Global-Stabilizing Near-Optimal Control Design for Nonholonomic Chained Systems

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***Abstract*— It is well known that a nonlinear optimal control requires the solution to a two-point boundary value problem or to a nonlinear partial differential equation and that such a solution can only be obtained off line by numerical iteration. In this paper, a new and near-optimal control design framework is proposed for controlling any nonholonomic system in the chained form. The proposed design is based upon thorough study of uniform complete controllability of the corresponding linear time varying nominal system. It is shown that, no matter whether the first component $u_{1d}(t)$ of reference input vector is uniformly nonvanishing or simply nonconvergent to zero or vanishing or identically zero, uniform complete controllability of the (nominal) system can be recovered by employing the proposed time-folding/unfolding technique. Upon establishing the common property of uniform complete controllability, the proposed framework can be used to design both trajectory tracking control and regulation control in a systematic and unified manner. Using duality, uniform complete observability can also be established, a closed-form and exponentially convergent observer can be synthesized, and the controls designed using the proposed framework can be either state-feedback or output-feedback. The tracking controls are designed using the same 3-step process. That is, design of the proposed controls starts with optimal control solutions to two linear nominal subsystems, one time-invariant and the other time varying. The two solutions combined together render a globally stabilizing suboptimal control for the overall system. Then, the optimality condition is invoked to determine the distance between the suboptimal control and the optimal one. Consequently, an improved control can be obtained by introducing a nonlinear additive control term in such a way that the distance aforementioned is minimized as much as possible in closed form. An example is used to show that regulation control can be designed similarly. All the controls designed are in simple closed forms and hence real-time implementable, they are time varying and smooth, globally and exponentially/asymptotically stabilizing, and they are near optimal since their closeness to the optimal control (attainable only off-line) can be measured, monitored on line, and has been minimized.**

***Index Terms*— Nonholonomic chained systems, near-optimal control, state feedback, output feedback, tracking control.**

I. Introduction

Control of nonholonomic systems has received a great deal of attention [1], and many designs have been proposed. It has been shown in [2] that mechanical systems with nonholonomic constraints can be either locally or globally converted to the so-called chained form under a coordinate transformation and a control mapping. As a result, the chained form has been used as a canonical form in analysis and control design for nonholonomic systems. The chained form is also equivalent to the so-called power canonical form [3] and skew-symmetric chained form [4], and their dynamic extension has been explored in [5].

One of the main reasons of continuing research interests is that, by Brockett's theorem [6], nonholonomic systems cannot be asymptotically stabilized around a fixed point under any smooth (or even continuous) time-independent state feedback control law. Consequently, there has been a divide between the control design of making the system track a desired trajectory and the design of stabilizing the system around a point, and different approaches have been used to tackle each of the two problems or their combination.

The problem of regulation control (or posture stabilization) is to stabilize a constrained system at any given point in the state space. One line of the research efforts is to devise time-implicit but discontinuous feedback control laws, and the most notable among them is the time-invariant coordinates and control transformations (also known as the σ-process) proposed in [7]. The transformations are well defined everywhere except on the hyperplane of $x_1 = 0$ and, off the singularity hyperplane, they map the chained form into a linear time-invariant system by which stabilization can easily be achieved. If the initial condition happens to be on the hyperplane, a separate control law is designed to drive the state off the plane. Hence, the final control contains two separate laws and is discontinuous. The switching control guarantees exponential stability but, besides being discontinuous, contains terms such as $x_i(t)/x_1^{i-2}(t)$ which may be excessively large around the singularity hyperplane. In [8], the switching control law is re-designed using an algebraic Riccati equation related to the time-invariant linear system after the transformations. To avoid excessively large value in the neighborhood of the singularity hyperplane, the control law is revised in [9] to be explicitly bounded, and the resulting stability becomes semi-global. Instead of using the σ-process, discontinuous stabilizing control can also be designed using invariant manifold and sliding mode techniques, with acceleration feedback [10], for a third order system [11], and for high-order systems [12].

Posture stabilization can also be achieved under time-varying continuous controls. In particular, time-varying center manifold, averaging transformation, and control saturation are

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used in [3]; skew-symmetric chained form, Barbalat lemma and Lyapunov-like argument are employed in [4]; periodic systems and its Lyapunov argument are utilized in [13]; and a system argumentation and a modified σ transformation are developed in [14]; all to achieve global asymptotic stabilization. It is shown in [15], [16] that asymptotic stabilizing control can be made to be so-called ρ-exponentially stabilizing. A stabilizing control is also developed for a third-order model of wheeled robots in [17]. In addition, hybrid designs of combining time-varying and switching control laws are pursued in [18], [5], [19]. Recently, a robust control design is proposed in [20] to ensure practical stability for perturbed systems whose nominal systems are driftless. Besides its ability of dealing with perturbation terms, the design in [20] is novel and interesting because controllability is used to select the so-called bounded transverse functions whose trajectories lie in a neighborhood of the origin and to which the system trajectory converges. In essence, the result in [20] bridges tracking and stabilization problems by exploring controllability and by choosing transverse functions (while adopting the requirement of practical stability), and the general ideas of exploring controllability and trajectory and of bridging the design problems are very relevant to this paper.

The problem of trajectory tracking is generally different from the regulation problem as the reference input vector (in particular, its first element $u_{1d}(t)$) is not or does not converge to zero in general. Most of the existing results explore this property to avoid the loss of linear controllability at the origin, and hence control design for the tracking problem is somewhat less challenging than that for regulation. In [21], a locally exponentially stabilizing control is proposed for trajectory tracking using the standard linearization technique and under the assumption that the linearized system is uniformly completely controllable along the desired trajectory. Using the backstepping method [22], trajectory tracking control can be designed to ensure semi-global asymptotic stability by employing a high-gain feedback [23], and stability can made global for a 3-order model of wheeled robot [24] or for line tracking [23]. Recently, tracking control designed using the backstepping method is shown to also ensure global asymptotic stability [25], whereas exponential stability is established only for slowly-changing reference trajectories. It is also shown in [26] that a linear time varying control can ensure global exponential stability if $u_{1d}(t)$ is continuously differentiable, non-vanishing, and Lipschitz with respect to time. For a third-order model of wheeled robot, local stability is shown under persistent reference motion in both x and y directions [27]; global exponential stability is established under the assumption that the reference trajectory satisfies a persistent excitation like condition [28]; and global asymptotic stability is obtained under three conditions on reference linear and angular velocities [29].

In practice, output feedback control is preferred to state feedback control, and there have been a few results available so far. In [26], a linear time varying output feedback tracking control is proposed to ensure global exponential stability again under the conditions that reference input $u_{1d}(t)$ is continuously differentiable, non-vanishing, and Lipschitz with respect to time. In [30], [25], an output feedback tracking control is designed under the conditions that $u_{1d}(t)$ satisfies a persistent excitation condition and is differentiable up to $(n-2)$ order, the control becomes a switch control if $u_{1d}(t)$ changes its sign, and the stability is claimed to be almost everywhere. There is no result available on time-varying smooth output-feedback control for regulation.

The results aforementioned present significant advances, but there are several fundamental issues that have not been adequately addressed. First, several sufficient conditions have been proposed for tracking control design, but there has not been any report on thorough study of (uniform complete) controllability for chained systems along a desired trajectory or a system trajectory. It is necessary to determine whether uniform complete controllability can be ensured for various types of trajectories. The issue becomes much more acute for the regulation control problem as it is well known that chained systems are nonlinearly controllable but not linearly controllable at the origin. The interesting question is whether the intrinsic nonlinear controllability of chained systems can be explicitly revealed and retained somehow in terms of linear controllability (through transformation) in order to make regulation control design parallel to tracking control design. Second, because of the lack of sufficient understanding in controllability, there is no unifying framework by which various controls (tracking and regulation controls as well as state feedback and output feedback controls) can be designed in a systematic manner in order to achieve asymptotic stability. For observer-based output feedback control designs, uniform complete observability is required, and little has been done to analyze the property for chained systems. Third, for both theoretical and practical reasons, it is desirable to obtain controls that are smooth, have simple closed-form expressions, and ensure best performance possible. It follows from Brockett's theorem [6] that time varying smooth control laws would be the only type of choices. It also well known that optimal control laws are generally time varying and smooth. Thus, it is fundamentally interesting to study whether closed-form time varying smooth controls can be designed for chained systems to yield the best performance achievable real time for both tracking and regulation, which is the main thrust of the proposed near optimal control design framework.

In this paper, a unifying design framework is proposed based on both Lyapunov direct method and nonlinear optimal control theory. In order to find an appropriate Lyapunov function for both state and output feedback designs, uniform complete controllability of chained systems along a desired trajectory is investigated, and the simple condition of $u_{1d}(t)$ being uniformly nonvanishing (which by itself is already less restrictive than those in the existing results such as [26]) is found. More importantly, it is shown that, using the so-called time-folding/unfolding technique, uniform complete controllability can be retained by transformation if $u_{1d}(t)$ is merely nonconvergent to zero or even vanishing. For the tracking problem, uniform complete controllability can always be ensured, and hence a Lyapunov function is found in terms of a differential Riccati equation and for both cases of state feedback and output feedback. For the stabilization problem, while uniform complete controllability is inherently absent for the original (linear time varying nominal) system, intrinsic controllability of chained systems is exposed by judiciously designing the input component $u_1(t)$ and by applying time/state transforms so that the transformed system becomes uniformly completely controllable in order to proceed with control design. This innovative transformation makes the proposed design framework applicable in a systematic and parallel way to both tracking and stabilization problems. On the other hand, the proposed design framework is to generate time-varying smooth controls in simple closed forms and to guarantee the so-called near optimal performance in addition to global and asymptotic/exponential stability. The basic idea of near-optimality is that, although optimal control for nonlinear systems such as those in the chained form can only be solved iteratively and off-line, many suboptimal controls can be found, that the standard optimality condition can be restated as a distance measure (called the optimality residue) between the suboptimal control and the (*unattainable*) optimal one, and that the residue can be minimized. Specifically, the framework applies to the tracking designs in three steps. First, utilizing the structure of chained systems, two linear optimal controls are designed for the two linear nominal subsystems of corresponding error dynamics. In the second step, it is shown that combining the two individually-optimal controls yields a globally stabilizing suboptimal control for the overall chained system. In the third step, a nonlinear additive control term is analytically synthesized so that the resulting control becomes near optimal in the sense that the corresponding optimality residue is minimized. In short, the proposed framework can be used to design asymptotically stabilizing controls for tracking and stabilization and for the cases of state feedback and output feedback, and the resulting controls are in simple closed forms, time varying and smooth, globally asymptotically stabilizing, and of near-optimal performance.

The paper is organized as follows. In section II, the problems of tracking and regulation are formulated together, the basic idea of near optimal control is motivated, necessary properties such as structural property of chained systems, uniform complete controllability, and uniform complete observability are discussed and established, and design steps of the proposed near optimal framework are provided. An example is used to show that, by designing $u_1(t)$ first and then applying appropriate transformation to recover uniform complete controllability, stabilization control design becomes parallel to trajectory tracking control design. In section III, the framework is illustrated by the design of a state feedback near optimal tracking control. In section IV, the framework is applied to synthesize an output feedback near optimal control by incorporating a closed-form exponentially convergent observer. In section V, simulation results of a car-like mobile robot are presented to illustrate the proposed near optimal controls, and their superior performance is validated through comparisons. In section VI, brief conclusions are drawn.

II. Problem Formulation

The class of nonholonomic chained systems studied in this paper are of form:

$$\dot{x}_1 = u_1, \;\; \dot{x}_2 = x_3 u_1, \;\; \cdots \;\; \dot{x}_{n-1} = x_n u_1, \;\; \dot{x}_n = u_2, \quad (1)$$

where $x = [x_1, \cdots, x_n]^T \in \Re^n$ is the state, $u = [u_1, u_2]^T \in \Re^2$ is the control input, and $y = [x_1, x_2]^T \in \Re^2$ is the output. For trajectory tracking, the desired trajectory to be followed is given by:

$$\begin{cases} \dot{x}_{1d} = u_{1d}, \\ \dot{x}_{2d} = x_{3d} u_{1d}, \\ \vdots \\ \dot{x}_{(n-1)d} = x_{nd} u_{1d}, \\ \dot{x}_{nd} = u_{2d}, \end{cases} \quad (2)$$

where $x_d = [x_{1d}, \cdots, x_{nd}]^T \in \Re^n$, $y_d = [x_{1d}, x_{2d}]^T \in \Re^2$, $u_d(t) = [u_{1d}(t), u_{2d}(t)]^T \in \Re^2$ is the time-varying reference input (i.e., open-loop steering control) that is assumed to be uniformly bounded. In the tracking control design, property of $u_d(t)$ is explored and utilized. For regulation/stabilization, reference input u_d is identically zero, and the corresponding analysis and design can be done directly in terms of $u_1(t)$ (as will be shown by example 4 in section II-F).

Chained system (1) has the nice property that its vector fields are left-invariant with respect to a Lie group operation. It is given in [20] that, for any vectors $\vartheta, \varrho \in \Re^n$, their operation of Lie group product is $\vartheta * \varrho$, where $(\vartheta * \varrho) \in \Re^n$,

$$(\vartheta * \varrho)_1 = \vartheta_1 + \varrho_1, \quad (\vartheta * \varrho)_n = \vartheta_n + \varrho_n,$$

and, for $i = n-1, n-2, \cdots, 2$,

$$(\vartheta * \varrho)_i = \vartheta_i + \varrho_i + \sum_{j=i+1}^{n} \frac{\varrho_1^{j-i}}{(j-i)!} \vartheta_j .$$

It is elementary to verify that the identity element in the Lie group is $\emptyset \stackrel{\triangle}{=} 0 \in \Re^n$ in the sense that $\emptyset * \varrho = \varrho * \emptyset$ holds for all ϱ. Accordingly, the group inverse of x_d, denoted by $x_d^{-1} \stackrel{\triangle}{=} \left[\begin{array}{cccc} x_{1d}^{-1} & x_{2d}^{-1} & \cdots & x_{nd}^{-1} \end{array} \right]^T$ and defined by $x_d^{-1} * x_d = x_d * x_d^{-1} = \emptyset$, is found to be

$$x_{1d}^{-1} = -x_{1d}, \quad x_{nd}^{-1} = -x_{nd},$$

and, for $i = n-1, \cdots, 2$,

$$x_{id}^{-1} = -x_{id} - \sum_{j=i+1}^{n} \frac{x_{1d}^{j-i}}{(j-i)!} x_{jd}^{-1} .$$

Using the group operation, we can define the state tracking error between (1) and (2) as $x_e \stackrel{\triangle}{=} x_d^{-1} * x$, that is, $x_e = [x_{1e}, \cdots, x_{ne}]^T$, $x_{1e} = -x_{1d} + x_1$, $x_{ne} = -x_{nd} + x_n$, and for $i = n-1, \cdots, 2$,

$$x_{ie} = -x_{id} + x_i + \sum_{j=i+1}^{n} \frac{x_1^{j-i}}{(j-i)!} x_{jd}^{-1} - \sum_{j=i+1}^{n} \frac{x_{1d}^{j-i}}{(j-i)!} x_{jd}^{-1} .$$

In addition, let's denote the output tracking error by $y_e = [x_{1e}, x_{2e}]^T$ and the feedback control (to be designed) by $v =$

$[v_1, v_2]^T \triangleq u - u_d$. Then, it is straightforward to verify that the corresponding error system is

$$\begin{aligned} \dot{x}_e &= A(u_{1d}(t))x_e + [B + G(x_e)]v + F(x_{1e}, u_{2d})x_{1e}, \\ y_e &= Cx_e, \end{aligned} \quad (3)$$

where

$$A(u_{1d}(t)) = \begin{bmatrix} 0 & 0 & 0 & \cdots & & 0 \\ 0 & 0 & u_{1d}(t) & 0 & \cdots & 0 \\ 0 & 0 & 0 & u_{1d}(t) & \cdots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & & u_{1d}(t) \\ 0 & 0 & 0 & \cdots & & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad G(x_e) = \begin{bmatrix} 0 & 0 \\ x_{3e} & 0 \\ \vdots & \vdots \\ x_{ne} & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$F(x_{1e}, u_{2d}) = \begin{bmatrix} 0 \\ -(n-2)!u_{2d}x_{1e}^{n-3} \\ -(n-3)!u_{2d}x_{1e}^{n-4} \\ \vdots \\ -u_{2d} \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix}.$$

In the paper, two types of controls will explicitly be constructed: state-feedback trajectory tracking control and output-feedback tracking control. In addition, the design process of regulation/stabilization control is outlined by an example. The design objective is that, for system (3), all three control strategies are in closed-form for real-time implementation, achieve near optimality (the best achievable real-time), and ensure global asymptotic stability of x_e. In short, a new near-optimal control design framework is developed.

