Design and Analysis of Algorithms

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Lecture 7:
Lower Bound of Comparison Sorts

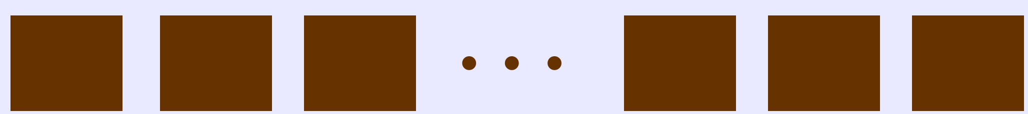
Slides modified from Dr. Hon, with permission

About this lecture

- Lower bound of any comparison sorting algorithm
  - applies to insertion sort, selection sort, merge sort, heapsort, quicksort, …
  - does not apply to counting sort, radix sort, bucket sort
- Based on Decision Tree Model

Comparison Sort

- Comparison sort only uses comparisons between items to gain information about the relative order of items
- It's like the elements are stored in boxes, and we can only pick two boxes at a time to compare which one is larger, without knowing their values



Worst-Case Running Time

Merge sort and heapsort are the "smartest" comparison sorting algorithms we have studied so far:

worst-case running time is $\Theta(n \log n)$

Question: Do we have an even smarter algorithm? Say, runs in $o(n \log n)$ time?

Answer: No! (main theorem in this lecture)

Lower Bound

Theorem: Any comparison sorting algorithm requires $\Omega(n \log n)$ comparisons to sort n distinct items in the worst case

Corollary: Any comparison sorting algorithm runs in $\Omega(n \log n)$ time in the worst case

Corollary: Merge sort and Heapsort are (asymptotically) optimal comparison sorts

Proof of Lower Bound

The main theorem only counts comparison operations, so we may assume all other operations (such as moving items) are for free

Consequently, any comparison sort can be viewed as performing in the following way:

1. Continuously gather relative ordering information between items
2. In the end, move items to correct positions

We use the above view in the proof

Decision Tree of an Algorithm

Consider the following algorithm to sort 3 items A, B, and C:

Step 1: Compare A with B

Step 2: Compare B with C

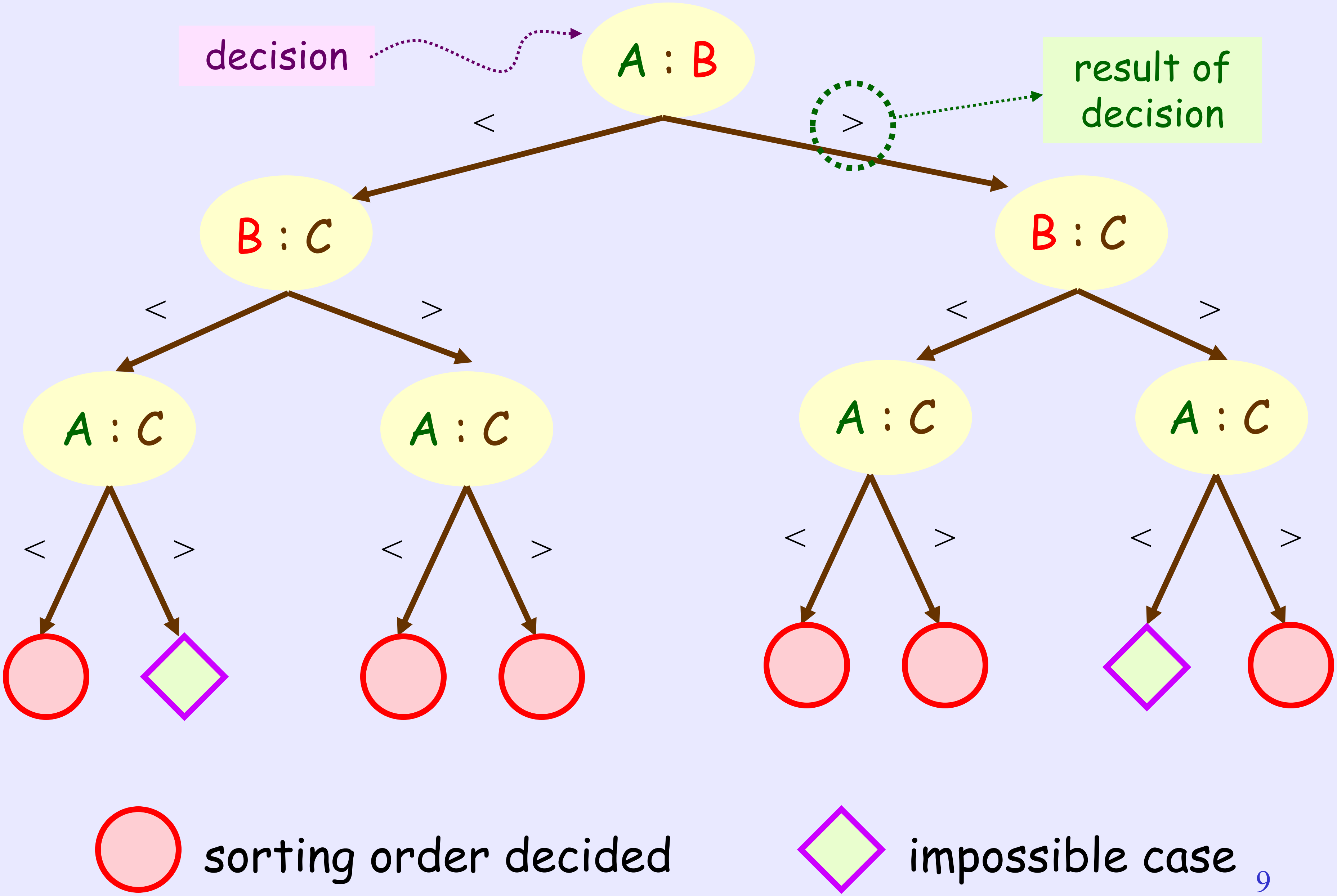
Step 3: Compare A with C

Afterwards, decide the sorting order of the 3 items

Decision Tree of an Algorithm

- The previous algorithm always use 3 comparisons, and can sort the 3 items

- In particular, the comparisons used in different inputs (i.e., permutations) can be captured in a decision tree, as shown in the next slide:


decision
A : B
result of decision
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B : C
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sorting order decided
impossible case

Decision Tree of an Algorithm

A cleverer algorithm may sort the 3 items, sometimes, using at most 2 comparisons:

