Design and Analysis of Algorithms

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Lecture 5: Quicksort

Slides modified from Dr. Hon, with permission

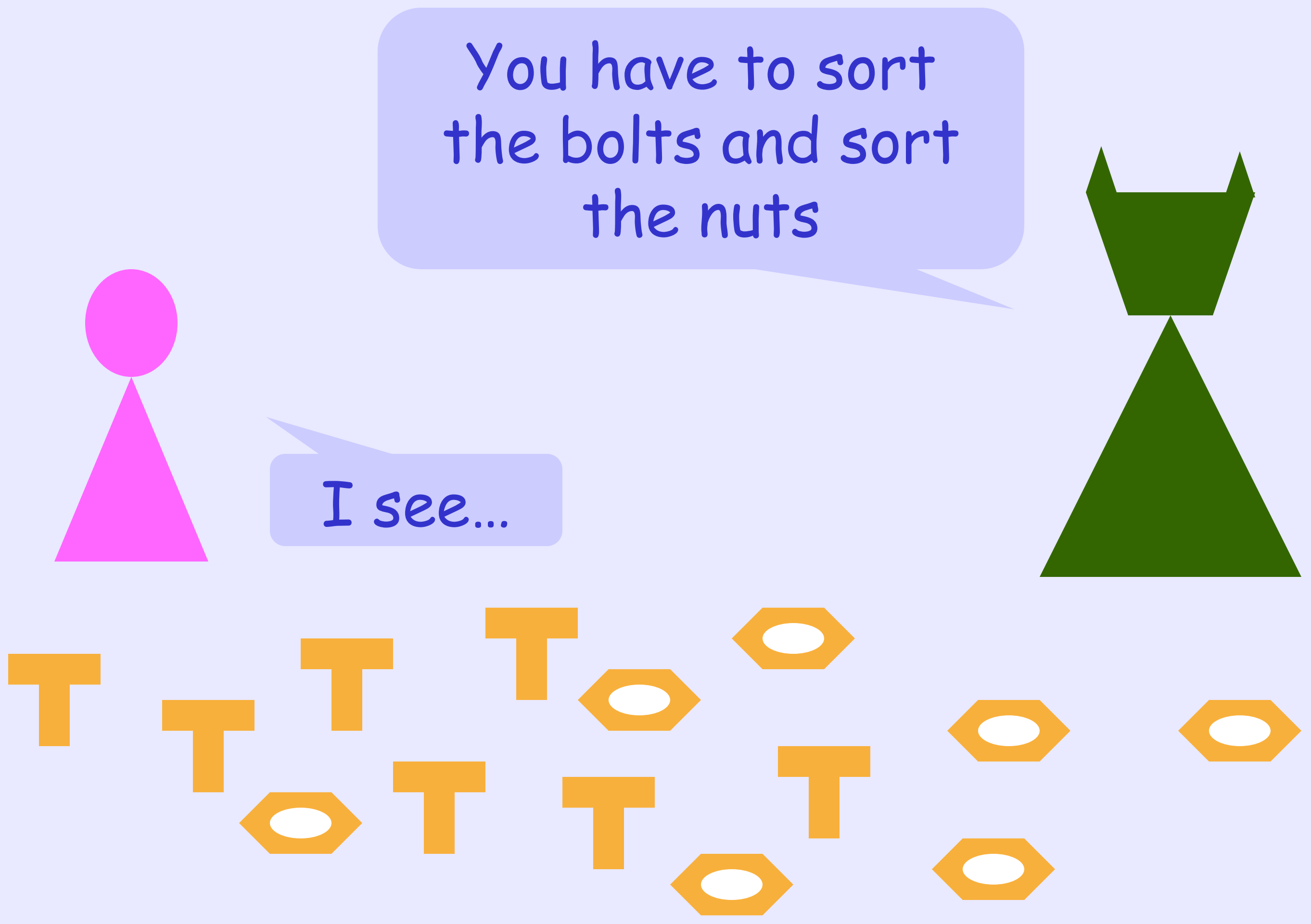
About this lecture

- Introduce Quicksort
  - Cinderella's Problems
- Running time of Quicksort
  - Worst-Case
  - Average-Case