The technical development in the rest of this section is as follows. In subsection II-A, several facts related to the chained form and its error dynamics are summarized. In subsection II-B, the basic idea of near-optimal control is introduced. Structural properties of error system (3) are discussed in subsection II-C to illustrate the main steps of near-optimal control design. Controllability needed for the trajectory tracking control design is studied in subsection II-D. Observability needed for the design of output feedback control design is briefed in subsection II-E. In subsection II-F, the condition for uniform complete controllability/observability is shown to hold under various choices of $u_{1d}(t)$ (or $u_1(t)$) for both trajectory tracking and stabilization problems.

A. Models of Nonholonomic Systems and Their Tracking Error Dynamics

It is well known that many nonholonomic systems can be transformed into the chained form by coordinate transformations [2]. In order to ensure wide applicability of our proposed control design, we choose to demonstrate the proposed control design framework using the $(n, 2)$ chained form in (1). It is straightforward to see that all the results apply directly to the class of (n, m) chained systems:

$$\begin{aligned} &\dot{x}_1 = u_1; \;\; \dot{x}_{j,n_j} = u_{j+1}, \\ &\dot{x}_{j,i} = u_1 x_{j,i+1}, \;\; 2 \le i \le n_j - 1, \; 1 \le j \le m-1, \end{aligned}$$

where $[x_1, X_2^T, \ldots, X_m^T]^T \in \Re^n$ with $X_j = [x_{j-1,2}, \ldots, x_{j-1,n_{j-1}}]^T$ are sub-states for $2 \le j \le m$, and $u = [u_1, u_2, \ldots, u_m]^T$ is the input vector. The only difference is that, analogous to the decomposition into two subsystems to be developed in subsection II-C, the resulting error system (3) of the above (m, n) chained model contains m subsystems.

Alternative models equivalent to the chained form can be employed for nonholonomic systems. For instance, it is shown in [3] that chained form (1) can be transformed to the so-called power form:

$$\dot{\varphi}_1 = u_1, \; \dot{\varphi}_2 = u_2, \; \dot{\varphi}_3 = \varphi_1 u_2, \; \cdots \; \dot{\varphi}_n = \frac{1}{(n-2)!}\varphi_1^{n-2}u_2,$$

which also has its own dynamic extensions [5].

If the definition of tracking error is properly modified, the resulting error dynamics are different but retain all the important properties so that the proposed design can be applied successfully. Should the conventional choice of tracking error $x_e \triangleq x - x_d$ is made, the error system would be the same as (3) except that $F = 0$ and

$$G = \begin{bmatrix} 0 & x_{3e} + x_{3d} & \cdots & x_{ne} + x_{nd} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

Since the above matrix G explicitly depends on x_{3d} up to x_{nd}, the subsequent control design and stability analysis would require their uniform boundedness. The use of Lie group operation removes this requirement.

Another advantage of using the group operation is that, for left-invariant control systems, tracking error dynamics of form (3) can be derived directly from their original equations and that transformation to the chained form is not necessary [20]. This means that the proposed design is not confined to the chained form and can be applied directly to nonholonomic systems of left-invariant vector fields. Nonetheless, a transformation is generally needed to render error system (3) and, by using the inverse of the transformation, performance measures used in the control design (such as index (4) in subsection II-B) can be expressed in terms of physical variables and hence have explicit physical meanings.

In applications to robotic vehicles, both kinematic constraints and robotic dynamics need to be considered in the control design. It is straightforward to show that, upon taking care of nonholonomic constraints, dynamic equations can be transferred into a reduced-order vector differential equation of v and $\dot{v}$. Then, the proposed near optimal design for v can be extended to a torque-level control by using standard methods such as backstepping design. Since those standard designs are available and effective for dealing with unconstrained dynamic equations as shown in [31] and references therein, we choose in this paper to focus upon the kinematic control problem of constrained systems.

B. Necessity of Near-Optimal Control

To synthesize a performance-oriented control for chained system (3), we can begin with the nonlinear optimal control theory [32], [33]. Consider the following cost functional:

$$J(t, x_e(t), v(t)) = \frac{1}{2}\int_{t_0}^{\infty} [x_e^T C^T Q(t) C x_e + v^T R(t) v] dt, \quad (4)$$

where matrices $Q(t)$ and $R(t)$ can be freely chosen by the designer as long as they are uniformly bounded as $0 < \underline{q} I \leq Q(t) \leq \overline{q} I$ and $0 < \underline{r} I \leq R(t) \leq \overline{r} I$. Then, conditions for optimality can be found using the calculus of variations. That is, given Hamiltonian H as

$$\begin{aligned} H &= \frac{1}{2} x_e^T C^T Q(t) C x_e + \frac{1}{2} v^T R(t) v \\ &\quad + \lambda^T \{A(t) x_e + [B + G(x_e)] v + F(x_{1e}, u_{2d}) x_{1e}\}, \end{aligned}$$

where $\lambda \in \Re^n$ is the Lagrangian multiplier, necessary conditions for optimality are [32]:

$$\dot{x}_e = \frac{\partial H}{\partial \lambda}, \quad \frac{\partial H}{\partial v} = 0, \quad \text{and} \quad \dot{\lambda} = -\frac{\partial H}{\partial x_e}.$$

It follows that condition $\dot{x}_e = \partial H / \partial \lambda$ is always satisfied and that condition $\partial H / \partial v = 0$ is guaranteed by the optimal control candidate

$$v^* = -R^{-1}(t) [B + G(x_e)]^T P^*(t, x_e) x_e, \quad (5)$$

where function $P^*(t, x_e)$ is a matrix parameterization of the optimal Lagrangian multiplier as $\lambda^* = P^*(t, x_e) x_e$. Thus, control (5) meets all first-order necessary conditions if matrix $P^*(t, x_e)$ is selected according to

$$\left.\frac{d\lambda}{dt}\right|_{\lambda=\lambda^*} = -\left.\frac{\partial H}{\partial x_e}\right|_{P=P^*},$$

or simply,

$$E^*(x_e) \triangleq E(x_e)\Big|_{P=P^*} = 0, \quad (6)$$

where $\text{vec}[\eta_i] \triangleq \begin{bmatrix} \eta_1^T & \cdots & \eta_n^T \end{bmatrix}^T$, and $E(x_e)$ is given by the double-column formula (7). Consequently, equation (6) is often referred to as the optimality condition, and $\|E(x_e)\|^2$ will be called *residue* from the optimality condition. Since the optimal value of residue is zero (i.e., $E^*(x_e) = 0$), the optimal control design can be interpreted as the problem of minimizing the residue.

If optimal control (5) were pursued, matrix $P^*(t, x_e)$ would have to be solved from (nonlinear) differential equation (6) with boundary conditions $x_e(t_0)$ and $P(\infty, x_e(\infty)) > 0$, which could be done only off-line through numerical iterations. Thus, the resulting optimal control (5) is not solvable in real time or practical for most applications. To overcome this fundamental limitation of optimal control and to achieve better performance, we propose a near-optimal control design which, according to the aforementioned discussion on (6), can be characterized as the problem of finding a closed-form control similar to (5) such that its associated residue is minimized. Steps of the proposed near-optimal control design will be presented in section II-C by exploiting properties of the error system.

C. Structural Properties of the Error System

It is obvious from (3) that

$$A = \text{diag}\{A_1, A_2\}, \; B = \text{diag}\{B_1, B_2\}, \; C = \text{diag}\{C_1, C_2\},$$

$$G = \begin{bmatrix} 0 & 0 \\ G_2 & 0 \end{bmatrix}, \text{ and } F = \begin{bmatrix} 0 \\ F_2 \end{bmatrix},$$

where

$$A_1 = 0, \;\; B_1 = 1, \;\; C_1 = 1, \;\; B_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T,$$

$$C_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}, \;\; A_2(u_{1d}(t)) = u_{1d}(t) A_2^*,$$

$$A_2^* \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & & 1 \\ 0 & 0 & \cdots & & 0 \end{bmatrix}, \quad G_2(z) = \begin{bmatrix} z_2 \\ z_3 \\ \vdots \\ z_{n-1} \\ 0 \end{bmatrix},$$

$$\text{and} \quad F_2(x_{1e}, u_{2d}) = \begin{bmatrix} -(n-2)! u_{2d} x_{1e}^{n-3} \\ -(n-3)! u_{2d} x_{1e}^{n-4} \\ \vdots \\ -u_{2d} \\ 0 \end{bmatrix}. \quad (8)$$

Therefore, error dynamics in (3) can be partitioned into the following two subsystems:

$$\dot{x}_{1e} = A_1 x_{1e} + B_1 v_1, \quad y_{1e} = C_1 x_{1e}, \quad (9)$$

and

$$\begin{aligned} \dot{z} &= A_2(u_{1d}(t)) z + B_2 v_2 + G_2(z) v_1 \\ &\quad + F_2(x_{1e}, u_{2d}) x_{1e}, \\ y_{2e} &= C_2 z, \end{aligned} \quad (10)$$

where $z = [z_1, \cdots, z_{n-1}]^T \triangleq [x_{2e}, \cdots, x_{ne}]^T \in \Re^{n-1}$. The decomposition into subsystems (9) and (10) yields two useful properties. First, subsystem (9) is of first order, linear, time-invariant, and independent of subsystem (10). Subsystem (10) is nonlinear but has a linear time varying nominal system defined by

$$\dot{z} = A_2(u_{1d}(t)) z + B_2 v_2, \quad y_{2e} = C_2 z. \quad (11)$$

Second, coupling from subsystems (9) to (10) is through $[G_2(z) v_1 + F_2(x_{1e}, u_{2d}) x_{1e}]$, the nonlinear terms in the system. Utilizing these structural properties, the proposed near-optimal control design will be carried out by the following three steps:

Step 1: Determine closed-form optimal controls $v_{l,1}^*$ and $v_{l,2}^*$ for linear subsystem (9) and linear time varying nominal system (11), respectively.

Step 2: Design a sub-optimal control v_{so} to ensure exponential stability of the overall system (3). Specifically, $v_{so,1} = v_{l,1}^*$ and $v_{so,2} = v_{l,2}^*$ are shown to be the proper choices.

$$E(x_e) \triangleq \frac{d\lambda}{dt} + \frac{\partial H}{\partial x_e} = \frac{\partial P}{\partial t} x_e + \text{vec}\left[x_e^T \frac{\partial P}{\partial x_{ie}}\right] \{[A - (B+G)R^{-1}(B+G)^T P]x_e + F x_{1e}\} + [P^T A + A^T P - P^T (B+G) R^{-1} (B+G)^T P + C^T Q C] x_e + P F x_{1e} - \text{vec}\left[x_e^T P^T (B+G)(R^{-1})^T \frac{\partial G^T}{\partial x_{ie}} P x_e\right] + \text{vec}\left[\frac{\partial (F x_{1e})^T}{\partial x_{ie}} P x_e\right]. \quad (7)$$

Step 3: Synthesize a near-optimal control of form $v_{no,i} = v_{so,i} + v_{na,i}$, where $i = 1, 2$ and $v_{na,i}$ are closed-form, nonlinear additive control terms chosen to minimize the corresponding residue from the optimality condition and to ensure exponential stability of the overall system.

In essence, the proposed near-optimal control design starts with a linear optimal control for linear dynamics and then chooses a nonlinear additive control to compensate for nonlinear dynamics, to minimize the optimality residue, and to ensure exponential stability. A design of linear optimal control $v_{l,2}^*$ calls for controllability and observability of linear time varying nominal system (11), and these two fundamental issues are the topics of subsections II-D and II-E, respectively.

D. Controllability of Linear Time Varying Nominal Subsystem

In order to solve an infinite-time state-feedback optimal control problem for linear time-varying system (11), uniform complete controllability of pair $\{A_2(u_{1d}(t)),\ B_2\}$ needs to be established. Below is the standard definition adopted from [34].

Definition 1: [34] Let $W_c(t_0, t_f)$ and $\Phi(t, t_0)$ denote controllability Grammian and open-loop state transition matrix of system (11), i.e.,

$$\begin{aligned} W_c(t_0, t_f) &= \int_{t_0}^{t_f} \Phi(t_0, \tau) B_2 B_2^T \Phi^T(t_0, \tau) d\tau, \\ \dot{\Phi}(t, t_0) &= A_2(u_{1d}(t)) \Phi(t, t_0). \end{aligned} \quad (12)$$

Then, system (11) is uniformly completely controllable if the following two inequalities hold for all t:

$$0 < \alpha_{c1}(\delta) I \le W_c(t, t+\delta) \le \alpha_{c2}(\delta) I,\ \|\Phi(t, t+\delta)\| \le \alpha_{c3}(\delta),$$

where $\delta > 0$ is a fixed constant, and $\alpha_{ci}(\cdot)$ are fixed positively valued functions.

The following simple assumption is introduced to establish all of the properties needed in the tracking control designs (including controllability and observability properties, solution of Lyapunov function, and global exponential stability). It should be noted that, for the trajectory tracking problem in general, the assumption can be made without loss of any generality and that, if $u_{1d}(t)$ vanishes over time, the tracking problem reduces to the regulation problem for which a smooth time varying control can be designed without any assumption (as will be illustrated by an example in section II-F). Should an open-loop reference input $u_d(t)$ be selected to be discontinuous, right continuous steering time functions are typical choices. It is straightforward to see that lemma 1 also holds if right continuity in assumption 1 is replaced by either left continuity or semi-continuity (either upper or lower). However, lemma 1 no longer holds if $u_{1d}(t)$ is piecewise continuous, uniformly bounded, and uniformly nonvanishing. For example, consider $u_{1d}(t) = \sum_i \delta(t - iT_s)$, where $T_s > 0$ is a constant, $\delta(t - T_s) = 1$ at $t = T_s$, and $\delta(t - T_s) = 0$ at $t \neq T_s$. In this case, it follows from the proof of lemma 1 that $W_c(t_0, t_f) = B_2 B_2^T (t_f - t_0)$ is singular.

