Design and Analysis of Algorithms

Instructor: Sharma Thankachan

Lecture 2: Growth of Function

About this lecture

- Introduce Asymptotic Notation
  - $\Theta(\ )$, $O(\ )$, $\Omega(\ )$, $o(\ )$, $\omega(\ )$

Dominating Term

Recall that for input size n,

- Insertion Sort 's running time is:

  $An^2 + Bn + C$, (A,B,C are constants)

- Merge Sort 's running time is:

  $Dn \log n + En + F$, (D,E,F are constants)

- To compare their running times for large n, we can just focus on the dominating term (the term that grows fastest when n increases)
  - An^2 vs $Dn \log n$

Dominating Term

- If we look more closely, the leading constants in the dominating term does not affect much in this comparison
  - We may as well compare n^2 vs $n \log n$ (instead of An^2 vs $Dn \log n$)
- As a result, we conclude that Merge Sort is better than Insertion Sort when n is sufficiently large

Asymptotic Efficiency

- The previous comparison studies the asymptotic efficiency of two algorithms
- If algorithm P is asymptotically faster than algorithm Q, P is often a better choice
- To aid (and simplify) our study in the asymptotic efficiency, we now introduce some useful asymptotic notation

Big-O notation

Definition: Given a function $g(n)$, we denote $O(g(n))$ to be the set of functions

$\{ f(n) \mid$ there exists positive constants c and n_0 such that

$$0 \leq f(n) \leq c\, g(n)$$

for all $n \geq n_0 \}$

Rough Meaning: $O(g(n))$ includes all functions that are upper bounded by $g(n)$

Big-O notation (example)

- $4n \in O(5n)$ [proof: c = 1, n ¸ 1]
- $4n \in O(n)$ [proof: c = 4, n ¸ 1]
- $4n + 3 \in O(n)$ [proof: c = 5, n ¸ 3]
- $n \in O(0.001n^2)$ [proof: c = 1, n ¸ 100]
- $\log_e n \in O(\log n)$ [proof: c = 1, n ¸ 1]
- $\log n \in O(\log_e n)$ [proof: c = log e, n ¸ 1]

Remark: Usually, we will slightly abuse the notation, and write $f(n) = O(g(n))$ to mean $f(n) \in O(g(n))$

Big-Omega notation

Definition: Given a function $g(n)$, we denote $\Omega(g(n))$ to be the set of functions

$\{ f(n) \mid$ there exists positive constants c and n_0 such that

$$0 \leq c\, g(n) \leq f(n)$$

for all $n \geq n_0 \}$

Rough Meaning: $\Omega(g(n))$ includes all functions that are lower bounded by $g(n)$

Big-O and Big-Omega

- Similar to Big-O, we will slightly abuse the notation, and write $f(n) = \Omega(g(n))$ to mean

  $f(n) \in \Omega(g(n))$

Relationship between Big-O and Big-Ω :

$$f(n) = \Omega(g(n)) \Leftrightarrow g(n) = O(f(n))$$

Big-Ω notation (example)

- $5n = \Omega(4n)$ [proof: c = 1, n ¸ 1]
- $n = \Omega(4n)$ [proof: c = 1/4, n ¸ 1]
- $4n + 3 = \Omega(n)$ [proof: c = 1, n ¸ 1]
- $0.001n^2 = \Omega(n)$ [proof: c = 1, n ¸ 100]
- $\log_e n = \Omega(\log n)$ [proof: c = 1/log e, n ¸ 1]
- $\log n = \Omega(\log_e n)$ [proof: c = 1, n ¸ 1]

Θ notation (Big-O \ Big-Ω)

Definition: Given a function $g(n)$, we denote $\Theta(g(n))$ to be the set of functions

$\{\ f(n) \mid$ there exists positive constants c_1, c_2, and n_0 such that

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$$

for all $n \geq n_0\ \}$

Meaning: Those functions which can be both upper bounded and lower bounded by of $g(n)$

Big-O, Big-Ω, and Θ

- Similarly, we write $f(n) = \Theta(g(n))$ to mean $f(n) \in \Theta(g(n))$

Relationship between Big-O, Big-Ω, and Θ:

$$f(n) = \Theta(g(n))$$

$$\Leftrightarrow$$

$$f(n) = \Omega(g(n)) \text{ and } f(n) = O(g(n))$$

Θ notation (example)

- $4n = \Theta(n)$ [$c_1 = 1, c_2 = 4, n$ ¸ 1]
- $4n + 3 = \Theta(n)$ [$c_1 = 1, c_2 = 5, n$ ¸ 3]
- $\log_e n = \Theta(\log n)$ [$c_1 = 1/\log e, c_2 = 1, n$ ¸ 1]
- Running Time of Insertion Sort = $\Theta(n^2)$
  - If not specified, running time refers to the worst-case running time
- Running Time of Merge Sort = $\Theta(n \log n)$

Little-o notation

Definition: Given a function $g(n)$, we denote $o(g(n))$ to be the set of functions

$\{ f(n) \mid$ for any positive c, there exists positive constant n_0 such that

$$0 < f(n) < c\, g(n)$$

for all $n \geq n_0 \}$

Note the similarities and differences with Big-O

Little-o (equivalent definition)

Definition: Given a function $g(n)$, $o(g(n))$ is the set of functions

$$\{ f(n) \mid \lim_{n \to \infty} \left(f(n)/g(n)\right) = 0 \}$$

Examples:

- $4n = o(n^2)$
- $n \log n = o(n^{1.000001})$
- $n \log n = o(n \log^2 n)$

Little-omega notation

Definition: Given a function $g(n)$, we denote $\omega(g(n))$ to be the set of functions

$$\{\, f(n) \mid \text{for any positive } c \text{, there exists positive constant } n_0 \text{ such that}$$

$$0 < c\,g(n) < f(n)$$

$$\text{for all } n \geq n_0 \,\}$$

Note the similarities and differences with the Big-Omega definition

Little-omega (equivalent definition)

Definition: Given a function $g(n)$, $\omega(g(n))$ is the set of functions

$$\{ f(n) \mid \lim_{n \to \infty} \left(g(n)/f(n) \right) = 0 \}$$

Relationship between Little-o and Little-ω :

$$f(n) = \omega(g(n)) \Leftrightarrow g(n) = o(f(n))$$

To remember the notation:

O is like $\leq$: $f(n) = O(g(n))$ means $f(n) \leq cg(n)$

Ω is like $\geq$: $f(n) = \Omega(g(n))$ means $f(n) \geq cg(n)$

Θ is like $=$: $f(n) = \Theta(g(n)) \Leftrightarrow g(n) = \Theta(f(n))$

o is like $<$: $f(n) = o(g(n))$ means $f(n) < cg(n)$

ω is like $>$: $f(n) = \omega(g(n))$ means $f(n) > cg(n)$

Note: Not any two functions can be compared asymptotically (E.g., $\sin x$ vs $\cos x$)

What's wrong with it?

Your friend, after this lecture, has tried to prove $1+2+\ldots+n = O(n)$

- His proof is by induction:
- First, $1 = O(n)$
- Assume $1+2+\ldots+k = O(n)$
- Then, $1+2+\ldots+k+(k+1) = O(n) + (k+1)$

$$= O(n) + O(n) = O(2n) = O(n)$$

So, $1+2+\ldots+n = + O(n)$ [where is the bug??]