Cinderella's Problem

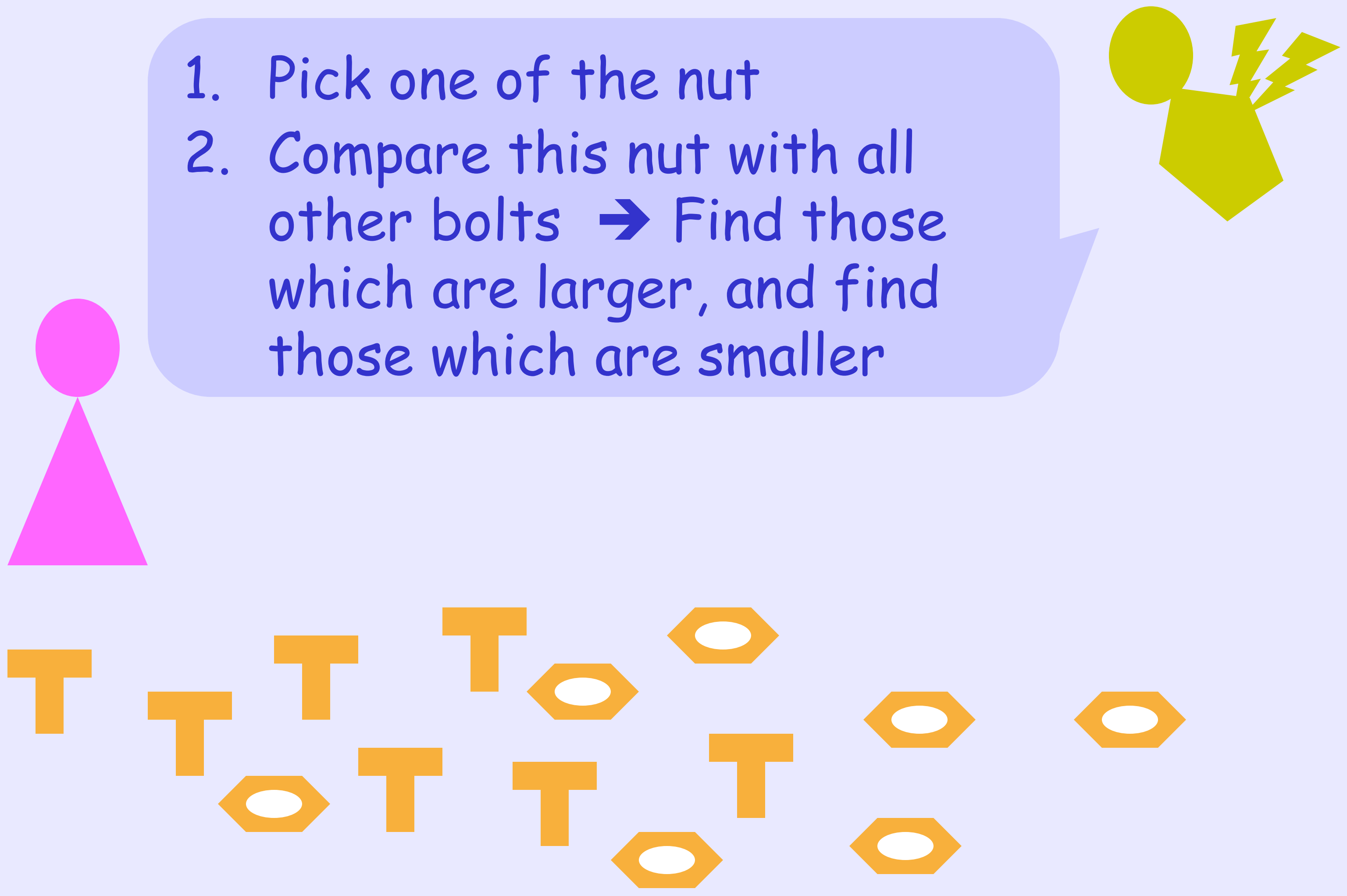
You have to find the largest bolt and the largest nut
I see...

Cinderella's New Problem

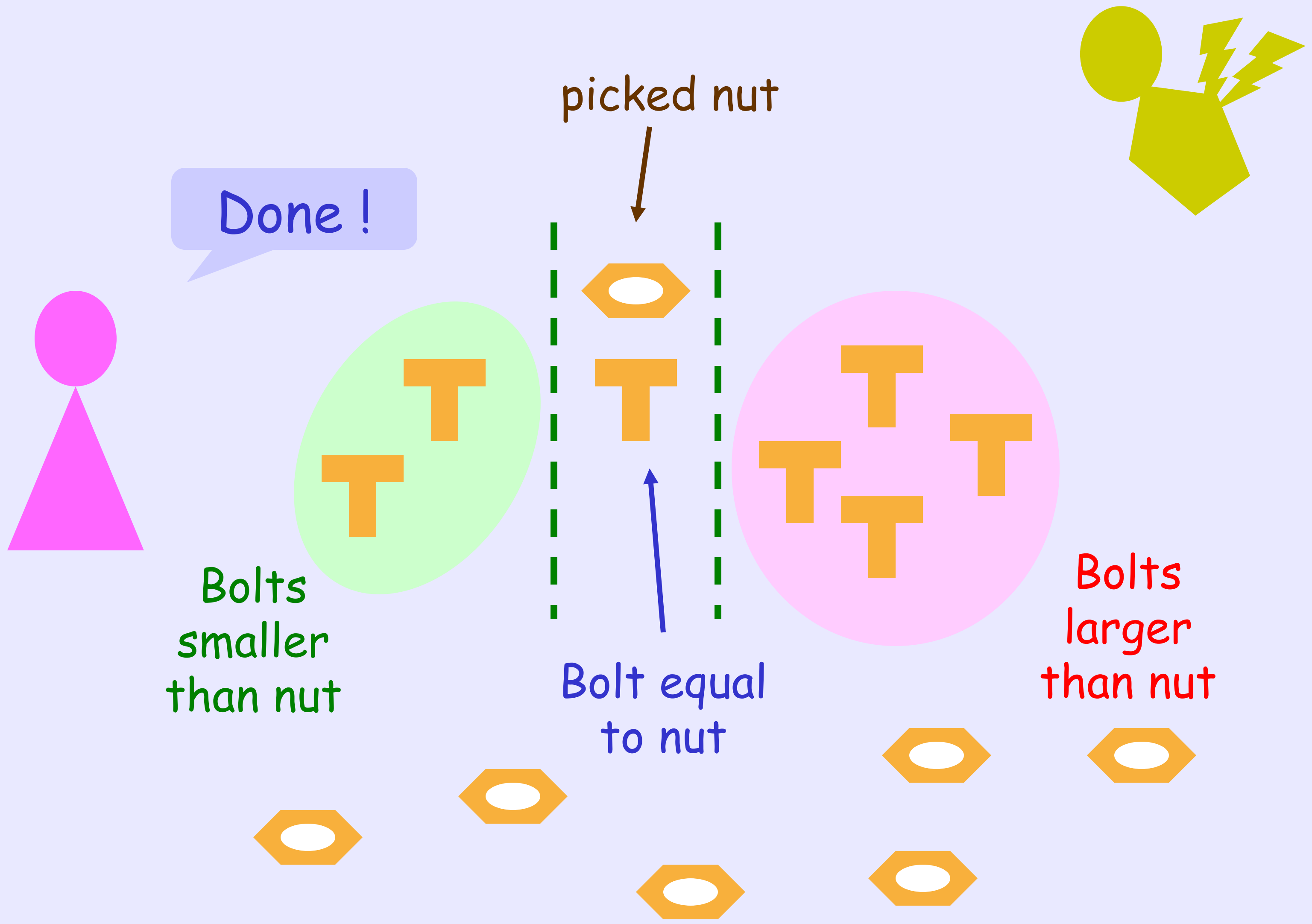


Fairy Godmother's Proposal

1. Pick one of the nut
2. Compare this nut with all other bolts ➔ Find those which are larger, and find those which are smaller

