theta_x) W(y, \Theta_y) cos\theta_y d\omega_y$

Figure 1. Direct and Indirect Components of the Potential Function.

the environment. Suppose we place a light detector in the environment. Let the detector be hypothetical in the sense, it does not in any way affect the flow of light in the environment. Emission from a given point, x, along a given direction, Θ_x, may result in some light being detected by the detector. The light can be detected directly from the emission point and/or indirectly from other points of the environment due to reflection, transmission etc. (see Figure 1). The quantity of light detected will vary depending on the point of emission and at a given point depending on the direction of emission. This quantity, may be termed as potential, W, of a point and direction to contribute light into the detector. Further more as W is a function of the surface position, x and the direction, Θ around x, we have in the process defined a potential function $W(x, \Theta_x)$.

We shall now derive an expression for such a function. Light on being emitted from (x, Θ_x) can enter the detector directly if the path (x, Θ_x) leads to the detector. So to represent the direct component we will use a function $g(x, \Theta_x)$ which has a value 1 if the light path starting from x along Θ_x reaches the detector unhidered, 0 otherwise.

The quantity of light entering the detector due to one or more reflections may be expressed recursively as follows. The emission from any (x, Θ_x) will reach the nearest surface point y and then possibly be reflected. If we take the probability of the whole amount of flux getting reflected in any one of the hemispherical directions Θ_y around y as $\rho_y(\Theta_y, \Theta_x) cos\theta_y d\omega_y$, where the symbols used are as in Figure 2, then the quantity of light entering the detector due to this reflection will be this probability times the potential of the point y along Θ_y, i.e. $\rho_y(\Theta_y, \Theta_x) cos\theta_y d\omega_y W(y, \Theta_y)$. Then the cumulative result of the reflection in the hemispherical direction around y will be

$$\int_{\Omega_y} \rho_y(\Theta_y, \Theta_x) W(y, \Theta_y) cos\theta_y d\omega_y$$

The complete expression for the potential function is therefore given by:

$$W(x, \Theta_x) = g(x, \Theta_x) + \int_{\Omega_y} \rho_y(\Theta_y, \Theta_x) W(y, \Theta_y) cos\theta_y d\omega_y \qquad (1)$$

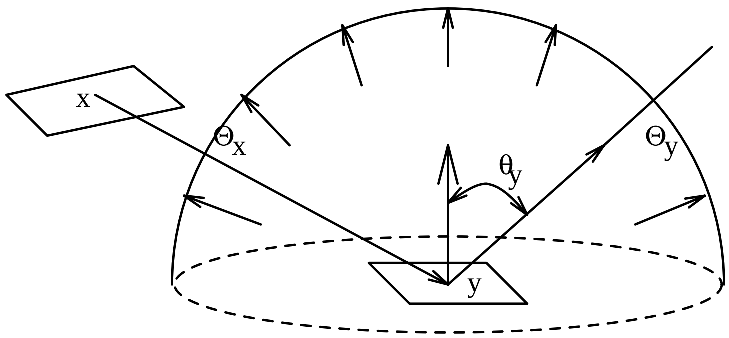


Figure 2. Hemispherical Directions for the Outgoing illumination.

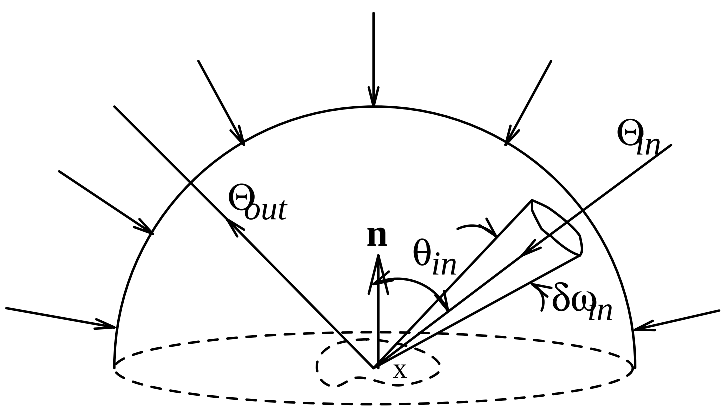


Figure 3. Hemispherical Directions for the Incoming illumination.