Definition 2: A time function $w(t) : [t_0, \infty) \to R$ is said to be uniformly right continuous if, for every $\epsilon > 0$, there exists $\eta > 0$ such that $t \le s \le t + \eta$ implies $|w(s) - w(t)| < \epsilon$ for all $t \in [t_0, \infty)$.

Definition 3: A time function $w(t) : [t_0, \infty) \to R$ is said to be uniformly nonvanishing if there exist constants $\delta > 0$ and $\underline{w} > 0$ such that, for any value of t, $|w(s)| \ge \underline{w}$ holds somewhere within the interval $[t, t + \delta]$.

Assumption 1: Desired reference control input, $u_{1d}(t) : [t_0, \infty) \to R$, is uniformly right continuous, uniformly bounded, and uniformly nonvanishing.

Lemma 1: Under assumption 1, system (11) is uniformly completely controllable (i.e., there exists a constant $\delta_c^* > 0$ such that definition 1 holds for all choices of constant δ satisfying $\delta \ge \delta_c^*$).

Proof: It is straightforward to show that the state transition matrix is

$$\begin{aligned} \Phi(t, \tau) &= e^{\int_\tau^t A_2(s) ds} = e^{A_2^* \int_\tau^t u_{1d}(s) ds} \\ &= \sum_{k=0}^{n-2} \frac{1}{k!} (A_2^*)^k \beta^k(t, \tau), \end{aligned} \quad (13)$$

where $\beta(t, \tau) = \int_\tau^t u_{1d}(s) ds$. In deriving the last equation of (13), the property that $(A_2^*)^{(n-1)} = 0$ is used.

It follows from assumption 1 that there exists constant $\overline{u}_1$ such that $|u_{1d}(t)| \le \overline{u}_1$ for all t. Thus, we know from (13) that

$$\|\Phi(t, t+\delta)\| \le e^{\|A_2^*\| \cdot |\beta(t, t+\delta)|} \le e^{\overline{u}_1 \|A_2^*\| \delta} \triangleq \alpha_3(\delta).$$

Similarly, we have that, for any unit vector c,

$$\begin{aligned} c^T W_c(t, t+\delta) c &\le \int_t^{t+\delta} \|\Phi(t, \tau)\|^2 d\tau \\ &\le \int_t^{t+\delta} e^{2\overline{u}_1 \|A_2^*\| (\tau - t)} d\tau \\ &\le \int_0^\delta e^{2\overline{u}_1 \|A_2^*\| s} ds \triangleq \alpha_2(\delta). \end{aligned}$$

On the other hand, according to assumption 1, there exist constants $\delta > 0$ and $\underline{u}_1 > 0$ such that, for any t, $|u_{1d}(s)| \ge \underline{u}_1$ holds for some $s(t) \in [t, t + \delta]$. In addition, by uniform right continuity and uniform boundedness, $u_{1d}(\tau)$ is uniformly bounded in magnitude, has a fixed sign, and is uniformly bounded away from zero within the subintervals $[s(t), s(t) + \sigma(\delta, \underline{u}_1)] \subset [t, t+\delta]$, where function $\sigma(\cdot)$ is independent of t. It follows from (13) that, for any unit vector c,

$$\begin{aligned} c^T W_c(t, t+\delta)c &\geq c^T W_c(s(t), s(t) + \sigma(\delta, \underline{u}_1))c \\ &= \int_{s(t)}^{s(t)+\sigma(\delta,\underline{u}_1)} \left| c^T e^{A_2^* \int_{s(t)}^{\tau} u_{1d}(\eta)d\eta} B_2 \right|^2 d\tau \\ &= \int_0^{\sigma(\delta,\underline{u}_1)} \left| c^T e^{A_2^* \int_{s(t)}^{s(t)+\phi} u_{1d}(\eta)d\eta} B_2 \right|^2 d\phi \\ &= \int_0^{\sigma(\delta,\underline{u}_1)} \left| c^T e^{A_2^* \int_0^{\phi} u_{1d}(s(t)+\varrho)d\varrho} B_2 \right|^2 d\phi. \end{aligned} \quad (14)$$

Now, let $\theta(\phi) = \int_0^\phi |u_{1d}(s(t) + \varrho)|d\varrho$ for $\phi \in [0, \sigma(\delta, \underline{u}_1)]$. It follows from $|u_{1d}(s(t)+\varrho)| \geq \underline{u}_1$ that function $\theta(\phi)$ is strictly monotonically increasing over $[0, \sigma(\delta, \underline{u}_1)]$ and uniformly for all t, that

$$\theta(\phi) = \begin{cases} \int_0^\phi u_{1d}(s(t) + \varrho)d\varrho & \text{if } u_{1d}(s(t)) > 0 \\ -\int_0^\phi u_{1d}(s(t) + \varrho)d\varrho & \text{if } u_{1d}(s(t)) < 0 \end{cases},$$

and that, since $d\theta/d\phi \neq 0$, function $\theta(\phi)$ has a well defined inverse with

$$d\phi = \frac{d\theta}{|u_{1d}(s(t) + \phi)|} \geq \frac{d\theta}{\overline{u}_1} > 0.$$

Therefore, we know that there exists a positive constant α_1 such that

$$\begin{aligned} &\int_0^{\sigma(\delta,\underline{u}_1)} \left| c^T e^{A_2^* \int_0^{\phi} u_{1d}(s(t)+\varrho)d\varrho} B_2 \right|^2 d\phi \\ \geq & \begin{cases} \frac{1}{\overline{u}_1} \int_0^{\sigma(\delta,\underline{u}_1)} \left| c^T e^{A_2^* \theta} B_2 \right|^2 d\theta, & \text{if } u_{1d}(s(t)) > 0 \\ \frac{1}{\overline{u}_1} \int_0^{\sigma(\delta,\underline{u}_1)} \left| c^T e^{-A_2^* \theta} B_2 \right|^2 d\theta, & \text{if } u_{1d}(s(t)) < 0 \end{cases} \\ \geq & \alpha_1(\delta, \overline{u}_1, \underline{u}_1) > 0. \end{aligned} \quad (15)$$

In (15), the property of both time invariant pairs $\{\pm A_2^*, B_2\}$ being controllable is used. The proof is completed by combining (14) and (15). □

E. Observability of Linear Time Varying Nominal Subsystem

For the ease of applying the proposed near-optimal framework to both state-feedback and output-feedback designs, output matrix C has already been embedded into performance index (4). As a result, observability property of system (11) is required for design and stability analysis in both cases.

Definition 4: [34] System (11) is uniformly completely observable if its observability Grammian

$$W_o(t, t+\delta) = \int_t^{t+\delta} \Phi^T(\tau, t) C_2^T C_2 \Phi(\tau, t) d\tau \quad (16)$$

and state transition matrix satisfy the following two inequalities: for all t,

$$0 < \alpha_{o1}(\delta)I \leq W_o(t, t+\delta) \leq \alpha_{o2}(\delta)I, \ \|\Phi(t, t+\delta)\| \leq \alpha_{c3}(\delta),$$

where $\delta > 0$ is a fixed constant, $\alpha_{oi}(\cdot)$ are fixed positively valued functions.

Comparing definitions 1 and 4, we know that uniform complete observability of pair $\{A_2, C_2\}$ is equivalent to uniform complete controllability of pair $\{-A_2^T, C_2\}$. In other words, system (11) is uniformly completely observable if and only if its dual system

$$\dot{z}' = -A_2^T(u_{1d}(t))z' + C_2^T v_2' \quad (17)$$

is uniformly completely controllable. Under state transformation $\xi \stackrel{\triangle}{=} [\xi_1, \cdots, \xi_{n-1}]^T = [z_{n-1}', \cdots, z_1']^T$, system (17) is transformed into $\dot{\xi} = A_2(-u_{1d}(t))\xi + B_2 v_2'$. Invoking lemma 1, we have the following result.

Lemma 2: Under assumption 1, system (11) is uniformly completely observable (that is, there exists a constant $\delta_o^* > 0$ such that definition 4 holds for all choices of constant δ satisfying $\delta \geq \delta_o^*$).

F. Relaxation and Removal of Assumption 1

As summarized in the introduction, existing results on tracking control design all require certain non-vanishing conditions. It is clear from the proof of lemma 1 that, for uniform complete controllability, $u_{1d}(t)$ being both uniformly bounded and uniformly nonvanishing is necessary and that certain uniform continuity (such as uniform right continuity, or uniform left continuity, or uniform semi-continuity) is also necessary. In fact, closest to assumption 1 is the assumption 2.12 in [26], but that assumption requires that $u_{1d}(t)$ is continuously differentiable and global Lipschitz with respect to t. Thus, assumption 1 provides the least restrictive condition for uniform complete controllability of system (11).

Nonetheless, it is necessary to show that the proposed design framework is not confined to systems satisfying assumption 1. In what follows, three classes of $u_{1d}(t)$ are considered: it is convergent to zero (that is, vanishing); it is nonvanishing but not uniformly nonvanishing; and it is zero. Under those choices, system (11) may not be uniformly completely controllable. Examples are used to illustrate that assumption 1 can be relaxed or even removed by using the so-called time folding/unfolding technique. The basic idea here is to ensure assumption 1 in a transformed domain/space. Hence, despite of the loss of uniform complete controllability in the original domain/space, the proposed control design framework can be readily applied.

Example 1: Consider nominal system (11) with

$$u_{1d}(t) = \frac{1}{\kappa(t)} \cos(wt),$$

$w \geq 0$, $\kappa(t) > 0$ for any finite time $t \geq 0$, and $\lim_{t\to\infty} \kappa(t) = +\infty$ but $1/\kappa(t) \notin \mathcal{L}_1$. Obviously, signal $u_{1d}(t)$ is vanishing, and assumption 1 is not satisfied.

Let us introduce the following time and control transformations:

$$\tau = \int_0^t \frac{1}{\kappa(s)} ds, \quad \text{and} \quad v_2(t) = \frac{1}{\kappa(t)} v_2'(\tau).$$

The first transformation unfolds the time and is differentiable, and both transformations are one-to-one and onto. Under the transformations, nominal system (11) is mapped into

$$\frac{dz(\tau)}{d\tau} = u_{1d}'(\tau) A_2^* z(\tau) + B_2 v_2', \quad (18)$$

where $u'_{1d}(\tau) = \cos(wt)$ with t being replaced by the inverse of the above time transformation (which can be found once $\kappa(t)$ is specified). Clearly, $u'_{1d}(\tau)$ in system (18) satisfies assumption 1 in the transformed time domain of τ.

As will be shown in the subsequent sections, several types of controls can be designed (for v'_2) to exponentially stabilize system (18) (and its corresponding nonlinear systems) in the domain of τ. Once v'_2 is found, v_2 is found and it is well defined. For the class of $\kappa(t)$ considered in this example, the resulting stability in the domain of t is at least asymptotic stability, and additional stability claim may be drawn based on the property of $\kappa(t)$. For instance, if $\kappa(t) = t + 1$, the result in the domain of t is only asymptotic (but not uniform asymptotic) stability; and if $\kappa(t) = \sqrt{t+1}$, the result in the domain of t is uniform asymptotic (but not exponential) stability. ◇

Example 2: Consider nominal system (11) with

$$u_{1d}(t) = \begin{cases} \frac{w_0}{(t+1)^2}\cos(w_1 t), & t \in [2^{2n}-1,\ 2^{2n+1}-1), \\ \cos(w_2 t), & t \in [2^{2n+1}-1,\ 2^{2n+2}-1), \end{cases}$$

where $w_0, w_1, w_2 \geq 0$, $n \in \aleph$, and $\aleph = \{0, 1, 2, \cdots\}$ is the set of non-negative integers. It is apparent that $u_{1d}(t)$ is unvanishing but not uniformly unvanishing to satisfy assumption 1.

Now, let us define the time-folding transformation:

$$\tau = \begin{cases} \ln(1+t), & t \in [2^{2n}-1,\ 2^{2n+1}-1), \\ (2n+1)\ln 2 + \frac{\ln 2}{2^{2n+2}-2^{2n+1}}(t - 2^{2n+1}+1), & \\ & t \in [2^{2n+1}-1,\ 2^{2n+2}-1), \end{cases}$$

which is one-to-one and onto and has right-continuous first-order derivative. Under the transformation, nominal system (11) can be transformed into (18), where $u'_{1d}(\tau)$ is given by the double-column expression in equation (19). It is obvious that, no matter whether $w_0 = 0$ or not, $u'_{1d}(\tau)$ is uniformly unvanishing (as defined in assumption 1) in the domain of τ. The rest of developments can be carried out as did in example 1. ◇

Example 3: In the event that $u_{1d}(t)$ in nominal system (11) is vanishing and $|u_{1d}(t)| \in \mathcal{L}_1$, simple time folding/unfolding mappings from t to τ defined in examples 1 and 2 would no longer be adequate. In this case, our technique calls for a time-dependent state transformation through which time folding/unfolding is accomplished and assumption 1 is satisfied in the transformed state space. For instance, consider nominal system (11) with

$$u_{1d}(t) = e^{-t}, \quad A_2^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Now, let us define the time-unfolding state and control transformations:

$$z = \begin{bmatrix} e^{-t} & 0 \\ 0 & 1 \end{bmatrix} z', \quad \text{and} \quad v'_2 = v_2,$$

under which nominal system (11) is transformed into $\dot{z}'_1 = z'_1 + z'_2$ and $\dot{z}'_2 = v'_2$. This transformed system is time-invariant and obviously satisfies assumption 1, and control v'_2 could be designed and calculated. ◇

All the above examples deal with nominal system (11) naturally arising from the trajectory tracking problem. For the stabilization (or regulation) problem, the conventional choice is $u_{1d} \equiv 0$, in which case the above discussion can be applied directly to $u_1(t)$, as illustrated by the following example.