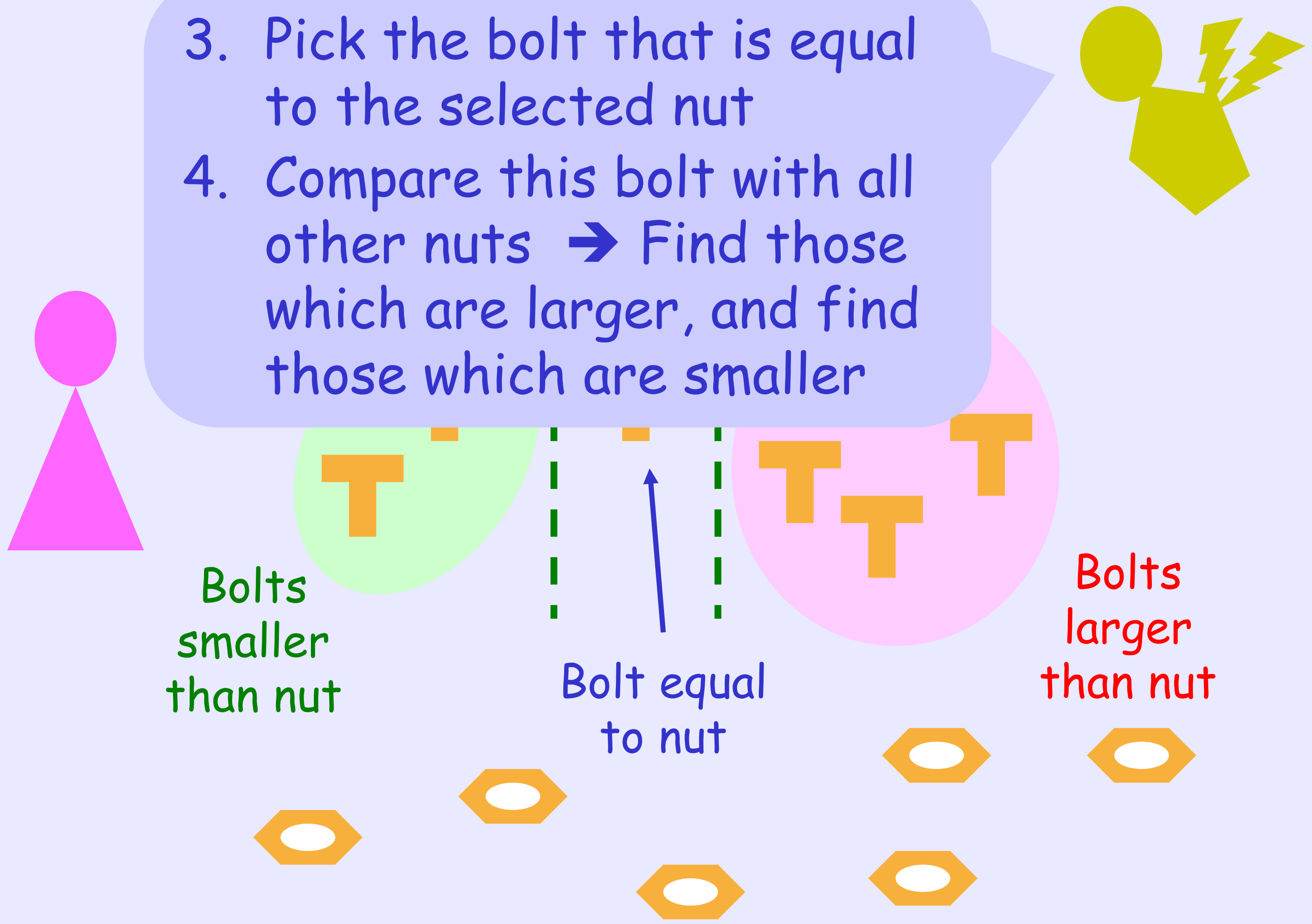


Fairy Godmother's Proposal

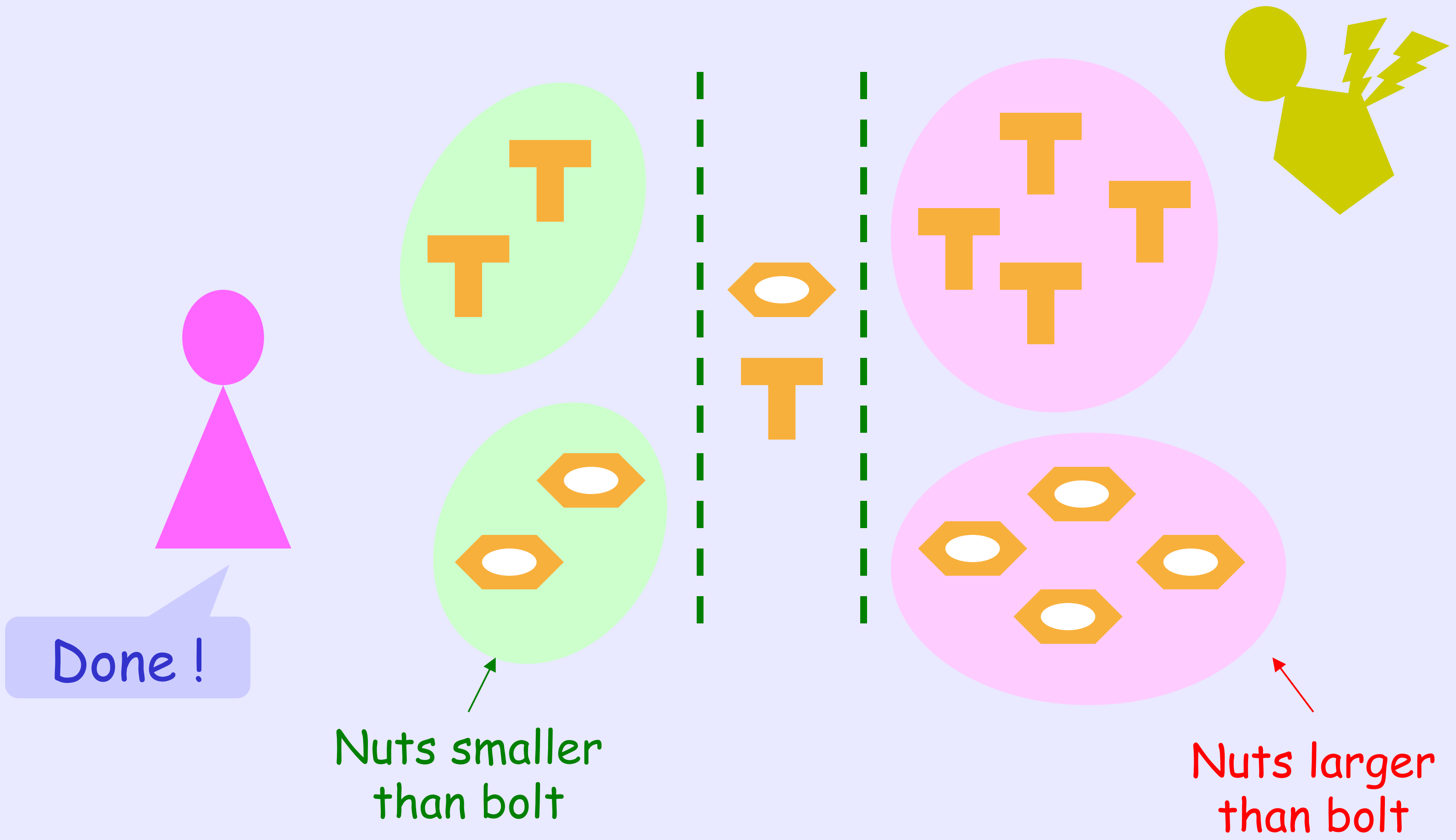


Fairy Godmother's Proposal

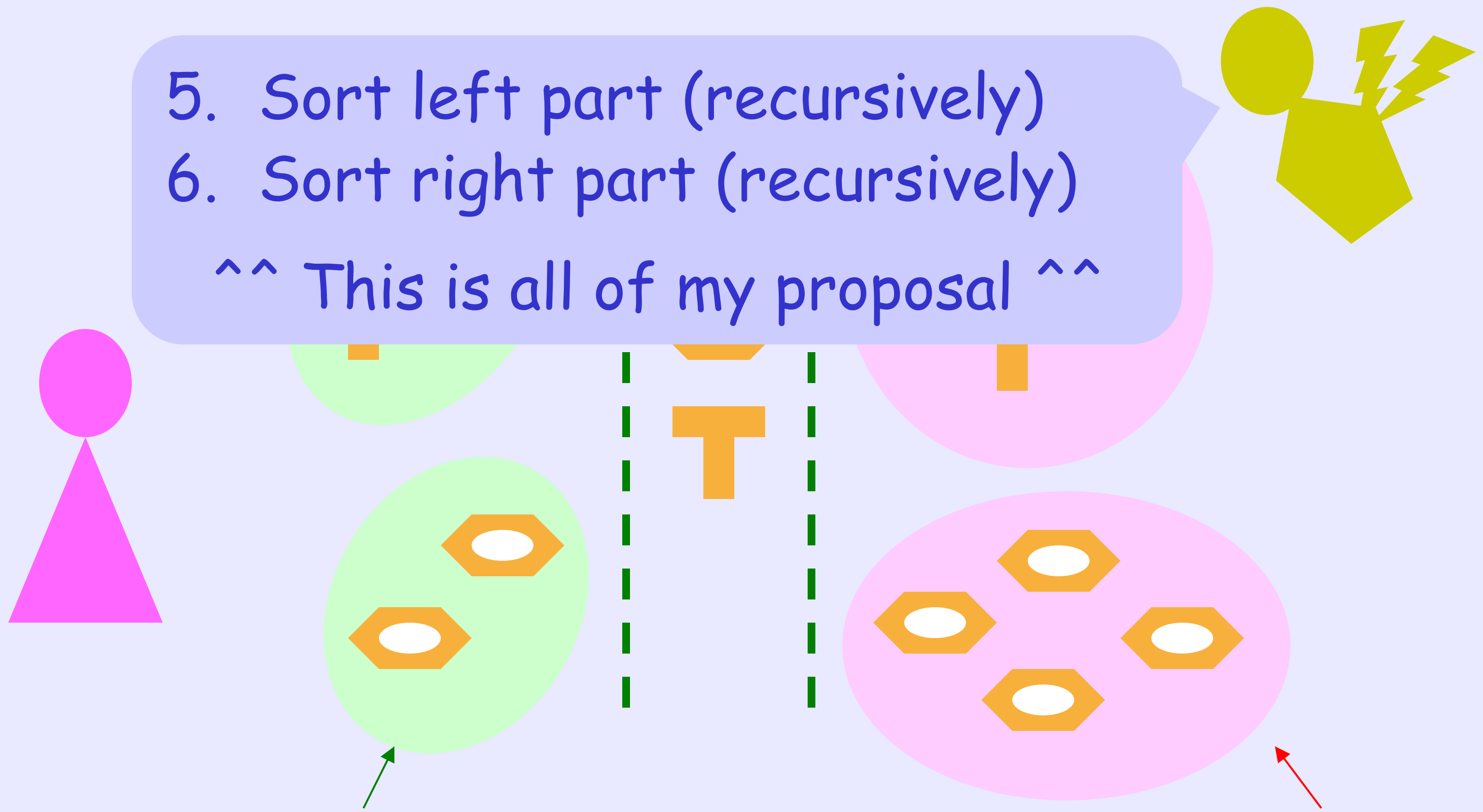
3. Pick the bolt that is equal to the selected nut
4. Compare this bolt with all other nuts ➔ Find those which are larger, and find those which are smaller