3. Luminance Equation

From the definition of surface bidirectional reflectance function [5,page64], the outgoing luminance (L) at any point x of a surface in the environment in any direction Θ_{out}, due to the luminance incident at x from direction Θ_{in} can be given by

$$L_{out}(x, \Theta_{out}) = \rho_x(\Theta_{out}, \Theta_{in}) L_{in}(x, \Theta_{in}) cos\theta_{in} d\omega_{in}$$

where θ_{in} and $d\omega_{in}$ are as shown in the Figure 3. Taking into account incoming luminance from all the directions in the incoming hemisphere around the point x, the outgoing luminance can be expressed as

$$L_{out}(x, \Theta_{out}) = \int_{\Omega_x} \rho_x(\Theta_{out}, \Theta_{in}) L_{in}(x, \Theta_{in}) cos\theta_{in} d\omega_{in}$$

where the integration range Ω_x represents the hemisphere around x. If we include emitting surfaces also in the general expression for the outgoing luminance then it takes the form:

$$L_{out}(x, \Theta_{out}) = \epsilon_{out}(x, \Theta_{out}) + \int_{\Omega_x} \rho_x(\Theta_{out}, \Theta_{in}) L_{in}(x, \Theta_{in}) cos\theta_{in} d\omega_{in}$$

For a nonparticipating environment consisting of mainly opaque solids the luminance in any incoming direction at x must be due to the outgoing luminance from some surface point y in an outgoing direction Θ_y where Θ_y is defined by the vector joining the point x to y. If we now wish to rewrite the luminance equation in terms of outgoing luminance and outgoing directions only, then by representing the outgoing directions at x and y as Θ_x and Θ_y we get:

$$L(x,\Theta_x) = \epsilon(x,\Theta_x) + \int_{\Omega_x} \rho_x(\Theta_x,\Theta_y) L(y,\Theta_y) cos\theta_x d\omega_x \tag{2}$$

If we look back at the potential equation (Equation 1), we find a striking similarity in the forms of these equations. In both the cases there is an implicit assumption that y represents a surface point visible to x. However it must be noted that in Equation 2 the integration is over the incoming hemisphere around x whereas in Equation 1 the integration is over the outgoing hemisphere around y.

4. Discrete Equations for a Diffuse Environment

Because of their inherently complex nature it is difficult to solve Equations 1 and 2. However, simplified forms of these have been made amenable to analytical solutions. A simplified discrete formulation of the Equation 2, widely known as radiosity equation[2], is

$$\phi_i = E_i + k_i \sum_{j=1}^{N} \phi_j F_{ij}$$

where ϕ_i is the radiosity, i.e. flux per unit area, E_i is the emission flux density, k_i is the hemispherical diffuse reflectance of patch i and F_{ij} is the form factor between patch i and patch j.

This discrete formulation is derived from the continuous Equation 2 by making the following assumptions:

1. The environment is a collection of a finite number, say N, of small diffusely reflecting patches with uniform radiosity.

2. As the luminance from any point of any such uniformly diffuse patch is $1/\pi$ times the flux per unit area we shall compute this total flux from any patch leaving that patch in all the hemispherical directions.

3. The solution is carried out in an enclosure, i.e. the hemispherical direction around any point, x, in the environment is assumed to be covered by one or more of the patches of that environment and every patch, j, may be assumed to occupy a solid angle, ω_{xj} (which may be zero) in the hemisphere over a surface point.

Using this equation we arrive at a system of equations for the whole environment of the form:

$L\phi = E$

where L is

$$\begin{bmatrix} 1-k_1F_{11} & \dots & -k_1F_{1i} & \dots & -k_1F_{1N} \\ . & \dots & . & \dots & . \\ . & \dots & . & \dots & . \\ -k_iF_{i1} & \dots & 1-k_iF_{ii} & \dots & -k_iF_{iN} \\ . & \dots & . & \dots & . \\ . & \dots & . & \dots & . \\ -k_NF_{N1} & \dots & -k_NF_{Ni} & \dots & 1-k_NF_{NN} \end{bmatrix}$$

which may be solved to arrive at the equilibrium state radiosity for each patch.

