Design and Analysis of Algorithms

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Supplement of Lecture 5: Probability & Expectation

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About this lecture

• What is Probability?
• What is an Event?
• What is a Random Variable?
• What is Expectation or “Average Value” of a Random Variable?
• Useful Thm: Linearity of Expectation
Experiment and Sample Space

• An experiment is a process that produces an outcome
• A random experiment is an experiment whose outcome is not known until it is observed
  - Exp 1: Throw a die once
  - Exp 2: Flip a coin until Head comes up
Experiment and Sample Space

- A **sample space** $\Omega$ of a random experiment is the set of all outcomes
  - Exp 1: **Throw a die once**
    - Sample space:  $\{ 1, 2, 3, 4, 5, 6 \}$
  - Exp 2: **Flip a coin until Head comes up**
    - Sample space:  ??

- Any subset of sample space $\Omega$ is called an **event**
Probability

- Probability studies the chance of each event occurring
- Informally, it is defined with a function $\Pr$ that satisfies the following:

1. For any event $E$, $0 \leq \Pr(E) \leq 1$
2. $\Pr(\Omega) = 1$
3. If $E_1$ and $E_2$ do not have common outcomes,
   \[ \Pr(E_1 \mid E_2) = \Pr(E_1) + \Pr(E_2) \]
Example

Questions:
1. Suppose the die is a fair die, so that
   \( Pr(1) = Pr(2) = \ldots = Pr(6). \)
   What is \( Pr(1) \)? Why?

2. Instead, if we know
   \( Pr(1) = 0.2, \ Pr(2) = 0.3, \ Pr(3) = 0.4, \)
   \( Pr(4) = 0.1, \ Pr(5) = Pr(6) = 0. \)
   What is \( Pr(\{1,2,4\}) \)?
Random Variable

Definition: A random variable $X$ on a sample space $\Omega$ is a function that maps each outcome of $\Omega$ into a real number. That is, $X: \Omega \to \mathbb{R}$.

Ex: Suppose that we throw two dice

$\Rightarrow \Omega = \{ (1,1), (1,2), \ldots, (6,5), (6,6) \}$

Define $X = \text{sum of outcome of two dice}$

$\Rightarrow X$ is a random variable on $\Omega$
Random Variable

• For a random variable $X$ and a value $a$, the notation

  \[ X = a \]

  denotes the set of outcomes $\omega$ in the sample space such that $X(\omega) = a$

  \[ X = a \] is an event

• In previous example,

  \[ X = 10 \] is the event \{(4,6), (5,5), (6,4)\}
Expectation

Definition: The expectation (or average value) of a random variable $X$, is

$$E[X] = \sum_i i \Pr(X=i)$$

Question:

- $X = \text{sum of outcomes of two fair dice}$
  
  What is the value of $E[X]$?

- How about the sum of three dice?
Let $X$ = sum of outcomes of two dice. The value of $X$ can vary from 2 to 12. So, we calculate:

$\Pr(X=2) = \frac{1}{36}, \Pr(X=3) = \frac{2}{36}, \Pr(X=4) = \frac{3}{36}, \ldots, \Pr(X=12) = \frac{2}{36},$

$$E[X] = 2*Pr(X=2) + 3*Pr(X=3) + \ldots + 11*Pr(X=11) + 12*Pr(X=12) = 7$$
Linearity of Expectation

Theorem: Given random variables $X_1, X_2, \ldots, X_k$, each with finite expectation, we have

$$E[X_1 + X_2 + \ldots + X_k] = E[X_1] + E[X_2] + \ldots + E[X_k]$$

Let $X =$ sum of outcomes of two dice.
Let $X_i =$ the outcome of the $i^{th}$ dice
What is the relationship of $X, X_1, \text{ and } X_2$?
Can we compute $E[X]$?
Let $X = \text{sum of outcomes of two dice.}$
Let $X_i = \text{the outcome of the } i^{\text{th}} \text{ dice}$

$\Rightarrow X = X_1 + X_2$

$\Rightarrow E[X] = E[X_1 + X_2] = E[X_1] + E[X_2]$

$= \frac{7}{2} + \frac{7}{2} = 7$

Can you compute the expectation of the sum of outcomes of three dice?