Example 4: To make system (1) globally asymptotically stable, we can recursively design two dynamic feedback control components u_1 and u_2. First, let dynamic feedback control u_1 be of form

$$\dot{u}_1 = -\frac{1}{t - t_0 + 1}u_1 - \left[\omega^2 - \frac{1}{4(t-t_0+1)^2}\right]x_1, \tag{20}$$

where $\omega > 0$ is a design parameter whose value is arbitrary, $u_1(t_0) = c_u\|x(t_0)\|$, and c_u is also a design parameter arbitrarily chosen by the designer so long as $c_u \neq 0$ whenever $x_1(0) = 0$. It follows from (20) and equation $\dot{x}_1 = u_1$ in (1) that the closed loop subsystem is

$$\ddot{x}_1 + \frac{1}{t-t_0+1}\dot{x}_1 + \left[\omega^2 - \frac{1}{4(t-t_0+1)^2}\right]x_1 = 0.$$

It is not difficult to verify that closed loop solutions are

$$\begin{aligned} x_1(t) &= \frac{1}{\sqrt{t-t_0+1}}\left\{x_1(t_0)\cos(\omega t - \omega t_0) + \frac{u_1(t_0)+0.5x_1(t_0)}{\omega}\sin(\omega t - \omega t_0)\right\}, \\ u_1(t) &= -\frac{1}{2(t-t_0+1)^{3/2}}\left\{x_1(t_0)\cos(\omega t - \omega t_0) + \frac{u_1(t_0)+0.5x_1(t_0)}{\omega}\sin(\omega t - \omega t_0)\right\} \\ &\quad + \frac{\omega}{\sqrt{t-t_0+1}}\left\{-x_1(t_0)\sin(\omega t - \omega t_0) + \frac{u_1(t_0)+0.5x_1(t_0)}{\omega}\cos(\omega t - \omega t_0)\right\}. \end{aligned}$$

It is obvious that signal $u'_1(t) \triangleq \sqrt{t-t_0+1}\cdot u_1(t)$ satisfies assumption 1 unless $\|x(t_0)\| = 0$. On the other hand, the rest of system dynamics in (1) can be expressed as

$$\dot{z}_s = u_1 A_2^* z_s + B_2 u_2, \tag{21}$$

where $z_s \triangleq [x_2, \cdots, x_n]^T$, matrices A_2^* and B_2 are those defined in (8). Letting $\tau = 2\sqrt{t-t_0+1} - 2$ and $u'_2 = \sqrt{t-t_0+1}\cdot u_2$, we know from (21) that

$$\frac{dz_s}{d\tau} = u'_1 A_2^* z_s + B_2 u'_2, \tag{22}$$

which is uniformly completely controllable. Hence, as will be shown in sections III and IV, control u'_2 (and in turn u_2) can be designed to make z_s asymptotically stable (and solution $x_1(t)$ is already asymptotically stable). ◇

Summarizing the above results, we know that assumption 1 does not pose any limitation to the proposed control design framework, and an important consequence is that uniform complete controllability can be recovered for stabilization of nonholonomic systems. Upon fully recovering uniform complete controllability and utilizing it, the proposed control design framework becomes applicable to not only trajectory tracking but also regulation and stabilization.

$$u'_{1d}(\tau) = \begin{cases} w_0 e^{-2\tau}\cos(w_1 e^{\tau} - w_1) & \tau \in [2n\ln 2,\ (2n+1)\ln 2) \\ \cos\left(w_2((2^{2n+2} - 2^{2n+1})(\frac{\tau}{\ln 2} - 2n - 1) + 2^{2n+1} - 1)\right) & \tau \in [(2n+1)\ln 2,\ (2n+2)\ln 2) \end{cases} \quad (19)$$

III. Design of State-Feedback Near-Optimal Control

In this section, state-feedback near-optimal control v_{sfno} will be synthesized by following the design steps outlined in section II-C. The design will then be extended in section IV to the case of output feedback.

A. Optimal Controls for Individual Linear Subsystems

In this subsection, optimal controls are individually designed for linear subsystem (9) and linear time-varying nominal system (11). Since linear optimal control design is well known, the focus is placed upon finding an appropriate Lyapunov function that will be used in the near-optimal control framework (in the cases of both state feedback and output feedback but not regulation) for nonlinear error system (3). To this end, choose

$$Q(t) = \text{diag}\{q_1,\ q_2(t)\},\ \text{and}\ R(t) = \text{diag}\{r_1,\ r_2(t)\}. \quad (23)$$

It follows from (4) that performance index can be rewritten as $J = J_1 + J_2$ where

$$\begin{cases} J_1(t, x_{1e}, v_1) = \frac{1}{2}\int_{t_0}^{\infty}[q_1 x_{1e}^2 + r_1 v_1^2]dt, \\ J_2(t, z, v_2) = \frac{1}{2}\int_{t_0}^{\infty}[q_2(t) z^T C_2^T C_2 z + r_2(t) v_2^2]dt. \end{cases} \quad (24)$$

Now, consider the Lyapunov function

$$V(x_e, t) = x_e^T P x_e, \quad (25)$$

where matrix P is the solution to the time-varying state-independent Riccati equation:

$$\dot{P} + [PA + A^T P - PBR^{-1}B^T P + C^T QC] = 0. \quad (26)$$

It follows from block property of matrices A, B, C, Q and R that $P(t) = \text{diag}[p_1, P_2(t)]$, $p_1 = \sqrt{q_1 r_1}$,

$$V(x_e, t) = p_1 x_{1e}^2 + z^T P_2 z \stackrel{\triangle}{=} V_1(x_{1e}) + V_2(z, t), \quad (27)$$

and matrix $P_2(t)$ is the solution to the following reduced-order differential Riccati equation: for some $P_2(\infty) > 0$,

$$\begin{aligned} 0 \quad = \quad & \dot{P}_2(t) + P_2(t)A_2(t) + A_2^T(t)P_2(t) \\ & -P_2(t)B_2 r_2^{-1}(t) B_2^T P_2(t) + C_2^T q_2(t) C_2. \end{aligned} \quad (28)$$

It should be noted that P_2 can be pre-computed by integrating backwards and off line and then stored with an adequate sampling period. If $u_{1d}(t)$ is periodic (and so are the choices of $r_2(t)$ and $q_2(t)$), solution $P_2(t)$ (hence $P(t)$) is also periodic. Finding solution of P requires that the history of $u_{1d}(t)$ be known. In some applications such as the target-tracking and leader-follower problems, the goal point for the tracker/follower need to be updated periodically and according to observation of target/leader's current position and velocity. In this case, it would be better to treat the problem not as a tracking problem but as the set point regulation problem with the set point being updated periodically.

The following lemma summarizes all the useful results. It should be noted that, in the seminal paper of [34], optimal control (such as the ones in (29)) is derived and uniform asymptotic stability of the closed loop system is shown. It is also shown in [35] that, for linear systems with uniformly bounded matrices, uniform asymptotic stability is equivalent to exponential stability. Although Lyapunov converse theorem ensures the existence of at least one Lyapunov function corresponding to exponential stability, the proof of the following lemma is mainly intended to show V in (25) is such a Lyapunov function. In fact, the two inequalities in (30) are critical to the subsequent developments of near-optimal control.

Lemma 3: Consider subsystems (9) and (11) under performance indices in (24), respectively. Then, under assumption 1, the linear optimal control vector is

$$v_l^*(x_e, t) = -R^{-1}(t) B^T P(t) x_e, \quad (29)$$

or equivalently,

$$\begin{cases} v_{l,1}^*(x_{1e}, t) = -r_1^{-1} p_1 x_{1e}, \\ v_{l,2}^*(x_{2e}, t) = -r_2^{-1}(t) B_2^T P_2(t) z, \end{cases}$$

where p_1 is given by that in (27) and $P_2(t)$ is defined by (28). Moreover, the closed loop system is globally exponentially stable, and Lyapunov function V in (25) satisfies the following two inequalities:

$$\begin{aligned} \gamma_1 \|x_e\|^2 \le V(x_e, t) \le \gamma_2 \|x_e\|^2, \\ \frac{d}{dt}[\Phi_{cl}^T(t, t_0) P(t) \Phi_{cl}(t, t_0)] \le -\gamma_3 \Phi_{cl}^T(t, t_0)\Phi_{cl}(t, t_0), \end{aligned} \quad (30)$$

where γ_j (for $j = 1, 2, 3$) are some positive constants, and $\Phi_{cl}(t, t_0)$ is the closed-loop state transition matrix defined by $d\Phi_{cl}(t, t_0)/dt = [A(t) - BR^{-1}(t)B^T P(t)]\Phi_{cl}(t, t_0)$.

Proof: In [34], existence of optimal control is shown for both linear time-invariant and time varying systems, and (as shown by (6) with $G \equiv 0$ and $F \equiv 0$) the linear optimal control is given by (29). In stability theorem of (6.10) in [34], asymptotic stability is established. Then, by invoking theorem 3 in [35], exponential stability is concluded. As a part of the proof of stability theorem of (6.10) in [34], solution P to Riccati equation (26) is shown to be positive definite and uniformly bounded. Thus, the first inequality in (30) is established with $\gamma_1 = \inf_{t \ge t_0} \lambda_{\min}(P(t))$ and $\gamma_2 = \sup_{t \ge t_0} \lambda_{\max}(P(t))$, where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ are the minimum and maximum eigenvalues, respectively. The rest of the proof is to establish the second inequality in (30).

It follows from (27) that, for subsystem (9) under control $v_{l,1}^*$ in (29),

$$\begin{cases} \dot{V}_1 = -q_1 x_{1e}^2 = -\frac{q_1}{p_1} V_1 = -\sqrt{\frac{q_1}{r_1}} V_1, \\ x_{1e}(t) = x_{1e}(t_0) e^{-\frac{p_1}{r_1}(t - t_0)} = x_{1e}(t_0) e^{-\sqrt{\frac{q_1}{r_1}}(t - t_0)}. \end{cases} \quad (31)$$

For subsystem (11), differentiating $V_2(t) \stackrel{\triangle}{=} V_2(z(t), t)$ (defined in (27)) along the trajectory of (11), (29) and (28) yields

$$\begin{aligned}\dot{V}_2 &= -q_2(t)z^TC_2^TC_2z - z^TP_2B_2r_2^{-1}B_2^TP_2z\\ &\le -\underline{q}z^TC_2^TC_2z - \underline{r}\|v_{l,2}^*\|^2. \qquad (32)\end{aligned}$$

On the right hand side of inequality (32), both terms are negative semi-definite with respect to $\|z\|$. Integrating $\dot{V}_2$ over an interval $[t, t+\delta]$ for any $t \ge t_0$ and for any $\delta \ge \max\{\delta_c^*, \delta_o^*\}$ (where δ_c^* and δ_o^* are defined in lemmas 1 and 2, respectively), we have

$$\begin{aligned}&V_2(t) - V_2(t+\delta)\\ \ge\ & \underline{q}\int_t^{t+\delta}\|C_2z(s)\|^2ds + \underline{r}\int_t^{t+\delta}\|v_{l,2}^*(s)\|^2ds. \qquad (33)\end{aligned}$$

It can be assumed without loss of any generality that, along the trajectory of (11), under control $v_{l,2}^*$ in (29), and for some function $\xi(z(t),t) \ge 0$, $\int_t^{t+\delta}\|v_{l,2}^*(s)\|^2ds = \xi(z(t),t)\|z(t)\|^2$. In what follows, the right hand side of inequality (33) is shown to be positive definite with respect to $\|z(t)\|$ by investigating two cases of function $\xi(z(t),t)$. The first case is that, for all $t > t_0$, $\xi(z(t),t) \ge \underline{\xi}$ holds for some constant $\underline{\xi} > 0$. In this case, the claim is obvious as $V_2(t) - V_2(t+\delta) \ge \underline{r}\int_t^{t+\delta}\|v_{l,2}^*(s)\|^2ds \ge \underline{r}\underline{\xi}\|z(t)\|^2$. The second case is that $\xi(z(t),t) = 0$ for some finite t or, as t approaches infinity (together with whatever change the state $z(t)$ has), $\xi(z(t),t)$ approaches zero. Thus, for those values of t (including possibly $t = +\infty$), the inequality $\int_t^{t+\delta}\|v_{l,2}^*(s)\|^2ds \le \epsilon\|z(t)\|^2$ holds for any choice of $\epsilon > 0$. On the other hand, the solution to system (11) under control $v_{l,2}^*$ in (29) is $z(s) = \Phi(s,t)\left[z(t) + \int_t^s\Phi(t,\tau)B_2v_{l,2}^*(\tau)d\tau\right]$. Hence, we know from uniform complete controllability and uniform complete observability of $\{A_2(t),\ B_2,\ C_2\}$ that double-column expressions of inequality (34) hold. In the derivations of (34), inequality $\|a+b\|^2 \ge (\|a\| - \|b\|)^2 \ge \frac{1}{2}\|a\|^2 - \|b\|^2$ and Schwarz (integral) inequality are applied. The right hand side of (34) is positive definite with respect to $\|z(t)\|$ as ϵ can be chosen to be arbitrarily small. Therefore, we have $V_2(t) - V_2(t+\delta) \ge \underline{q}[0.5\alpha_{o1}(\delta) - n^2\alpha_{o2}(\delta)\alpha_{c2}(\delta)\epsilon]\|z(t)\|^2$.

Summarizing the two cases and reconsidering inequality (33), we know that, for all $t \ge t_0$ and for some constant $\lambda > 0$,

$$V_2(t) - V_2(t+\delta) \ge \lambda\|z(t)\|^2 \ge \frac{\lambda}{\gamma_2}V_2(t). \qquad (35)$$

Recalling from (32) that $\dot{V}_2 \le 0$, we can rewrite the above inequality as

$$V_2(t) - V_2(t+\delta) \ge \frac{\lambda}{\gamma_2}V_2(t+\delta),$$

that is,

$$V_2(t+\delta) \le \frac{1}{1+\frac{\lambda}{\gamma_2}}V_2(t) \triangleq \frac{1}{\sigma}V_2(t). \qquad (36)$$

Therefore, for any $t \ge t_0$, N can be chosen to be the smallest positive integer such that $t_0 + (N-1)\delta < t \le t_0 + N\delta$. It follows from the second inequality in (36) that

$$\begin{aligned}V_2(t) &\le V_2(t_0+(N-1)\delta) \le \frac{1}{\sigma}V_2(t_0+(N-2)T)\\ &\le \cdots \le \frac{1}{\sigma^{N-1}}V_2(t_0) \le \sigma V_2(t_0)e^{-\frac{\ln\sigma}{\delta}(t-t_0)}, \qquad (37)\end{aligned}$$

which directly shows exponential convergence of $V_2(t)$. Thus, without loss of any generality, we can assume that, for any $\tau \ge t$ and $t \ge t_0$ and for some constant $\beta_{z,1}$ and time function $\beta_{z,2}(\cdot)$, $V_2(\tau) = V_2(t)\beta_{z,2}(\tau-t)e^{-\beta_{z,1}(\tau-t)}$, where $\beta_{z,2}(0) = 1$, $0 < \beta_{z,2}(t) \le \sigma$, and $\beta_{z,1} \ge \ln\sigma/\delta$ according to inequality (37) and to the fact that $V_2(t) \ge 0$. It follows that, given any $\delta \ge \max\{\delta_c^*, \delta_o^*\}$,

$$\begin{aligned}\int_t^{t+\delta}V_2(\tau)d\tau &= V_2(t)\int_t^{t+\delta}\beta_{z,2}(\tau-t)e^{-\beta_{z,1}(\tau-t)}d\tau\\ &= V_2(t)\int_0^{\delta}\beta_{z,2}(s)e^{-\beta_{z,1}s}ds\\ &\triangleq \beta_z(\delta)V_2(t),\end{aligned}$$

or equivalently, $V_2(t) = \beta_z^{-1}(\delta)\int_t^{t+\delta}V_2(\tau)d\tau$, where $\beta_z(\delta)$ is a finite positive number. Taking time derivative on both sides of the last equation and invoking (35) yield

$$\begin{aligned}\dot{V}_2(t) &= \frac{1}{\beta_z(\delta)}[V_2(t+\delta) - V_2(t)] \le -\frac{\lambda}{\gamma_2\beta_z(\delta)}V_2(t)\\ &\le -\frac{\gamma_1\lambda}{\gamma_2\beta_z(\delta)}\|z\|^2. \qquad (38)\end{aligned}$$

Combining (31) and (38) yields that, for some $\gamma_3 > 0$, $\dot{V} \le -\gamma_3\|x_e\|^2$. Thus, the second inequality in (30) can be concluded by noting that $V = x_e^TP(t)x_e$ and $x_e(t) = \Phi_{cl}(t,t_0)x_e(t_0)$. □

B. Designs of Suboptimal and Near-Optimal Controls

In this subsection, we first show that linear optimal control (29) is globally-exponentially-stabilizing and suboptimal for system (3) and that its performance can be quantified (against the optimal performance under the unattainable nonlinear optimal control in (5)) by the residue from the optimality condition, as summarized by the following theorem. This result leads naturally to the proposed design of near-optimal control which, to be stated shortly, selectively minimizes the optimality residue.