Fairy Godmother's Proposal



Fairy Godmother's Proposal



5. Sort left part (recursively)
6. Sort right part (recursively)
^^ This is all of my proposal ^^

Nuts smaller than bolt

Nuts larger than bolt

Fairy Godmother's Proposal

- Can you see why Fairy Godmother's proposal is a correct algorithm?
- What is the running time ?
  - Worst-case: $\Theta(n^2)$ comparisons
  - No better than the brute force approach !!
- Though worst-case runs badly, the average case is good: $\Theta(n \log n)$ comparisons

Quicksort uses Partition

The previous algorithm is exactly Quicksort, which makes use of a Partition function:

```
Partition(A,p,r)     /* to partition array A[p..r] */
    1. Pick an element, say A[t] (called pivot)
    2. Let q = #elements less than pivot
    3. Put elements less than pivot to A[p..p+q-1]
    4. Put pivot to A[p+q]
    5. Put remaining elements to A[p+q+1..r]
    6. Return q
```

More on Partition

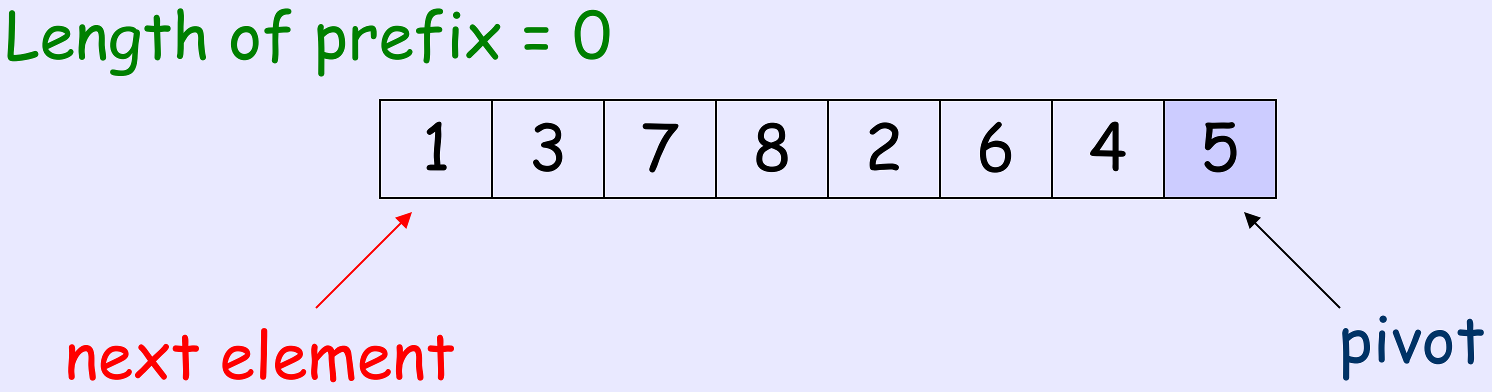
- After Partition(A,p,r), we obtain the value q, and know that
  - Pivot was A[p+q]
  - Before A[p+q] : smaller than pivot
  - After A[p+q] : larger than pivot
- There are many ways to perform Partition. One way is shown in the next slides
  - It will be an in-place algorithm (using O(1) extra space in addition to the input array)

Ideas for In-Place Partition

- Idea 1: Use A[r] (the last element) as pivot
- Idea 2: Process A[p..r] from left to right
  - The prefix (the beginning part) of A stores all elements less than pivot seen so far
  - Use two counters:
    - One for the length of the prefix
    - One for the element we are looking

In-Place Partition in Action

before running

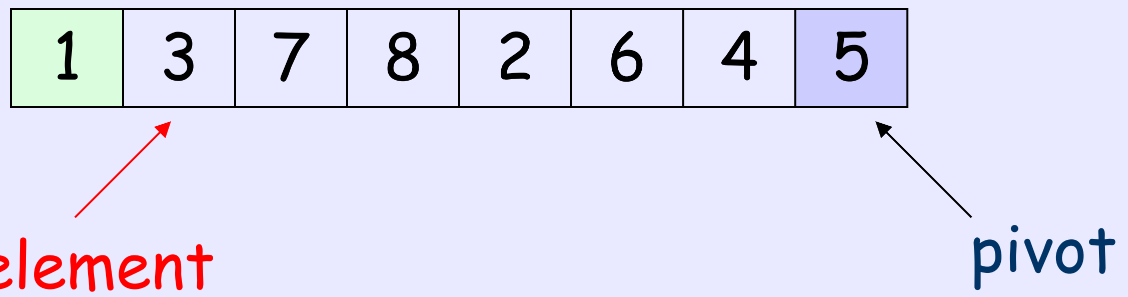


Because next element is less than pivot, we shall extend the prefix by 1

In-Place Partition in Action

after 1 step

Length of prefix = 1

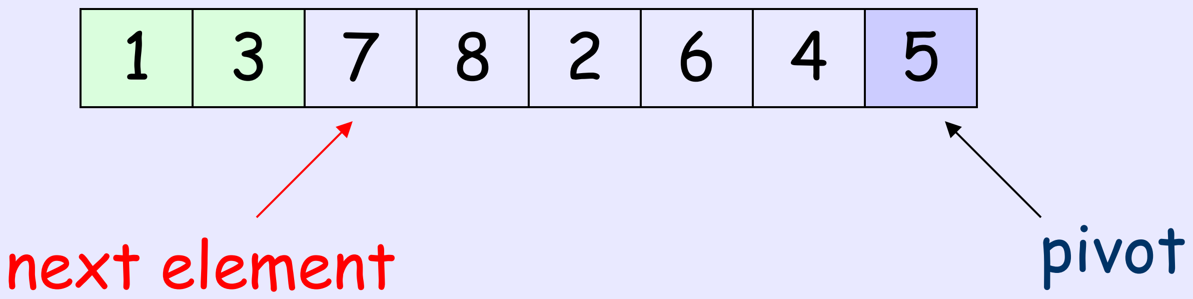


Because next element is smaller than pivot, and is adjacent to the prefix, we extend the prefix

