Design and Analysis of Algorithms

Instructor: Sharma Thankachan

Lecture 1: Getting Started

Slides modified from Dr. Hon, with permission

About this lecture

- Study a few simple algorithms for sorting
  - Insertion Sort
  - Selection Sort
  - Merge Sort
- Show why these algorithms are correct
- Try to analyze the efficiency of these algorithms (how fast they run)

The Sorting Problem

Input: A list of n numbers

Output: Arrange the numbers in increasing order

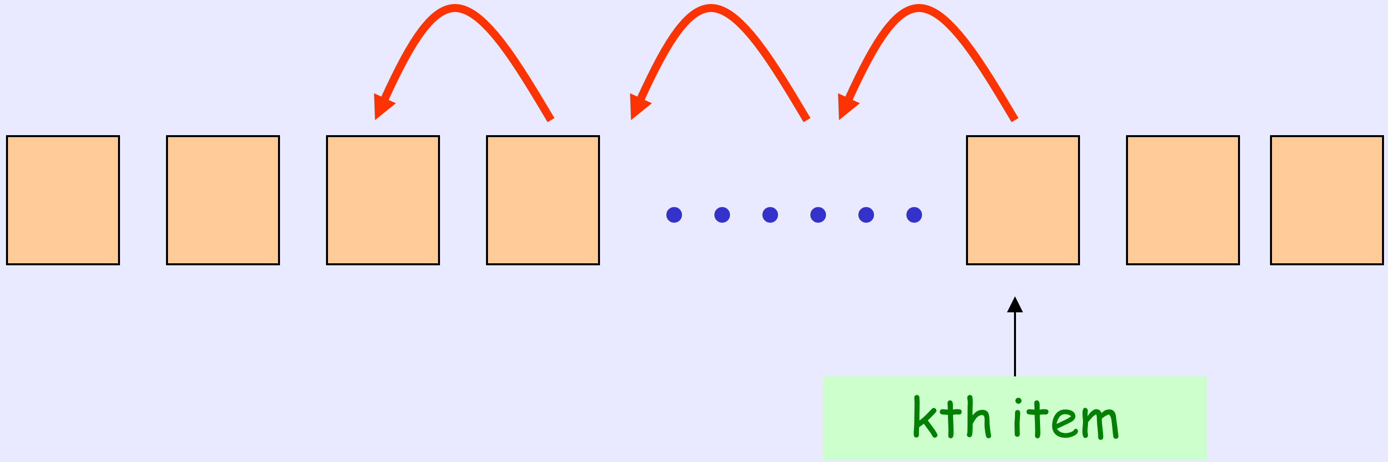
Remark: Sorting has many applications.

E.g., if the list is already sorted, we can search a number in the list faster

Insertion Sort

- Operates in n rounds
- At the k^{th} round,

Swap towards left side ;
Stop until seeing an item
with a smaller value.



Question: Why is this algorithm correct?

Selection Sort

- Operates in n rounds
- At the k^{th} round,
  - Find minimum item after $(k-1)^{th}$ position
  - Let's call this minimum item X
  - Insert X at k^{th} position in the list

Question: Why is this algorithm correct?

Divide and Conquer

- Divide a big problem into smaller problems
  → solve smaller problems separately
  → combine the results to solve original one
- This idea is called Divide-and-Conquer
- Smart idea to solve complex problems (why?)
- Can we apply this idea for sorting ?

Divide-and-Conquer for Sorting

- What is a smaller problem ?
  - ➔ E.g., sorting fewer numbers
  - ➔ Let's divide the list to two shorter lists
- Next, solve smaller problems (how?)
- Finally, combine the results
  - ➔ "merging" two sorted lists into a single sorted list (how?)

Merge Sort

- The previous algorithm, using divide-and-conquer approach, is called Merge Sort
- The key steps are summarized as follows:

Step 1. Divide list to two halves, A and B

Step 2. Sort A using Merge Sort

Step 3. Sort B using Merge Sort

Step 4. Merge sorted lists of A and B

Question: Why is this algorithm correct?