Theorem 1: Consider nonlinear tracking error system (3) under assumption 1 and under the control $v(x_e,t) = v_{sfso}(x_e,t)$, where

$$v_{sfso}(x_e,t) \triangleq \begin{bmatrix} v_{sfso,1} & v_{sfso,2}\end{bmatrix}^T = v_l^*(x_e,t), \qquad (39)$$

denotes the so-called state-feedback sub-optimal (sfso) control, and $v_l^*(x_e,t)$ is defined by (29). Then, the closed loop system is globally and exponentially stable. Furthermore, under performance index (4) with the choices of weighting matrices in (23), control (39) is suboptimal and its closeness to optimality can be measured by $\|E_s(x_e, v_{sfso})\|^2$, where

$$E_s(x_e, v_{sfso}) \triangleq \begin{bmatrix} E_{s1}(x_e, v_{sfso})\\ E_{s2}(x_e, v_{sfso})\end{bmatrix}, \qquad (40)$$

$$\begin{aligned}\int_t^{t+\delta}\|C_2 z(s)\|^2 ds &\geq \int_t^{t+\delta}\left|\|C_2\Phi(s,t)z(t)\| - \left\|C_2\Phi(s,t)\int_t^s \Phi(t,\tau)B_2 v_{l,2}^*(\tau)d\tau\right\|\right|^2 ds \\ &\geq \frac{1}{2}\int_t^{t+\delta}\|C_2\Phi(s,t)z(t)\|^2 ds - \int_t^{t+\delta}\|C_2\Phi(s,t)\|^2\left\|\int_t^s \Phi(t,\tau)B_2 v_{l,2}^*(\tau)d\tau\right\|^2 ds \\ &\geq \frac{1}{2}\int_t^{t+\delta}\|C_2\Phi(s,t)z(t)\|^2 ds - \int_t^{t+\delta}\|C_2\Phi(s,t)\|^2 ds \times \left\|\int_t^{t+\delta}\Phi(t,\tau)B_2 v_{l,2}^*(\tau)d\tau\right\|^2 \\ &\geq [0.5\alpha_{o1}(\delta) - n^2\alpha_{o2}(\delta)\alpha_{c2}(\delta)\epsilon]\|z(t)\|^2, \end{aligned} \quad (34)$$

$E_{s1}(x_e, v_{sfso}) = [\partial(F_2 x_{1e})^T/\partial x_{1e}]P_2 z$, and $E_{s2}(x_e, v_{sfso}) = -P_2 G_2 r_1^{-1} p_1 x_{1e} + P_2 F_2 x_{1e} - r_1^{-1} p_1 x_{1e} (A_2^*)^T P_2 z$.

Proof: As shown in section II-B, performance of control (39) can be quantified against optimal performance by checking optimality condition (6). That is, it follows from $E_s(x_e, v_{sfso}) = \dot{\lambda} + \partial H/\partial x_e$ with $\lambda = P(t)x_e$ that, under control (29),

$$\begin{aligned}E_s(x_e, v_{sfso}) &= \dot{P}x_e + PG(x_e)v_{sfso} + PF(x_{1e}, u_{2d})x_{1e} \\ &\quad +[PA + A^T P - PBR^{-1}B^T P + C^T QC]x_e \\ &\quad -\text{vec}\left[x_e^T PBR^{-1}\frac{\partial G^T(x_e)}{\partial x_{ie}}Px_e\right] \\ &\quad +\text{vec}\left[\frac{\partial(F(x_{1e}, u_{2d})x_{1e})^T}{\partial x_{ie}}Px_e\right]. \end{aligned} \quad (41)$$

Substituting both Riccati equation (26) and control (29) into equation (41) yields

$$\begin{aligned}E_s(x_e, v_{sfso}) &= -PGR^{-1}B^T Px_e + PFx_{1e} \\ &\quad -\text{vec}\left[x_e^T PBR^{-1}\frac{\partial G^T}{\partial x_{ie}}Px_e\right] \\ &\quad +\text{vec}\left[\frac{\partial(Fx_{1e})^T}{\partial x_{ie}}Px_e\right]. \end{aligned}$$

Thus, expression (40) can be obtained directly from the above equation by utilizing the special structures and functional dependence of matrices $G(x_e), F(x_{1e}, u_{2d}), B, P$ and R.

Exponential stability of the closed-loop system can be established using Lyapunov function $V(t) \stackrel{\triangle}{=} V(x_e, t)$ defined in (25). Although system (3) under control (39) is nonlinear, its closed loop dynamics can be rewritten as $\dot{x}_e = [A(t) - BR^{-1}(t)B^T P(t)]x_e + G(x_e)v_{sfso} + F(x_{1e}, u_{2d})x_{1e}$, or equivalently,

$$\begin{aligned}x_e(t) &= \Phi_{cl}(t, t_0)\left[x_e(t_0) + \int_{t_0}^t \Phi_{cl}(t_0, \tau)\left[G(x_e(\tau))v_{sfso}\right.\right. \\ &\quad \left.\left. +F(x_{1e}(\tau), u_{2d}(\tau))x_{1e}(\tau)\right]d\tau\right]. \end{aligned}$$

Hence, we have the expression of $\dot{V}$ as given by double-column equation (42). It follows from the second inequality in (30) that the time derivative of V along any trajectory of system (3) under control (39) can be expressed as:

$$\begin{aligned}\dot{V} &\leq -\gamma_3\|x_e\|^2 + 2x_e^T PG(x_e)v_{sfso} \\ &\quad +2x_e^T PF(x_{1e}, u_{2d})x_{1e} \\ &= -\gamma_3\|x_e\|^2 - 2z^T P_2 G_2 r_1^{-1} p_1 x_{1e} + 2z^T P_2 F_2 x_{1e}. \end{aligned} \quad (43)$$

It follows from (40) that inequality (43) can be rewritten as

$$\begin{aligned}\dot{V} &\leq -\gamma_3\|x_e\|^2 + 2z^T E_{s2}(x_e, v_{sfso}) \\ &\quad +2r_1^{-1}p_1 x_{1e} z^T (A_2^*)^T P_2 z \\ &\leq -\gamma_3\|x_e\|^2 + 2\|z\|\cdot\|E_{s2}(x_e, v_{sfso})\| \\ &\quad +2r_1^{-1}p_1 x_{1e} z^T (A_2^*)^T P_2 z, \end{aligned} \quad (44)$$

which will be referenced in the analysis of near optimal control to be designed.

Substituting (40) again into (44) yields *

$$\begin{aligned}\dot{V} &\leq -\gamma_3\|x_e\|^2 + 2\|z\|\left[r_1^{-1}p_1\|P_2\|\cdot\|G_2\| + \|P_2\|\cdot\|F_2\|\right. \\ &\quad \left. +2r_1^{-1}p_1\|(A_2^*)^T P_2\|\cdot\|z\|\right]|x_{1e}|. \end{aligned} \quad (45)$$

Recalling the structures of functions $G(\cdot)$ and $F(\cdot)$ in nonlinear tracking error system (3) as well as the structural of sfso control (39), we know that solution of $x_{1e}(t)$ remains to be that in (31) and that

$$\|G_2(z)\| \leq \|z\| \leq \|x_e\|. \quad (46)$$

Hence, it follows from boundedness of reference input u_{2d} that the following inequality holds for some constant $c_f > 0$:

$$\|F_2(x_{1e}, u_{2d})\| \leq c_f. \quad (47)$$

Therefore, using solution (31) and inequalities (46) and (47), we can rewrite inequality (45) as

$$\begin{aligned}\dot{V} &\leq -\gamma_3\|x_e\|^2 + (c_0|x_{1e}(t_0)|\|x_e\| + c_1\|x_e\|^2)e^{-c_2(t-t_0)} \\ &\leq \left[-2\beta_0 + 2\beta_2 c_2 e^{-c_2(t-t_0)}\right]V \\ &\quad +2\beta_1\sqrt{V}e^{-c_2(t-t_0)}|x_{1e}(t_0)|, \end{aligned} \quad (48)$$

where $c_0 = 2\gamma_2 c_f$, $c_1 = 6p_1 r_1^{-1}\gamma_2|x_{1e}(t_0)|$, $c_2 = \sqrt{q_1/r_1}$, $\beta_0 = \gamma_3/(2\gamma_2)$, $\beta_1 = c_0/(2\sqrt{\gamma_1})$, and $\beta_2 = c_1/(2c_2\gamma_1)$. The solution to inequality (48) is given by

$$\begin{aligned}&\sqrt{V(t)} \\ \leq\ &\sqrt{V(t_0)}e^{\int_{t_0}^t\left(-\beta_0 + \beta_2 c_2 e^{-c_2(\tau - t_0)}\right)d\tau} \\ &+\int_{t_0}^t e^{\int_s^t\left(-\beta_0+\beta_2 c_2 e^{-c_2(\tau-s)}\right)d\tau}\beta_1|x_{1e}(t_0)|e^{-c_2(s-t_0)}ds \\ \leq\ &\sqrt{V(t_0)}e^{\beta_2}e^{-\beta_0(t-t_0)} \\ &+\frac{\beta_1 e^{\beta_2}}{\beta_0 - c_2}\left[e^{-c_2(t-t_0)} - e^{-\beta_0(t-t_0)}\right], \end{aligned}$$

*Without the need of expression (44), one can obtain (45) directly from (43).

$$\begin{aligned}\dot{V} &= \left[x_e(t_0) + \int_{t_0}^{t} \Phi_{cl}(t_0,\tau)[G(x_e(\tau)) + F(x_{1e}(\tau), u_{2d}(\tau))x_{1e}(\tau)]v_{sfso}d\tau\right]^T \left\{\frac{d}{dt}[\Phi_{cl}^T(t,t_0)P(t)\Phi_{cl}(t,t_0)]\right\} \\ &\quad \times \left[x_e(t_0) + \int_{t_0}^{t} \Phi_{cl}(t_0,\tau)[G(x_e(\tau)) + F(x_{1e}(\tau), u_{2d}(\tau))x_{1e}(\tau)]v_{sfso}d\tau\right] + 2x_e^T P(t)G(x_e)v_{sfso} \\ &\quad +2x_e^T P(t)F(x_{1e}, u_{2d})x_{1e}. \end{aligned} \tag{42}$$

from which exponential stability is obvious. □

Theorem 1 provides not only closed-loop exponential stability but also a quantitative measure on the closeness of control (39) to nonlinear optimal control (5). One of the objectives of the proposed near optimal control design methodology is to find a closed-form control that minimizes the optimality residue. Clearly, control (39) is a good candidate to begin our search for the best among all the candidates that are both analytical and globally exponentially stabilizing. To this end, let the proposed state-feedback near-optimal (sfno) control be of form

$$v_{sfno}(x_e,t) = v_{sfso}(x_e,t) + v_{sfna}(x_e,t), \tag{49}$$

where $v_{sfso}(x_e,t)$ is given by (39), and $v_{sfna}(x_e,t) \triangleq [v_{sfna,1}, v_{sfna,2}]^T$ is a state-feedback nonlinear additive (sfna) control component to be determined. Given the residue of $E_s(x_e, v_{sfso})$ in (40), the residue corresponding to near optimal control (49) can be similarly derived from the optimality condition (6) under the constraint that $\lambda = P(t)x_e$. The constraint is necessary since matrix $P(t)$ from both Lyapunov function (25) and Riccati equation (26) is the best among available solutions. Therefore, it follows from optimality condition (6), from Riccati equation (26), and from the derivation of (40) that

$$\begin{aligned} & E_s(x_e, v_{sfno}) \\ =\; & \dot{P}x_e + [PA + A^T P - PBR^{-1}B^T P + C^T QC]x_e \\ & +\text{vec}\left[v_{sfso}^T \frac{\partial G^T(x_e)}{\partial x_{ie}} P x_e\right] + PBv_{sfna} \\ & +\text{vec}\left[v_{sfna}^T \frac{\partial G^T(x_e)}{\partial x_{ie}} P x_e\right] + PG(x_e)v_{sfso} \\ & +PG(x_e)v_{sfna} + PF(x_{1e}, u_{2d})x_{1e} \\ & +\text{vec}\left[\frac{\partial (Fx_{1e})^T}{\partial x_{ie}} P x_e\right] \\ =\; & \begin{bmatrix} \frac{\partial (F_2 x_{1e})^T}{\partial x_{1e}} P_2 z \\ -P_2 G_2 r_1^{-1} p_1 x_{1e} + P_2 F_2 x_{1e} - r_1^{-1} p_1 x_{1e} (A_2^*)^T P_2 z \end{bmatrix} \\ & + \begin{bmatrix} p_1 \\ (A_2^*)^T P_2 z + P_2 G_2 \end{bmatrix} v_{sfna,1} \\ & + \begin{bmatrix} 0 \\ P_2 B_2 \end{bmatrix} v_{sfna,2} \end{aligned} \tag{50}$$

$$\triangleq E_s(x_e, v_{sfso}) + M_1(z)v_{sfna,1} + M_2 v_{sfna,2}. \tag{51}$$

Symbol $E_s(x_e, v_{sfso})$ in expression (51) is used to denote a lumped sum of terms, and it is also good for intuitive comparison. Strictly speaking, while equation (50) is the expression for calculation and design, $E_s(x_e, v_{sfno})$ and $E_s(x_e, v_{sfso})$ is related by (51) only at the initial instant of time (when the state assumes the same value) after which the system trajectory becomes different under two different controls.

It is clear from (51) that the nonlinear additive control v_{sfna} should be designed to selectively minimizes the residue $\|E_s(x_e, v_{sfso} + v_{sfna})\|^2$. The following lemma provides the design of v_{sfna}, and v_{sfna} is solved analytically using the least-square minimization under the selection of $v_{sfna,1}(x_e,t) \equiv 0$ (which will be explained after theorem 2).

Lemma 4: Suppose that nonlinear tracking error system (3) satisfies assumption 1 and is under the state feedback near optimal control $v_{sfno}(x_e,t)$ in (49). Given performance index (4) with the choices of weighting matrices in (23), the following choice of $v_{sfna,2}(x_e,t)$ is near optimal under the selection of $v_{sfna,1}(x_e,t) \equiv 0$:

$$v_{sfna,2}(x_e,t) = -(P_2B_2)^+ Y_s, \tag{52}$$

where $(P_2B_2)^+ = [B_2^T P_2^T P_2 B_2]^{-1} B_2^T P_2^T$ is a pseudo-inverse of matrix P_2B_2, and

$$Y_s \triangleq -r_1^{-1} p_1 x_{1e}[(A_2^*)^T P_2 z + P_2 G_2] + P_2 F_2 x_{1e}. \tag{53}$$

Specifically, under the selection of $v_{sfna,1}(x_e,t) \equiv 0$, $\|E_s(x_e, v_{sfno})\|$ is minimized by control $v_{sfna,2}(x_e,t)$ in (52), and inequality

$$\begin{aligned} & \|E_s(x_e, v_{sfno})\|^2 \\ =\; & Y_s^T \left[I - P_2B_2(B_2^T P_2^T P_2 B_2)^{-1} B_2^T P_2^T\right] Y_s \\ & + \left[\frac{\partial (F_2 x_{1e})^T}{\partial x_{1e}} P_2 z\right]^2 \\ <\; & \|E_s(x_e, v_{sfso})\|^2 \end{aligned} \tag{54}$$

holds for all $x_e \in \Re^n$ but those at which $(P_2B_2)^+ Y_s = 0$ (and hence $\|E_s(x_e, v_{sfno})\| = \|E_s(x_e, v_{sfso})\|$ as $v_{sfna,2}(x_e,t) = 0$).