In-Place Partition in Action

after 2 steps

Length of prefix = 2

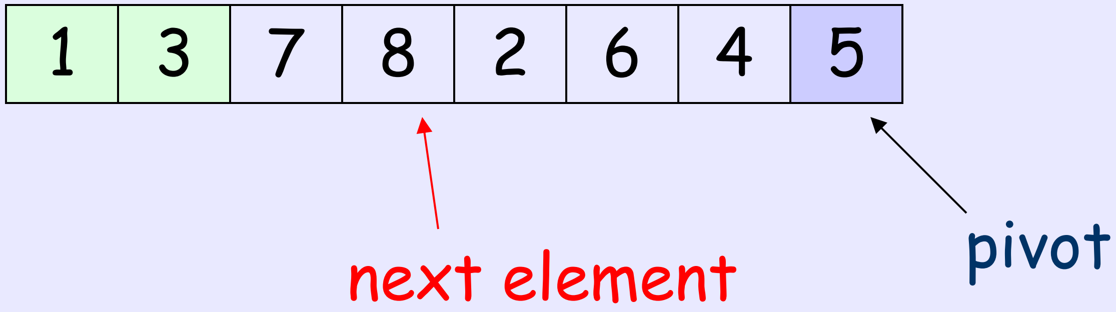


Because next element is larger than pivot, no change to prefix

In-Place Partition in Action

after 3 steps

Length of prefix = 2

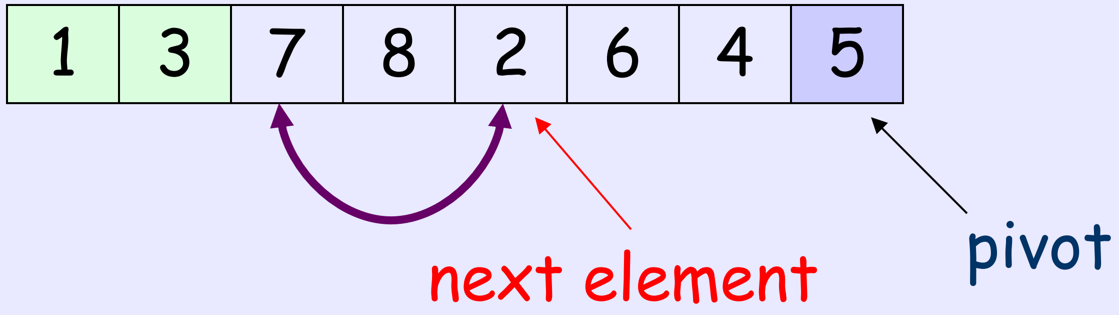


Again, next element is larger than pivot,
no change to prefix

In-Place Partition in Action

after 4 steps

Length of prefix = 2

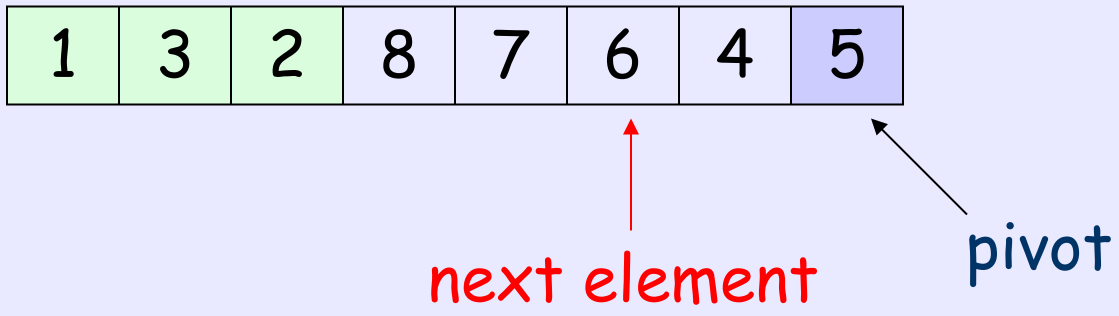


Because next element is less than pivot, we shall extend the prefix by swapping

In-Place Partition in Action

after 5 steps

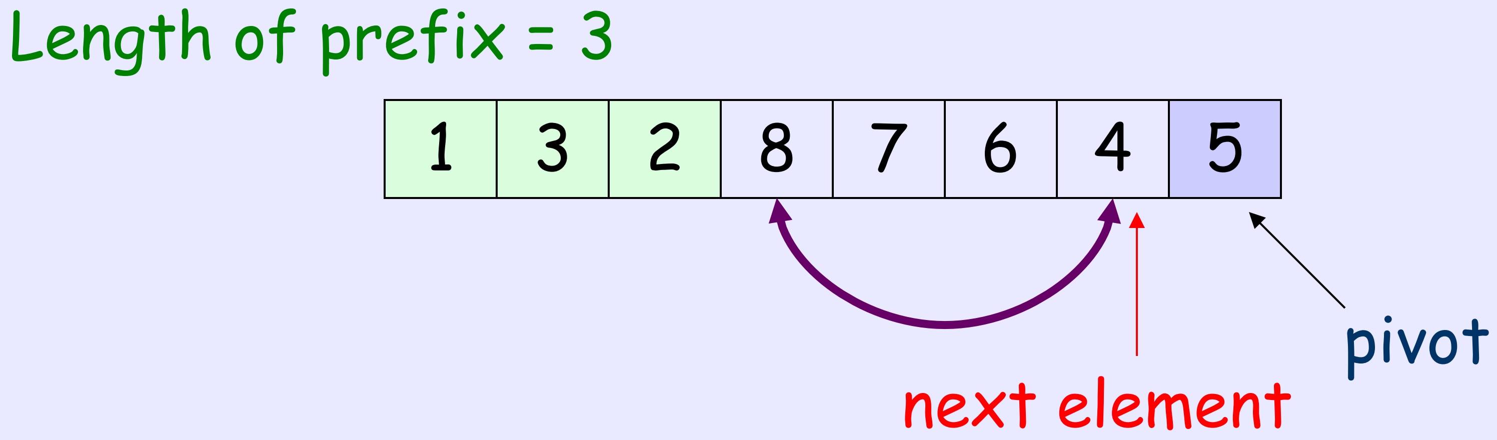
Length of prefix = 3



Because next element is larger than pivot, no change to prefix

In-Place Partition in Action

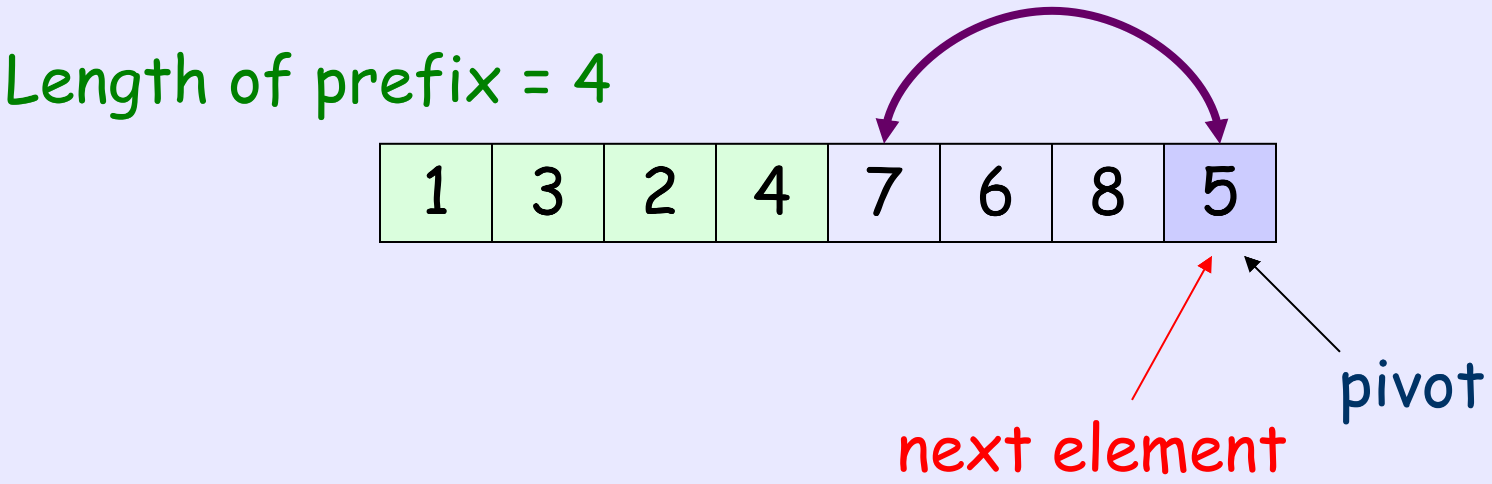
after 6 steps



Because next element is less than pivot, we shall extend the prefix by swapping

In-Place Partition in Action

after 7 steps

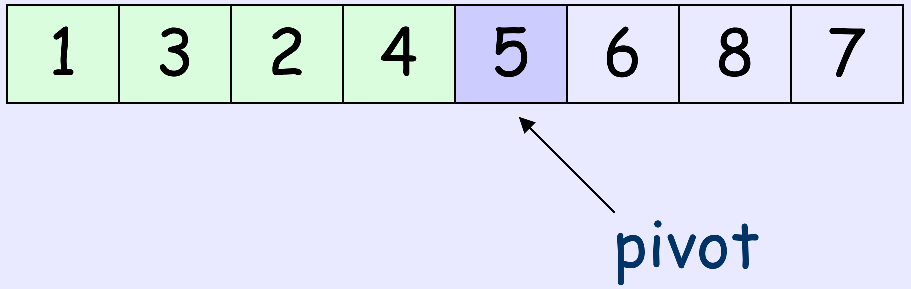


When next element is the pivot, we put it after the end of the prefix by swapping

In-Place Partition in Action