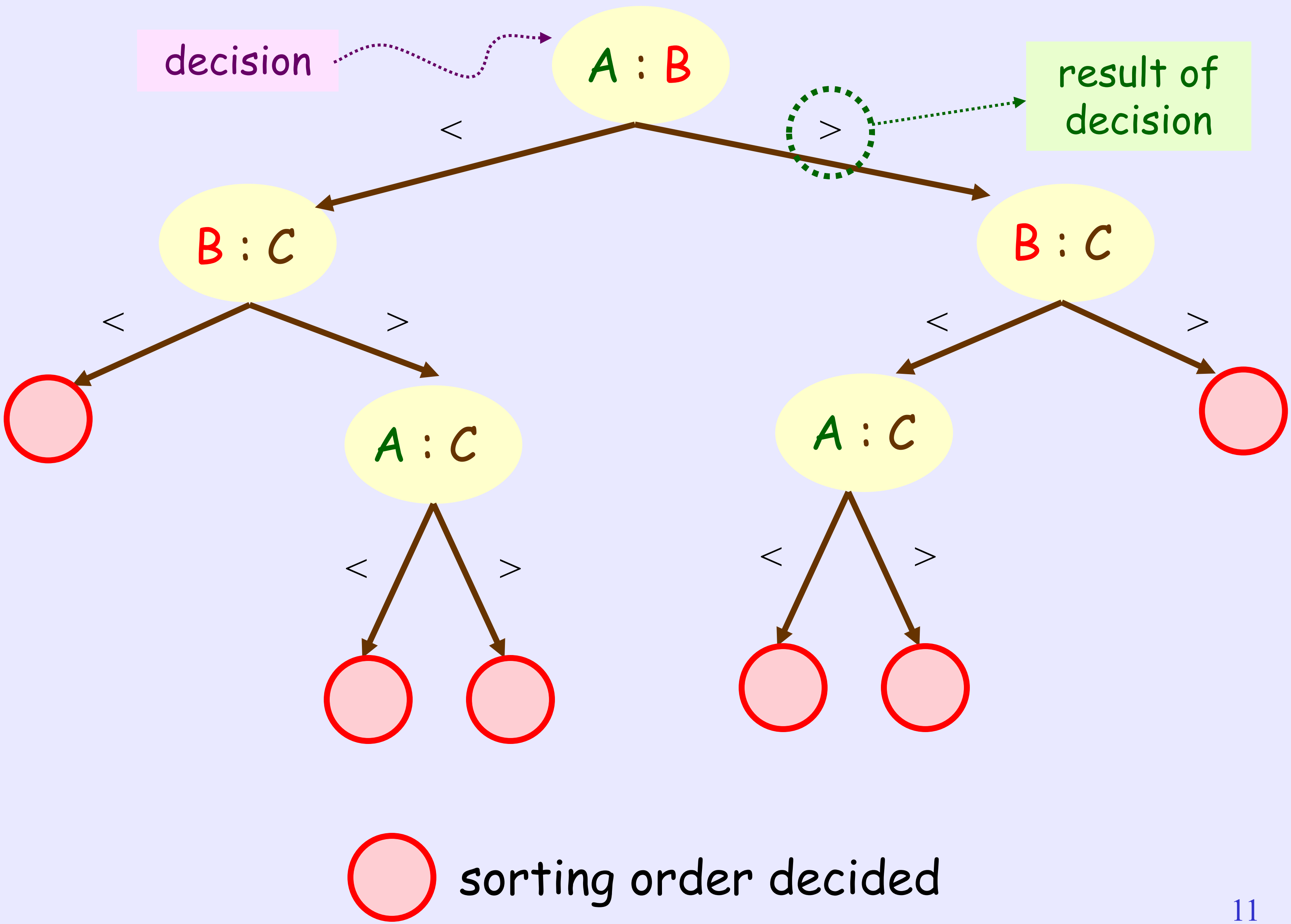
Step 1: Check if $A > B$

Step 2: Check if $B > C$

Step 3: Compare A with C if the result in Steps 1 and 2 are different

Afterwards, decide the sorting order

- Then, the decision tree becomes …


decision
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sorting order decided

Properties of Decision Tree

In general, assume the input has n items

Then, for ANY comparison sort algorithm:

- Each of the $n!$ permutations corresponds to a distinct leaf in the decision tree
- The height of the tree is the worst-case # of comparisons for any input

Question: What can be the height of the decision tree of the cleverest algorithm?

Lower Bound on Height

- There are n! leaves [for any decision tree]
- Degree of each node is at most 2
- Let h = node-height of decision tree

So, n! = total # leaves · 2^h

➔ h ¸ log (n!) = log n + log (n-1) + …

¸ log n + … + log (n/2)

¸ (n/2) log (n/2) = Ω(n log n)

We can also use Stirling's approximation:

$$n! = \sqrt{2\pi n}\,(n/e)^n\,(1+\Theta(1/n))$$

Proof of Lower Bound

Conclusion:

worst-case # of comparisons

= node-height of the decision tree

= $\Omega(n \log n)$ [for any decision tree]

➔ Any comparison sort, even the cleverest one, needs $\Omega(n \log n)$ comparisons in the worst case