Proof: The proof is to show that, given $v_{sfna,1}(x_e,t) \equiv 0$, function $\|E_s(x_e, v_{sfno})\|^2$ is minimized by the corresponding least square solution $v_{sfna,2}(x_e,t)$ in (52). It follows that, upon setting $v_{sfna,1} = 0$,

$$\|E_s(x_e, v_{sfno})\|^2 = \|E_{s2}(x_e, v_{sfno})\|^2 + \left[\frac{\partial (F_2 x_{1e})^T}{\partial x_{1e}} P_2 z\right]^2,$$

$$E_{s2}(x_e, v_{sfno}) = Y_s + P_2 B_2 v_{sfna,2},$$

and that, for all choices of $v_{sfna,2}$,

$$\begin{aligned}
& \|E_{s2}(x_e, v_{sfno})\|^2 \\
= \;& Y_s^T \left[I - P_2B_2(B_2^T P_2^T P_2 B_2)^{-1} B_2^T P_2^T\right] Y_s \\
& + \left[v_{sfna,2} + (B_2^T P_2^T P_2 B_2)^{-1} B_2^T P_2^T Y_s\right]^T \\
& \times B_2^T P_2^T P_2 B_2 \left[v_{sfna,2} + (B_2^T P_2^T P_2 B_2)^{-1} B_2^T P_2^T Y_s\right] \\
\geq \;& Y_s^T \left[I - P_2B_2(B_2^T P_2^T P_2 B_2)^{-1} B_2^T P_2^T\right] Y_s. \qquad (55)
\end{aligned}$$

Hence, inequality (54) can be readily concluded from (55). That is, the choice of $v_{sfna,2}$ in (52) minimizes $\|E_s(x_e, v_{sfno})\|^2$ under the choice of $v_{sfna,1} \equiv 0$. □

To justify the proposed design of nonlinear additive control, we must also show that the performance improvement quantified in lemma 4 is achieved uniformly over time by a comparative study of closed-loop stability. The following theorem shows an improvement of global exponential stability of the closed-loop system under near-optimal control (49).

Theorem 2: Consider system (3) that satisfies assumption 1. Then, under the near-optimal control (49) (which is in turn defined by (39) and (52)), the closed-loop system has a convergence rate of global exponential stability no less than that under suboptimal control (39).

Proof: To proceed with a comparative study of global and exponential stability, consider again Lyapunov function defined in (25). It follows from the discussion leading to (43) that, under control (49) (in terms of (39) and (29)) and along its resulting trajectory of (3),

$$\begin{aligned}
\dot V \;=\;& x_e^T \dot P x_e + 2x_e^T P[Ax_e + Bv_{sfno}] \\
& + 2x_e^T PGv_{sfno} + 2x_e^T PFx_{1e} \\
\leq \;& -\gamma_3\|x_e\|^2 - 2z^T P_2 G_2 r_1^{-1} p_1 x_{1e} + 2x_e^T PBv_{sfna} \\
& + 2x_e^T PGv_{sfna} + 2z^T P_2 F_2 x_{1e}. \qquad (56)
\end{aligned}$$

Substituting $v_{sfna,1}(x_e,t) \equiv 0$ into (56) yields

$$\begin{aligned}
\dot V \;\leq\;& -\gamma_3\|x_e\|^2 - 2z^T P_2 G_2 r_1^{-1} p_1 x_{1e} \\
& + 2z^T P_2 B_2 v_{sfna,2} + 2z^T P_2 F_2 x_{1e}.
\end{aligned}$$

It follows from (50) that the above inequality can be rewritten to be

$$\begin{aligned}
\dot V \;\leq\;& -\gamma_3\|x_e\|^2 + 2z^T E_{s2}(x_e, v_{sfno}) \\
& + 2r_1^{-1} p_1 x_{1e} z^T (A_2^*)^T P_2 z \\
\leq\;& -\gamma_3\|x_e\|^2 + 2\|z\| \cdot \|E_{s2}(x_e, v_{sfno})\| \\
& + 2r_1^{-1} p_1 x_{1e} z^T (A_2^*)^T P_2 z. \qquad (57)
\end{aligned}$$

Under control (52), inequality (54) holds. Consequently, the statement of the theorem can now be concluded by comparing (44) and (57) and by applying the comparison theorem in [33]. □

It is important to note that minimization of the optimality residue is first performed in lemma 4 for a given instant of time and is then shown to be uniform over time in the proof of theorem 2. Any further instantaneous reduction of making $\|E_s(x_e, v_{sfno})\|^2$ less than its value in (54) has to be done by minimizing not only $\|E_{s2}(x_e, v_{sfno})\|^2$ but also $\|E_{s1}(x_e, v_{sfno})\|^2$. Such a minimization is impossible unless control v_{sfno} is redesigned such that $v_{sfna,1} \neq 0$. So, why $v_{sfna,1} \equiv 0$ is selected? There are two reasons. First, as shown by (44), stability and convergence of the closed loop system is impacted not by $E_{s1}(x_e, v_{sfno})$ but only by $E_{s2}(x_e, v_{sfno})$. Thus, it is from stability argument that minimizing $\|E_{s2}(x_e, v_{sfno})\|^2$ is sufficient. Second, if the unconditional instantaneous least square minimum of $\|E_s(x_e, v_{sfno})\|^2$ is solved from equation (51), $\|E_s(x_e, v_{sfno}^*)\|^2$ being made pointwise smaller than that in (54) holds only for a very short period during the initial transient. Afterwards, the near-optimal performance is determined by whether the optimality condition is not only minimized instantaneously but also forced to diminish quickly and uniformly over time. In fact, it can be shown analytically that, under the least square solution with non-zero control term $v_{sfna,1}^*$, the closed loop exponential convergence rate will become much slower and consequently the value of optimality residue actually becomes larger soon after t_0. For this reason, in the proposed framework, nonlinear additive control (49) is designed under the choice of $v_{sfna,1} \equiv 0$, and such a control is indeed near-optimal.

IV. Design of Output-Feedback Near-Optimal Control

In this section, the framework of near-optimal tracking control design is extended to output feedback. In tracking error dynamics in (9) and (10), output tracking error is $y_e = [x_{1e}, x_{2e}]^T$. The following time-varying observer is to asymptotically estimate the unmeasured error state variables (i.e., z_2 up to z_{n-1} in subsystem (10)) from input-output information of y_e and v: for any initial condition $\hat z(t_0)$,

$$\begin{aligned}
\dot{\hat z} \;=\;& A_2(u_{1d}(t))\hat z + B_2 v_2(t) + G_2(\hat z) v_1(t) \\
& + F_2(x_{1e}, u_{2d}) x_{1e} + L(t)(z_1 - \hat z_1), \qquad (58)
\end{aligned}$$

where $L(\cdot)$ is a time-varying gain vector to be selected, $v_1(t) = v_{l,1}^*(x_{1e}, t)$ is defined in (29), and $v_2(t)$ is the observer-based control to be synthesized later. It follows from (10) and (58) that dynamics of estimation error $\tilde z \stackrel{\triangle}{=} z - \hat z$ are described by

$$\dot{\tilde z} = \mu(t) A_2^* \tilde z - L(t) C_2 \tilde z = [\mu(t) A_2^* - L(t) C_2]\tilde z, \qquad (59)$$

where $\mu(t) \stackrel{\triangle}{=} u_{1d}(t) + v_{l,1}^*(x_{1e}, t)$, and A_2^* is the matrix given in (8). The following lemma provides a closed-form design of observer (58).

Lemma 5: Under assumption 1, estimation error $\tilde z$ of equation (59) is globally and exponentially stable if gain vector $L(t)$ is set to be that in the double-column expression of (60), where $\beta_\mu(t,t_0) \stackrel{\triangle}{=} \int_{t_0}^t \mu(s)ds$, $\delta_o \geq \delta_{\mu,o}^*$ is a given constant, and $\delta_{\mu,o}^*$ is the value of δ_c^* resulting from the application of lemma 1 to pair $\{-\mu(t)(A_2^*)^T, C_2^T\}$.

Proof: It follows that, under control $v_1(t) = v_{l,1}^*(x_{1e}, t)$, the solution to subsystem (9) and given in (31) is exponentially convergent, and so is $v_1(t)$. Hence, we know from assumption 1 that time function $\mu(t) = u_{1d}(t) + v_1(t)$ also satisfies assumption 1. Now, consider the time varying "nominal system" of error dynamics (59): $\dot{\tilde z}' = \mu(t) A_2^* \tilde z'$ and $\tilde y' = C_2 \tilde z'$. For this fictitious system, let $\Phi_\mu(t,t_0)$ and $W_{o,\mu}(t-\delta_o, t)$

$$L(t)=\begin{bmatrix} \delta_o & \int_{t-\delta_o}^t \beta_\mu(s,t)ds & \cdots & \int_{t-\delta_o}^t \frac{\beta_\mu^{n-2}(s,t)}{(n-2)!}ds \\ \int_{t-\delta_o}^t \beta_\mu(s,t)ds & \int_{t-\delta_o}^t \beta_\mu^2(s,t)ds & \cdots & \int_{t-\delta_o}^t \frac{\beta_\mu^{n-1}(s,t)}{(n-2)!}ds \\ \vdots & \vdots & \cdots & \vdots \\ \int_{t-\delta_o}^t \frac{\beta_\mu^{n-2}(s,t)}{(n-2)!}ds & \int_{t-\delta_o}^t \frac{\beta_\mu^{n-1}(s,t)}{(n-2)!}ds & \cdots & \int_{t-\delta_o}^t \frac{\beta_\mu^{2n-4}(s,t)}{(n-2)!(n-2)!}ds \end{bmatrix}^{-1} C_2^T \tag{60}$$

denote its state transition matrix and observability Grammian, respectively. That is, $\Phi_\mu(t,s)=\sum_{k=0}^{n-2}\frac{1}{k!}(A_2^*)^k\beta_\mu^k(t,s)$, and

$$W_{o,\mu}(t-\delta_o,t)=\int_{t-\delta_o}^t \Phi_\mu^T(s,t-\delta_o)C_2^TC_2\Phi_\mu(s,t-\delta_o)ds. \tag{61}$$

Therefore, we know from lemmas 2 and 1 that, since the pair $\{\mu(t)A_2^*, C_2\}$ is uniformly completely observable, inequalities

$$\begin{array}{c} 0<\alpha_{\mu,1}(\delta_o)I\le W_{o,\mu}(t-\delta_o,t)\le\alpha_{\mu,2}(\delta_o)I, \\ \|\Phi_\mu(t-\delta_o,t)\|\le\alpha_{\mu,3}(\delta_o), \end{array} \tag{62}$$

hold for all $\delta_o\ge\delta_{\mu,o}^*$, where $\alpha_{\mu,i}(\cdot)$ are some positively valued functions.

To show global and exponential stability of estimation error dynamics (59), consider the following Lyapunov function candidate: $V_\mu(\tilde z,t)=\tilde z^TS_o(t-\delta_o,t)\tilde z$, where $\delta_o\ge\delta_{\mu,o}^*$, and

$$\begin{aligned} S_o(t-\delta_o,t) &= \Phi_\mu^T(t-\delta_o,t)W_{o,\mu}(t-\delta_o,t)\Phi_\mu(t-\delta_o,t) \\ &= \int_{t-\delta_o}^t \Phi_\mu^T(s,t)C_2^TC_2\Phi_\mu(s,t)ds. \end{aligned} \tag{63}$$

It follows from (63) and (62) that

$$\begin{array}{c} \gamma_{\mu,1}(\delta_o)\|\tilde z\|^2\triangleq\frac{\alpha_{\mu,1}(\delta_o)}{\alpha_{\mu,3}^2(\delta_o)}\|\tilde z\|^2\le V_\mu(\tilde z,t), \\ V_\mu(\tilde z,t)\le\alpha_{\mu,2}(\delta_o)\alpha_{\mu,3}^2(\delta_o)\|\tilde z\|^2\triangleq\gamma_{\mu,2}(\delta_o)\|\tilde z\|^2, \end{array} \tag{64}$$

which shows that Lyapunov function $V_\mu(\tilde z,t)$ is positive definite, decrescent, and radially-unbounded.

It follows that the time derivative of $V_\mu(\cdot)$ along the trajectory of (59) is

$$\begin{aligned} \dot V_\mu &= 2\tilde z^TS_o(t-\delta_o,t)\dot{\tilde z}+\tilde z^T\Phi_\mu^T(t,t)C_2^TC_2\Phi_\mu(t,t)\tilde z \\ &\quad +2\tilde z^T\left[\int_{t-\delta_o}^t \Phi_\mu^T(s,t)C_2^TC_2\frac{d\Phi_\mu(s,t)}{dt}ds\right]\tilde z \\ &\quad -\tilde z^T\Phi_\mu^T(t-\delta_o,t)C_2^TC_2\Phi_\mu(t-\delta_o,t)\tilde z \\ &= -2\tilde z^TS_o(t-\delta_o,t)L(t)C_2\tilde z+\tilde z^TC_2^TC_2\tilde z \\ &\quad -\tilde z^T\Phi_\mu^T(t-\delta_o,t)C_2^TC_2\Phi_\mu(t-\delta_o,t)\tilde z. \\ &= -\tilde z^TC_2^TC_2\tilde z-\tilde z^T\Phi_\mu^T(t-\delta_o,t)C_2^TC_2\Phi_\mu(t-\delta_o,t)\tilde z \\ &\le -\tilde z^TC_2^TC_2\tilde z, \end{aligned} \tag{65}$$

under the choice of feedback gain

$$L(t)=S_o^{-1}(t-\delta_o,t)C_2^T. \tag{66}$$

It is elementary to show that combining equations (61), (63), and (66) renders the closed form solution in (60) for observer gain $L(t)$. Furthermore, according to (64), gain matrix $L(t)$ has the property that, for any $\delta>0$,

$$\int_t^{t+\delta}\|L(\tau)\|^2d\tau\le\gamma_{\mu,1}^{-2}(\delta_o)\delta. \tag{67}$$

Integrating expression (65) of $\dot V_\mu$ over an interval $[t,t+\delta]$ for any $t\ge t_0$ and $\delta>0$, we have

$$\begin{aligned} & V_\mu(t)-V_\mu(t+\delta) \\ \ge\ & \int_t^{t+\delta}\tilde z^T(\tau)C_2^TC_2\tilde z(\tau)d\tau \\ =\ & \tilde z^T(t)\left[\int_t^{t+\delta}\Phi_{\mu,cl}^T(\tau,t)C_2^TC_2\Phi_{\mu,cl}(\tau,t)d\tau\right]\tilde z(t), \end{aligned} \tag{68}$$

where $V_\mu(t)\triangleq V_\mu(\tilde z(t),t)$, and $\Phi_{\mu,cl}(\tau,t)$ is the state transition matrix for close-loop estimation dynamics of (59). Recalling that $\{\mu(t)A_2^*, C_2\}$ is uniformly completely observable and that $L(t)$ satisfies inequality (67), we know by invoking theorem 4 in [36] that pair $\{[\mu(t)A_2^*-L(t)C_2], C_2\}$ is uniformly completely observable. Hence, there exists constants $\lambda_\mu>0$ and $\delta_{\mu,o,cl}^*>0$ such that, for any $\delta\ge\delta_{\mu,o,cl}^*$, $\int_t^{t+\delta}\Phi_{\mu,cl}^T(\tau,t)C_2^TC_2\Phi_{\mu,cl}(\tau,t)d\tau\ge\lambda_\mu I$. Substituting this inequality and (64) into (68) yields

$$V_\mu(t)-V_\mu(t+\delta)\ge\lambda_\mu\|\tilde z(t)\|^2\ge\frac{\lambda_\mu}{\gamma_{\mu,2}}V_\mu(t).$$

Exponential stability of $V_\mu(t)$ and in turn of $\|\tilde z(t)\|$ can be shown by comparing the above inequality and (35) and by duplicating that proof of $V_2(t)$ being exponentially convergent. □

Upon having the exponentially convergent observer (58), we can convert the state-feedback near-optimal control in (49) into an input-output near optimal control, as shown in the following theorem. Since the development of the input-output design is parallel to that of the state-feedback design, the proof of the theorem will focus upon providing key expressions and refer to the corresponding parts in the proofs of theorem 1, lemma 4, and theorem 2.