Analyzing the Running Times

- Which of previous algorithms is the best?
- Compare their running time on a computer
  - But there are many kinds of computers !!!

Standard assumption: Our computer is a RAM (Random Access Machine), so that

- each arithmetic (such as $+$, $-$, $\times$, $\div$), memory access, and control (such as conditional jump, subroutine call, return) takes constant amount of time

Analyzing the Running Times

- Suppose that our algorithms are now described in terms of RAM operations

  → we can count # of each operation used

  → we can measure the running time !

- Running time is usually measured as a function of the input size
  - E.g., n in our sorting problem

Insertion Sort (Running Time)

The following is a pseudo-code for Insertion Sort.
Each line requires constant RAM operations.

INSERTION-SORT(A)	*cost*	*times*
for $j \leftarrow 2$ **to** $length[A]$	c_1	n
do $key \leftarrow A[j]$	c_2	$n-1$
$\triangleright$ Insert $A[j]$ into the sorted sequence $A[1\,..\,j-1]$.	0	$n-1$
$i \leftarrow j-1$	c_4	$n-1$
while $i > 0$ and $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$
do $A[i+1] \leftarrow A[i]$	c_6	$\sum_{j=2}^{n} (t_j - 1)$
$i \leftarrow i-1$	c_7	$\sum_{j=2}^{n} (t_j - 1)$
$A[i+1] \leftarrow key$	c_8	$n-1$

t_j = # of times *key* is compared at round j

Insertion Sort (Running Time)

- Let $T(n)$ denote the running time of insertion sort, on an input of size n
- By combining terms, we have

$$T(n) = c_1 n + (c_2+c_4+c_8)(n-1) + c_5 \Sigma\, t_j + (c_6+c_7)\, \Sigma\, (t_j - 1)$$

- The values of t_j are dependent on the input (not the input size)

Insertion Sort (Running Time)

- Best Case:

  The input list is sorted, so that all $t_j = 1$

  Then, $T(n) = c_1n + (c_2+c_4+c_5+c_8)(n-1)$

  $= Kn + c$ ➔ linear function of n

- Worst Case:

  The input list is sorted in decreasing order, so that all $t_j = j-1$

  Then, $T(n) = K_1n^2 + K_2n + K_3$

  ➔ quadratic function of n

Worst-Case Running Time

- In our course (and in most CS research), we concentrate on worst-case time
- Some reasons for this:

1. Gives an upper bound of running time
2. Worst case occurs fairly often

Remark: Some people also study average-case running time (they assume input is drawn randomly)

Try this at home

- Revisit pseudo-code for Insertion Sort
  - make sure you understand what's going on
- Write pseudo-code for Selection Sort

Merge Sort (Running Time)

The following is a partial pseudo-code for Merge Sort.

```
MERGE-SORT(A, p, r)
  if p < r
     then q ← ⌊(p + r)/2⌋
          MERGE-SORT(A, p, q)
          MERGE-SORT(A, q + 1, r)
          MERGE(A, p, q, r)
```

The subroutine MERGE(A,p,q,r) is missing.

Can you complete it?

Hint: Create a temp array for merging

Merge Sort (Running Time)

- Let $T(n)$ denote the running time of merge sort, on an input of size n
- Suppose we know that Merge( ) of two lists of total size n runs in $c_1 n$ time
- Then, we can write $T(n)$ as:

$$T(n) = 2T(n/2) + c_1 n + c_2 \qquad \text{when } n > 1$$
$$T(n) = c_3 \qquad \text{when } n = 1$$

- Solving the recurrence, we have
- $T(n) = K_1 \, n \log n + K_2 \, n + K_3$

Which Algorithm is Faster?

- Unfortunately, we still cannot tell
  - since constants in running times are unknown
- But we do know that if n is VERY large, worst-case time of Merge Sort must be smaller than that of Insertion Sort
- Merge Sort is asymptotically faster than Insertion Sort