after 8 steps

Length of prefix = 4



Partition is done, and return length of prefix

Quicksort

The Quicksort algorithm works as follows:

```
Quicksort(A,p,r)     /* to sort array A[p..r]  */
   1.  if ( p ⍰ r )   return;
   2.  q = Partition(A,p,r);
   3.  Quicksort(A, p, p+q-1);
   4.  Quicksort(A, p+q+1, r);
```

To sort A[1..n], we just call Quicksort(A,1,n)

Worst-Case Running Time

The worst-case running time of Quicksort can be expressed by:

$$T(n) = \max_{q=0 \text{ to } n-1} \big(T(q) + T(n-q-1)\big) + \Theta(n)$$

We prove $T(n) = O(n^2)$ by substitution method:

1. Guess $T(n) \cdot cn^2$ for some constant c
2. Next, verify our guess by induction

Worst-Case Running Time

Inductive Case:

$$T(n) = \max_{q=0 \text{ to } n-1} \left(T(q) + T(n-q-1)\right) + \Theta(n)$$

$$\cdot \max_{q=0 \text{ to } n-1} \left(cq^2 + c(n-q-1)^2\right) + \Theta(n)$$

Maximized when q = 0
or when q = n-1

$$\cdot\ c(n-1)^2 + \Theta(n)$$

$$= cn^2 - 2cn + c + \Theta(n)$$

$$\cdot\ cn^2$$ when c is large enough

Inductive Case is OK now. How about Base Case?