Theorem 3: Consider tracking error system consisting of (9) and (10) which satisfies assumption 1. Given performance index (4) with the choices of weighting matrices in (23), choose the output feedback near optimal (ofno) control to be

$$\begin{aligned} v_{ofno}(\hat x_e,t) &\triangleq \begin{bmatrix} v_{ofno,1} & v_{ofno,2}\end{bmatrix}^T \\ &= v_{ofso}(\hat x_e,t)+v_{ofna}(\hat x_e,t), \end{aligned} \tag{69}$$

$$v_{ofso}(\hat x_e,t) = -R^{-1}(t)B^TP(t)\hat x_e, \tag{70}$$

where $P(t)$ is given by (27), $\hat x_e=[x_{1e},\hat z^T]^T$, $\hat z(t)$ is defined by (58), $v_{ofso}(\hat x_e,t)$ is the so-called output feedback sub-optimal (ofso) control, and $v_{ofna}(\hat x_e,t)$ is the so-called output feedback nonlinear additive (ofna) control term. Then,

(a) the closed-loop system is globally exponentially stable, and $v_{ofso}(\hat x_e,t)$ is suboptimal if $v_{ofna}(t)=0$;

(b) the closed-loop system is also globally exponentially stable if

$$v_{ofna} \triangleq \begin{bmatrix} v_{ofna,1} & v_{ofna,2} \end{bmatrix}^T = \begin{bmatrix} 0 & -(P_2B_2)^+Y_o \end{bmatrix}^T, \quad (71)$$

where

$$Y_o \triangleq -r_1^{-1}p_1x_{1e}[(A_2^*)^TP_2\hat{z}+P_2G_2(\hat{z})] +q_2C_2^T(z_1-\hat{z}_1)+P_2L(t)(z_1-\hat{z}_1) +P_2F_2x_{1e}. \quad (72)$$

Moreover, control (69) together with (71) is near optimal in the sense that the optimality residue $\left\|\dot{\lambda}+\partial H/\partial x_e\right\|^2_{\lambda=P(t)\hat{x}_e}$ is minimized not only at any fixed time instant under the selection of $v_{ofna,1}=0$ but also uniformly over time.

Proof: The proof consists of three parts. In the first part, control (69) together with (71) is shown to be instantaneously near optimal under performance index (4) and under the selection of $v_{ofna,1} = 0$. In the second part, exponential stability of the output feedback sub-optimal control (70) is established for system (9) and (10). Finally, in the third part, the closed loop system under control (69) and (71) is shown to be exponentially stable and uniformly near optimal over time.

Part I: It follows from (9) and (58) that

$$\dot{\hat{x}}_e = A(u_{1d}(t))\hat{x}_e+[B+G(\hat{x}_e)]v+F(x_{1e},u_{2d})x_{1e} + \begin{bmatrix} 0 & [L(t)(z_1-\hat{z}_1)]^T \end{bmatrix}^T. \quad (73)$$

It follows from the optimality condition that, under control (69) and by using $P(t)$ in (26) and setting $\lambda = P(t)\hat{x}_e$, the optimality residue is:

$$\begin{aligned} & E_o(\hat{x}_e,v_{ofno}) \\ &= \dot{P}\hat{x}_e+[PA+A^TP-PBR^{-1}B^TP \\ &\quad +C^TQC]\hat{x}_e+C^TQC\tilde{x}_e+\text{vec}\left[v_{ofso}^T\frac{\partial G^T(x_e)}{\partial x_{ie}}P\hat{x}_e\right] \\ &\quad +\text{vec}\left[v_{ofna}^T\frac{\partial G^T(x_e)}{\partial x_{ie}}P\hat{x}_e\right]+PBv_{ofna}+PG(\hat{x}_e)v_{ofso} \\ &\quad +PG(\hat{x}_e)v_{ofna}+P\begin{bmatrix} 0 \\ L(t)(z_1-\hat{z}_1) \end{bmatrix} \\ &\quad +PF(x_{1e},u_{2d})x_{1e}+\text{vec}\left[\frac{\partial(F(x_{1e},u_{2d})x_{1e})^T}{\partial x_{ie}}P\hat{x}_e\right], \end{aligned}$$

where $\tilde{x}_e = x_e-\hat{x}_e$ and $\tilde{z}=z-\hat{z}$. By noting $\tilde{x}_{1e}\equiv 0$ and by utilizing the special structures and functional dependence of matrices $G(\hat{x}_e)$, $F(x_{1e},u_{2d})$, B, C, P and R, one can show that, parallel to (50),

$$E_o(\hat{x}_e,v_{ofso}) = \begin{bmatrix} \frac{\partial(F_2x_{1e})^T}{\partial x_{1e}}P_2\hat{z} \\ Y_o \end{bmatrix}, \quad (74)$$

$$\begin{aligned} & E_o(\hat{x}_e,v_{ofno})-E_o(\hat{x}_e,v_{ofso}) \\ &= \begin{bmatrix} p_1B_1 & 0 \\ (A_2^*)^TP_2\hat{z}+P_2G_2(\hat{z}) & P_2B_2 \end{bmatrix}\begin{bmatrix} v_{ofna,1} \\ v_{ofna,2} \end{bmatrix}, \quad (75) \end{aligned}$$

where Y_o is defined by (72), $E_o(\hat{x}_e,v_{ofno}) = \begin{bmatrix} E_{o1}(\hat{x}_e,v_{ofno}) & E_{o2}^T(\hat{x}_e,v_{ofno}) \end{bmatrix}^T$ and $E_o(\hat{x}_e,v_{ofso}) = \begin{bmatrix} E_{o1}(\hat{x}_e,v_{ofso}) & E_{o2}^T(\hat{x}_e,v_{ofso}) \end{bmatrix}^T$ are the optimal residues of output feedback near-optimal control v_{ofno} and output feedback sub-optimal control v_{ofso}, respectively. It is straightforward to show as did lemma 4 that, upon fixing $v_{ofna,1}=0$, $\|E_o(\hat{x}_e,v_{ofno})\|^2$ is minimized by $v_{ofna}(t)$ in (71), that is,

$$\|E_{o2}(\hat{x}_e,v_{ofno})\| \le \|E_{o2}(\hat{x}_e,v_{ofso})\| \quad (76)$$

Part II: To show that control $v_{ofso}(\hat{x}_e,t) = -R^{-1}(t)B^TP(t)\hat{x}_e$ is globally exponentially stabilizing, we note that $v_{ofso}(\hat{x}_e,t) = v_{sfso}(x_e,t)+\tilde{v}(\tilde{x}_e,t)$, where $v_{sfso}(x_e,t)$ is given by (29), and $\tilde{v}(\tilde{x}_e,t) \triangleq R^{-1}(t)B^TP(t)\tilde{x}_e$.

Consider again the Lyapunov function in (25). It follows from the discussion leading to (43) that the time derivative of V along any trajectory of (3) under control $v_{ofso}(\hat{x}_e,t)$ is given by

$$\begin{aligned} \dot{V} &= x_e^T\dot{P}x_e+2x_e^TP[Ax_e+Bv_{sfso}]+2x_e^TPB\tilde{v} \\ &\quad +2x_e^TPG(x_e)v_{ofso}+2x_e^TPFx_{1e} \\ &\le -\gamma_3\|x_e\|^2+2x_e^TPB\tilde{v}+2x_e^TPFx_{1e} \\ &\quad +2x_e^TP[G(\tilde{x}_e)+G(\hat{x}_e)]v_{ofso} \\ &= -\gamma_3\|x_e\|^2+2z^TP_2B_2r_2^{-1}B_2^TP_2\tilde{z}+2z^TP_2F_2x_{1e} \\ &\quad -2z^TP_2[G_2(\tilde{z})+G_2(\hat{z})]r_1^{-1}p_1x_{1e}. \end{aligned} \quad (77)$$

It follows from (74) that inequality (77) can be expressed as

$$\begin{aligned} \dot{V} &\le -\gamma_3\|x_e\|^2+2z^TP_2B_2r_2^{-1}B_2^TP_2\tilde{z} \\ &\quad +2z^T[-P_2G_2(\tilde{z})+(A_2^*)^TP_2\hat{z}]r_1^{-1}p_1x_{1e} \\ &\quad -2z^T(q_2C_2^T+P_2L)\tilde{z}_1+2z^TE_{o2}(\hat{x}_e,v_{ofso}) \\ &\le -\gamma_3\|x_e\|^2+2z^TP_2B_2r_2^{-1}B_2^TP_2\tilde{z}+2z^T[-P_2G_2(\tilde{z}) \\ &\quad +(A_2^*)^TP_2z-(A_2^*)^TP_2\tilde{z}]r_1^{-1}p_1x_{1e}-2z^T(q_2C_2^T \\ &\quad +P_2L)\tilde{z}_1+2\|z\|\cdot\|E_{o2}(\hat{x}_e,v_{ofso})\|, \end{aligned} \quad (78)$$

which will be referenced in the part III for stability analysis of output feedback near optimal control (69).

Applying the expression of $E_{o2}(\hat{x}_e,v_{ofso})$ in (74) to inequality (78) and then taking bounds yield

$$\begin{aligned} \dot{V} &\le -\gamma_3\|x_e\|^2+2\|z\|\left[\|P_2B_2r_2^{-1}B_2^TP_2\|\cdot\|\tilde{z}\|\right. \\ &\quad \left.+r_1^{-1}p_1|x_{1e}|\cdot\|P_2\|(4\|\tilde{z}\|+3\|z\|)\right] \\ &\quad +4|\tilde{z}_1|\cdot\|z\|(q_2+\|P_2\|\cdot\|L\|) \\ &\quad +2\|z\|\cdot\|P_2\|\cdot\|F_2\|\cdot|x_{1e}|. \end{aligned} \quad (79)$$

Substituting (31) into inequality (79) and invoking the fact of $\tilde{z}$ being exponentially stable (which is stated in lemma 5; say, of rate $\gamma_{\mu,3}$) yield

$$\begin{aligned} \dot{V} &\le -\gamma_3\|x_e\|^2+c_3\|x_e\|e^{-c_4t}+(c_5\|x_e\| \\ &\quad +c_6\|x_e\|^2)e^{-c_2t}+c_7\|x_e\|e^{-(c_2+c_4)t} \\ &\le \left[-\beta_0+\beta_3e^{-c_2t}\right]V+\beta_4\sqrt{V}e^{-c_4t} \\ &\quad +\beta_5\sqrt{V}e^{-c_2t}+\beta_6\sqrt{V}e^{-(c_2+c_4)t}, \end{aligned} \quad (80)$$

where $c_3 = (2\gamma_2^2\underline{r}^{-1}+4\bar{q}+4\gamma_{\mu,1}^{-1}\gamma_2)\|\tilde{z}(t_0)\|\sqrt{\gamma_{\mu,3}\gamma_{\mu,2}/\gamma_{\mu,1}}$, $c_4 = \gamma_{\mu,3}$, $c_5 = 2\gamma_2c_f|x_{1e}(t_0)|$, $c_6 = 6p_1r_1^{-1}\gamma_2|x_{1e}(t_0)|$, $c_7 = 8p_1r_1^{-1}\gamma_2|x_{1e}(t_0)|\|\tilde{z}(t_0)\|\sqrt{\gamma_{\mu,3}\gamma_{\mu,2}/\gamma_{\mu,1}}$, $\beta_3 = c_6/\gamma_1$, $\beta_4 = c_3/\sqrt{\gamma_1}$, $\beta_5 = c_5/\sqrt{\gamma_1}$, and $\beta_6 = c_7/\sqrt{\gamma_1}$. Exponential stability is obvious by comparing the above inequality and (48) and by invoking the subsequent argument in the proof of theorem 1.