Similarly we shall derive the discrete formulation using the potential equation. For this we shall define a quantity called importance, say ψ_i, as the potential of the i-th patch over the full hemispherical direction. We can arrive at an expression for ψ_i by integrating $W(x, \Theta_x)$ for every point of the i-th patch and over the hemisphere around each point of the patch. Thus

$$\begin{aligned}
\psi_i &= \int_{x \in A_i} \int_{\Omega_x} W(x, \Theta_x) cos\theta_x d\omega_x dx \\
&= \int_{x \in A_i} \int_{\Omega_x} g(x, \Theta_x) cos\theta_x d\omega_x dx + \int_{x \in A_i} \int_{\Omega_x} \left[\int_{\Omega_y} \rho_y(\Theta_y, \Theta_x) W(y, \Theta_y) cos\theta_y d\omega_y \right] cos\theta_x d\omega_x dx
\end{aligned}$$

Using the earlier mentioned assumptions the equation for ψ_i may be simplified to

$$\begin{aligned}
\psi_i &= \int_{x \in A_i} \int_{\Omega_x} g(x, \Theta_x) cos\theta_x d\omega_x dx + A_i \int_{\Omega_x} \left[\int_{\Omega_y} \rho_y(\Theta_x, \Theta_y) W(y, \Theta_y) cos\theta_y d\omega_y \right] cos\theta_x d\omega_x \\
&= R_i + A_i \sum_{j=1}^{N} \rho_j \int_{\omega_{xj}} \left[\int_{\Omega_y} W(y, \Theta_y) cos\theta_y d\omega_y \right] cos\theta_x d\omega_x \\
&\approx R_i + A_i \sum_{j=1}^{N} \rho_j \int_{\omega_{xj}} \frac{\psi_j}{A_j} cos\theta_x d\omega_x \\
&= R_i + A_i \sum_{j=1}^{N} \rho_j \frac{\psi_j}{A_j} \int_{y \in A_j} Vis(x, y) \frac{cos\theta_x cos\theta_y}{D_{xy}^2} dA_y \\
&\approx R_i + \sum_{j=1}^{N} k_j \psi_j \frac{A_i}{A_j} F_{ij} \\
&= R_i + \sum_{j=1}^{N} k_j \psi_j F_{ji}
\end{aligned}$$

where $k_i = \pi \rho_i$, $Vis(x, y)$ is the visibility between two surface points x and y, D_{xy} is the distance between x and y, and $R_i = \int_{x \in A_i} \int_{\Omega_x} g(x, \Theta_x) cos\theta_x d\omega_x dx$. As $g(x, \Theta_x)$ is simply a visibility function between detector and the point x along direction Θ_x, R_i is related to the detector-to-patch form-factor.

Thus the system of equations for the whole environment may be written as

$$L^* \psi = R$$

where L^* is

$$\begin{bmatrix}
1 - k_1 F_{11} & \dots & -k_i F_{i1} & \dots & -k_N F_{N1} \\
. & \dots & . & \dots & . \\
. & \dots & . & \dots & . \\
-k_1 F_{1i} & \dots & 1 - k_i F_{ii} & \dots & -k_N F_{Ni} \\
. & \dots & . & \dots & . \\
. & \dots & . & \dots & . \\
-k_1 F_{1N} & \dots & -k_i F_{iN} & \dots & 1 - k_N F_{NN}
\end{bmatrix}$$

This set of equations may be solved to arrive at the equilibrium importance for the whole environment.

It is easy to see that the discrete transport operators L and L^* as described above are transpose of each other.

5. Concluding Remarks

The primary use of the adjoint equations has been to increase computational efficiency. In the area of neutron transport[6] adjoint equations have been used for the estimation of flux and to provide a good choice for importance function biasing. In [3] we have presented algorithms for efficiently computing the global illumination in the Monte Carlo simulation of the particle model of light. In a Monte Carlo simulation of the particle model of light[7, 8], particles, packets of energy, are shot out in different directions from different positions of the surface of the light source. The interaction of each particle in the environment is traced. Global illumination is then computed from the history of these interactions. The potential equation forms the basis for this algorithm. Furthermore, it is used to derive biasing functions that are used to carry out importance sampling of the emission and reflection functions. These biasing functions ensure that more and more particles are directed towards those parts of the environment which are of greater interest and need more accurate illumination computation. Similarly in [1] the importance of patches is used to increase the efficiency of the hierarchic radiosity algorithm [9] by subdividing patches depending on their importance. So far the application of the adjoint equations has been restricted to global illumination computations in diffuse environments. Techniques for dealing with global illumination in general environments with diffuse, specular and other more complex optical behaviour [10−−13] are still excessively demanding in terms of computational resources. We believe that the benefits of using the adjoint equations to devise new algorithms in such cases will be much more. Similarly a number of multi-pass algorithms[14−−16] have been devised for dealing with the problem of accurately reconstructing the luminance function with effects such as shadows, highlights, caustics, etc.. Once again these algorithms tend to be resource intensive and more optimal solutions could be sought by the use of these equations.

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