Worst-Case Running Time

Conclusion:

1. $T(n) = O(n^2)$
2. However, we can also show

$$T(n) = \Omega(n^2)$$

by finding a worst-case input

➔ $T(n) = \Theta(n^2)$

Average-Case Running Time

So, Quicksort runs badly for some input…

But suppose that when we store a set of n numbers into the input array, each of the $n!$ permutations are equally likely

➔ Running time varies on input

What will be the "average" running time ?

Average Running Time

Let X = # comparisons in all Partition

Later, we will show that

Running time = $O(n + \boxed{X})$ → varies on input

Finding average of X (i.e. #comparisons) gives average running time

Our first target: Compute average of X

Average # of Comparisons

We define some notation to help the analysis:

- Let $a_1, a_2, \ldots, a_n$ denote the set of n numbers initially placed in the array

- Further, we assume $a_1 < a_2 < \ldots < a_n$ (So, a_1 may not be the element in A[1] originally)

- Let X_{ij} = # comparisons between a_i and a_j in all Partition calls

Average # of Comparisons

Then, X = # comparisons in all Partition calls

$= X_{12} + X_{13} + \ldots + X_{n-1,n}$

➔ Average # comparisons

$= E[X]$

$= E[X_{12} + X_{13} + \ldots + X_{n-1,n}]$

$= E[X_{12}] + E[X_{13}] + \ldots + E[X_{n-1,n}]$

Average # of Comparisons

The next slides will prove: $E[X_{ij}] = 2/(j-i+1)$

Using this result,

$$E[X] = \Sigma_{i=1 \text{ to } n-1} \Sigma_{j=i+1 \text{ to } n} 2/(j-i+1)$$

$$= \Sigma_{i=1 \text{ to } n-1} \Sigma_{k=1 \text{ to } n-i} 2/(k+1)$$

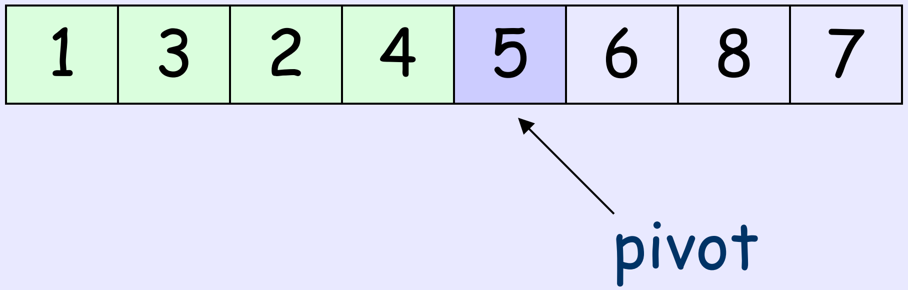
$$< \Sigma_{i=1 \text{ to } n-1} \Sigma_{k=1 \text{ to } n} 2/k$$

$$= \Sigma_{i=1 \text{ to } n-1} O(\log n) = O(n \log n)$$

Comparison between a_i and a_j

Question: # times a_i be compared with a_j ?

Answer: At most once, which happens only if a_i or a_j are chosen as pivot

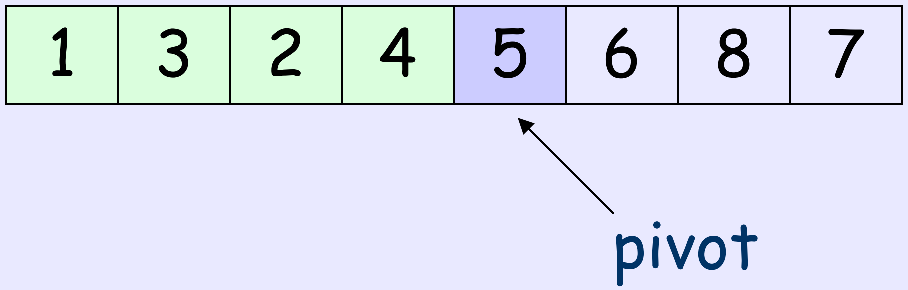


After that, the pivot is fixed and is never compared with the others

Comparison between a_i and a_j

Question: Will a_i always be compared with a_j ?

Answer: No. E.g., after Partition in Page 14:



we will separately Quicksort the first 4 elements, and then the last 3 elements

→ 3 is never compared with 8

Comparison between a_i and a_j

Observation:

Consider the elements $a_i, a_{i+1}, \ldots, a_{j-1}, a_j$

(i) If a_i or a_j is first chosen as a pivot,
then a_i is compared with a_j

(ii) Else, if any element of $a_{i+1}, \ldots, a_{j-1}$ is first chosen as a pivot,
then a_i is never compared with a_j

Comparison between a_i and a_j

When the n! permutations are equally likely to be the input,

$$\boxed{\begin{aligned}&\Pr(a_i \text{ compared with } a_j \text{ once}) = 2/(j-i+1)\\&\Pr(a_i \text{ not compared with } a_j) = (j-i-1)/(j-i+1)\end{aligned}}$$

$$\rightarrow E[X_{ij}] = 1 * 2/(j-i+1) + 0 * (j-i-1)/(j-i+1)$$
$$= 2/(j-i+1)$$

Consider a_i, a_{i+1}, …, a_{j-1}, a_j. Given a permutation, if a_i is chosen a pivot first, then by exchanging a_i with a_{i+1} initially, a_{i+1} will be chosen as a pivot first

Proof: Running time = $O(n+X)$

Observe that in the Quicksort algorithm:

- Each Partition fixes the position of pivot

  → at most n Partition calls

- After each Partition, we have 2 Quicksort
- Also, all Quicksort (except 1st one: Quicksort(A,1,n)) are invoked after a Partition

  → total $\Theta(n)$ Quicksort calls

Proof: Running time = $O(n+X)$

So, if we ignore the comparison time in all Partition calls, the time used = $O(n)$

Thus, we include back the comparison time in all Partition calls,

Running time = $O(n + X)$