Part III: To show that control (69) together with v_{ofna} in (71) is globally exponential stabilizing and uniformly near optimal over time, consider again Lyapunov function in (25) whose time derivative along any trajectory of (9) and (10) and under control (69) is

$$\begin{aligned}\dot{V} &= x_e^T\dot{P}x_e + 2x_e^T P[Ax_e + Bv_{sfso}] \\ &\quad +2x_e^T PB\tilde{v} + 2x_e^T PBv_{ofna} \\ &\quad +2x_e^T PG(x_e)[v_{ofso} + v_{ofna}] + 2x_e^T PFx_{1e}.\end{aligned}$$

Noting that $v_{ofna,1} \equiv 0$ implies $G(x_e)v_{ofna} = 0$. Thus, it follows from the discussion leading to (77) that,

$$\begin{aligned}\dot{V} &\le -\gamma_3\|x_e\|^2 + 2x_e^T PB\tilde{v} + 2x_e^T PBv_{ofna} \\ &\quad +2x_e^T P[G(\tilde{x}_e) + G(\hat{x}_e)]v_{ofso} + 2x_e^T PFx_{1e} \\ &= -\gamma_3\|x_e\|^2 + 2z^T P_2B_2r_2^{-1}B_2^TP_2\tilde{z} + 2z^TP_2B_2v_{ofna,2} \\ &\quad -2z^TP_2[G_2(\tilde{z}) + G_2(\hat{z})]r_1^{-1}p_1x_{1e} \\ &\quad +2z^TP_2F_2x_{1e}.\end{aligned} \tag{81}$$

Now, substituting expression (75) and control (71) into (81) yields

$$\begin{aligned}\dot{V} &\le -\gamma_3\|x_e\|^2 + 2z^TP_2B_2r_2^{-1}B_2^TP_2\tilde{z} \\ &\quad +2z^T[-P_2G_2(\tilde{z}) + (A_2^*)^TP_2\hat{z}]r_1^{-1}p_1x_{1e} \\ &\quad -2z^T(q_2C_2^T + P_2L)\tilde{z}_1 + 2z^TE_{o2}(\hat{x}_e, v_{ofno}) \\ &\le -\gamma_3\|x_e\|^2 + 2z^TP_2B_2r_2^{-1}B_2^TP_2\tilde{z} \\ &\quad +2z^T[-P_2G_2(\tilde{z}) + (A_2^*)^TP_2z - (A_2^*)^TP_2\tilde{z}]r_1^{-1}p_1x_{1e} \\ &\quad -2z^T(q_2C_2^T + P_2L)\tilde{z}_1 + 2\|z\|\|E_{o2}(\hat{x}_e, v_{ofno})\|,\end{aligned} \tag{82}$$

It follows from the result of (76) under control (71) that the conclusion of stability and uniformly improved performance can be drawn by comparing (78) and (82) and by invoking the comparison theorem [33]. □

V. Application to Control of a Mobile Robot

In this section, the proposed framework of near-optimal control design is applied to a car-like mobile robot. Like an automobile, front wheels of the robot are steering wheels, and rear wheels are driving wheels with a fixed straight forward orientation. As shown in [37], [38], kinematic model of the car-like robot is given by:

$$\begin{aligned}\dot{x}_c = \rho_c\cos(\theta_c)\omega_{c1}, \quad \dot{y}_c = \rho_c\sin(\theta_c)\omega_{c1}, \\ \dot{\theta}_c = \tfrac{\rho_c}{l_c}\tan(\phi_c)\omega_{c1}, \quad \dot{\phi}_c = \omega_{c2},\end{aligned} \tag{83}$$

where (x_c, y_c) are Cartesian coordinates of the guidepoint, θ_c is the orientation angle of the car body with respect to the x_c axis, ϕ_c is the steering angle, ρ_c is the driving wheel radius, l_c is the distance between the two wheel-axle centers, ω_{c1} is the angular velocity of the driving wheel, and ω_{c2} is the steering rate. Kinematic model (83) has singularity at $\phi_c = \pm\pi/2$, which fortunately does not occur either in practice or mathematically by limiting the range of ϕ_c within $(-\pi/2,\ \pi/2)$. The range of θ_c is also set within $(-\pi/2,\ \pi/2)$ to ensure an one-to-one mapping of following coordinate and control transformations:

$$x_1 = x_c, \quad x_2 = y_c, \quad x_3 = \tan(\theta_c), \quad x_4 = \frac{\tan(\phi_c)}{l_c\cos^3(\theta_c)},$$

and

$$\omega_{c1} = \frac{u_1}{\rho_c\cos(\theta_c)},$$

$$\omega_{c2} = -\frac{3\sin(\theta_c)}{l_c\cos^2(\theta_c)}\sin^2(\phi_c)u_1 + l_c\cos^3(\theta_c)\cos^2(\phi_c)u_2.$$

Under the above transformations, kinematic model (83) is mapped into chained form (1) with $n = 4$. It follows from the Lie group operation that the tracking errors are defined as $x_{1e} = x_1 - x_{1d}$, $x_{2e} = x_2 - x_{2d} + (x_1 - x_{1d})(x_{1d}x_{4d} - x_{3d}) + 0.5(x_{1d}^2 - x_1^2)x_{4d}$, $x_{3e} = x_3 - x_{3d} - x_{4d}(x_1 - x_{1d})$, and $x_{4e} = x_4 - x_{4d}$. In what follows, time varying smooth laws synthesized for tracking and regulation control are simulated for the car-like mobile robot.

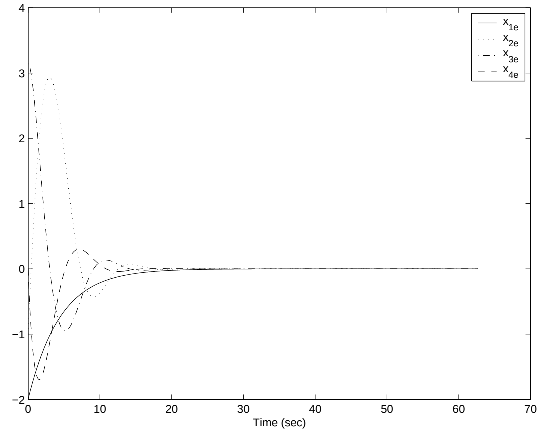
For trajectory tracking controls, let the reference trajectory be generated with zero initial conditions ($x_{1d}(0) = x_{2d}(0) = x_{3d}(0) = x_{4d}(0) = 0$) and under the two sinusoidal steering inputs:

$$u_{1d} = a_0 + a_1\sin(0.1t),\ u_{2d} = b_0 + b_1\cos(0.1t) + b_2\cos(0.2t),$$

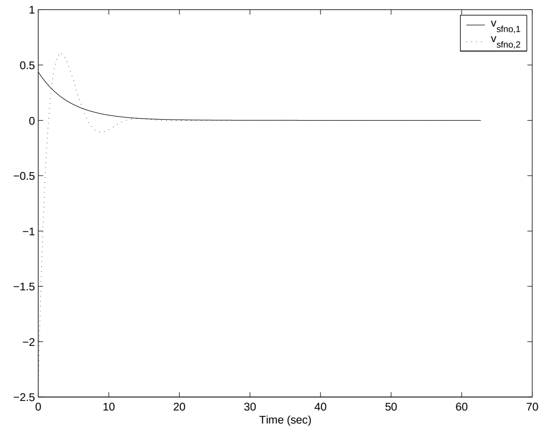
where $a_0 = 0.3183$, $a_1 = 1$, $b_0 = b_1 = 0$, and $b_2 = 0.0106$. Over the interval $[0, 20\pi]$, the desired trajectory moves from the initial position $[0,0,0,0]^T$ to the position $[20,10,0,0]^T$, and the segment is shown by the solid curve in figure (1f). Since the steering inputs are of period 20π, the reference trajectory for $t \ge 20\pi$ will continue its motion by repeating the same pattern of the segment defined in the interval $t \in [0, 20\pi]$.

In the simulation of state feedback near-optimal control, the following choices are made: (a) Initial conditions are set to be $x_1(0) = -2$, $x_2(0) = -1$, $x_3(0) = \pi$, and $x_4(0) = 0$; (b) Control design parameters are chosen to be $r_1 = r_2 = 20$ and $q_1 = q_2 = 1$. In figures (1b) and (1a), state-feedback near-optimal control (49) (consisting of (39) and (52)), and its corresponding closed loop tracking error state variables are provided, respectively. For the purpose of comparison, state feedback sub-optimal control (39) and its resulting error state trajectory are given by figures (1d) and (1c), respectively. In figure (1e), histories of the optimality residual values under the two state-feedback controls are plotted. In figure (1f), phase portraits in the Cartesian space are plotted. It is obvious that the proposed state-feedback near-optimal control (49) together with (52) achieves better performance than that under suboptimal control (39).

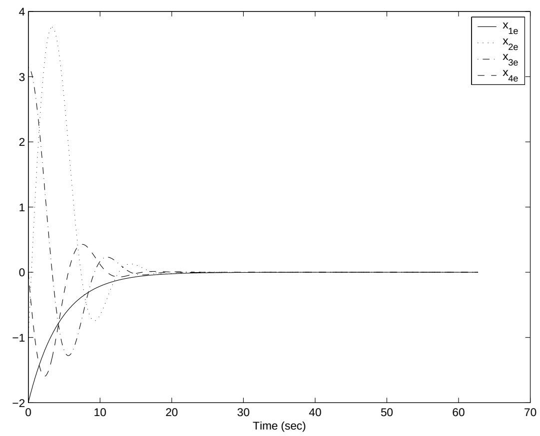
For output-feedback near-optimal tracking control (69), the same choices are made as those for the near optimal state-feedback tracking control, and the additional choices made for observer (58) include: initial condition $\hat{z}(t_0) = [\ -1 \quad 0 \quad 0\ ]^T$, $\delta_o = 2$, observer gain vector $L(t)$ in (60) and with $\beta_\mu(s,t) = a_0(s-t) + 10a_1(\cos(0.1t) - \cos(0.1s)) + r_1^{-1}p_1x_{1e}(0)\sqrt{r_1/q_1}\left(e^{-\sqrt{q_1/r_1}s} - e^{-\sqrt{q_1/r_1}t}\right)$. In figure 2, simulation results under the control are provided, including a comparison (figure (2c)) against to output-feedback sub-optimal control (70). In figure (2d), convergence of state estimation by the proposed observer is shown.



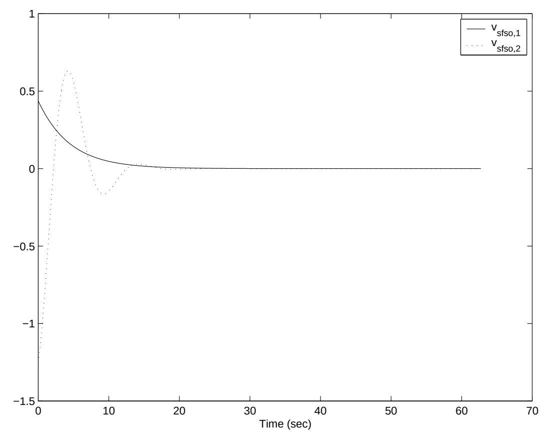
(1a) Tracking errors under near-optimal control



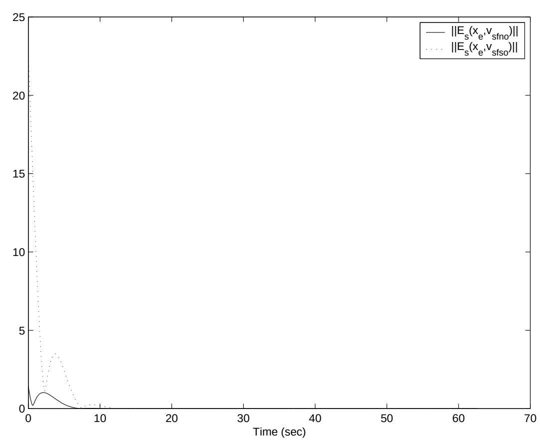
(1b) State-feedback near-optimal control



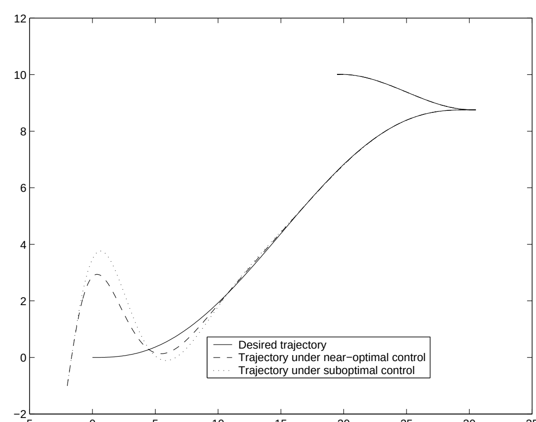
(1c) Tracking errors under suboptimal control



(1d) State-feedback suboptimal control

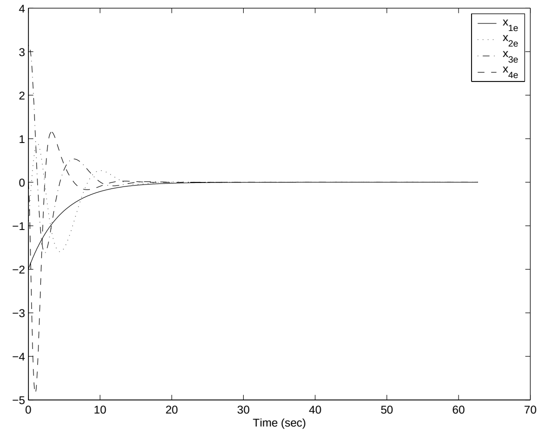


(1e) History of optimality residual value

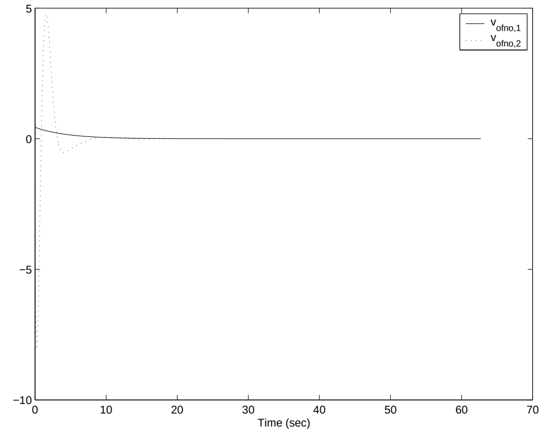


(1f) Phase portraits (x_c versus y_c)

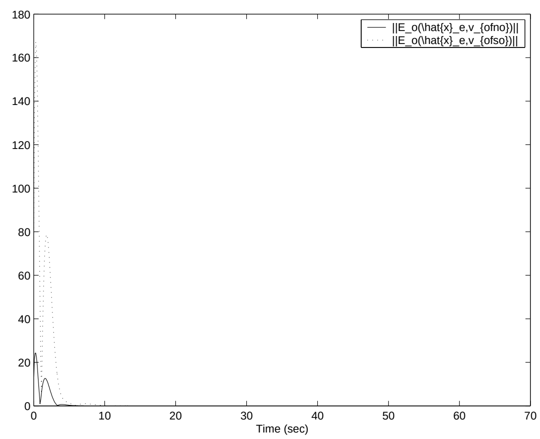
Fig. 1. Simulation results of state feedback controls



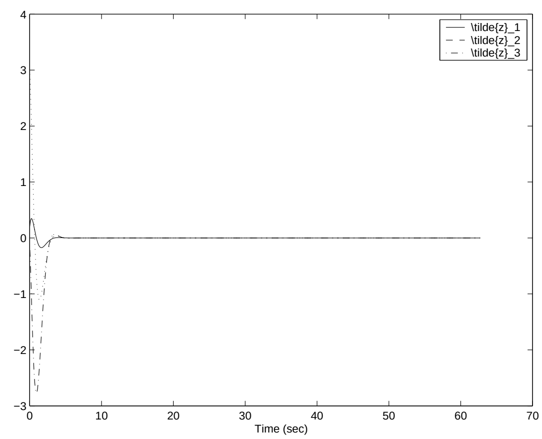
(2a) Tracking errors under near-optimal control



(2b) Output-feedback near optimal control



(2c) History of optimality residual value



(2d) Convergence of state estimation error

Fig. 2. Simulation results of output feedback controls

VI. Conclusion

In this paper, a new unifying design framework is proposed for controlling nonholonomic chained systems by investigating uniform complete controllability of time varying systems, by presenting a time-folding/unfolding technique, and by developing the concept of near optimal control. It is explicitly shown that, for both trajectory tracking and regulation of nonholonomic systems, uniform complete controllability can be retained by transformations no matter whether $u_{1d}(t)$ is uniformly nonvanishing or merely nonconvergent to zero or vanishing or identically zero (in the last case, $u_1(t)$ is directly analyzed). As a result of the common controllability property, tracking control and stabilizing control can be designed in a unified manner. In addition, near-optimal state and output tracking controls can be designed in three steps: two optimal control solutions are obtained first for two linear nominal subsystems, their combination is used to generate a stabilizing but suboptimal for the overall system, and a nonlinear additive control term is calculated using the optimality condition to minimize the distance between the suboptimal control and the unattainable optimal control. It is shown that all the proposed controls are globally asymptotically stabilizing, in simple closed forms, time varying and smooth, and near-optimal. Simulation study of a car-like robot shows effectiveness of the proposed methodology.

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