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Robust state observer and control design using command-to-state mapping ☆

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Abstract

In this paper, by introducing the concept of command-to-state/output mapping, it is shown that the state of an uncertain nonlinear system can robustly be estimated if command-to-state mapping of the system and that of an uncertainty-free observer converge to each other. Then, a global Jacobian system is defined to capture this convergence property for the dynamics of estimation error, and a set of general stability and convergence conditions are derived using Lyapunov direct method. It is also shown that the conditions are constructive and can be reduced to an algebraic Lyapunov matrix equation by which nonlinear feedback in the observer and its corresponding Lyapunov function can be searched in a way parallel to those of nonlinear control design. Case studies and examples are used to illustrate the proposed observer design method. Finally, observer-based control is designed for systems whose uncertainties are generated by unknown exogenous dynamics.

Keywords: Robust observer; Robust nonlinear control; Robust state estimation; Observer Lyapunov function; Lyapunov direct method; Lyapunov stability; Matrix Lyapunov equation; Generalized algebraic Lyapunov equation; Jacobian matrix

1. Introduction

Estimation of internal state variables has always been an integrated part of control design. For nonlinear systems, observer design to estimate the full state from input and output remains to be a challenging problem. If the system under consideration is subject to bounded disturbances, a successful observer design must be robust. Existing results of global convergent observers such as Tsinias (1989), Gauthier, Hammouri, and Othman (1992) are for particular types of systems and do not consider uncertainties except for Dawson, Qu, and Carroll (1992), Shim and Seo (2003). Closely related are high-gain but semi-globally and robustly convergent observer designs (Khalil, 1996; Khalil & Esfandiari, 1993). On the other hand, observer design can also be pursued by studying local behaviors of nonlinear systems, often through the use of first-order Taylor series expansion around the origin (Lohmiller & Slotine, 1998) or the adoption of a quadratic Lyapunov function (Tsinias, 1990; Praly, 2001) (which, in general for nonlinear systems, is valid only locally within some region). Although these results are important and significant, there is a lack of general understanding on what conditions nonlinear state estimation requires, on plant dynamics and on the reference input. Most importantly, there is no process reported so far for an engineer to constructively search for and design an observer.

In this paper, state estimation is considered for general nonlinear systems with uncertainties and measurement noises. Instead of starting with imposing various conditions on the plant to be estimated or on the observer, we study the robust estimation problem by investigating *command-to-state/output mapping* (CSM). The idea is that, when the command, uncertainty and noise are all present (in general), any successful observer design must make its CSM converge to that of

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the plant while attenuating the influence of uncertainty and noise. This prompts us to introduce the concept of a CSM being convergent. Using this concept, a set of conditions on synthesizing a general class of robust observers are obtained using the Lyapunov direct method.

It is worth pointing out that the concept of a CSM being convergent is different from Lyapunov stability concepts (Khalil, 1996) and input-to-state stability (Sontag & Wang, 1995) and that it is similar to incremental stability (Angeli, 2002). In other words, while a plant to be estimated and its observer must be uniformly bounded, they need not be stable or asymptotically stable or input-to-state stable. In fact, there is no need to estimate any asymptotically stable state variable(s). Another reason to use the concept of CSM is that, for nonlinear systems, observability may depend on specific properties of the command. That is, observability for all bounded values of the command is a requirement too restrictive to be met in certain applications. It is shown that, for the plant and its observer to have convergent CSM (in the presence of uncertainty and noises or not), the error dynamics between the observer and the plant (not just the observer itself or the plant) should have certain stability properties (such as Lyapunov asymptotic stability in the absence of uncertainty). It turns out that stability and convergence of the error dynamics are equivalent to those of a *nonlinearly-defined Jacobian system* (for both the observer and the plant), not only locally but also globally (i.e., everywhere) in the appropriate state space.

The concept of CSM and the innovative development/use of its globally-valid nonlinear Jacobian system enable us to convert the robust estimation problem to a stabilization problem. It is shown that, for many cases, algebraic state-dependent Lyapunov matrix equations are the conditions for designing globally convergent observers. In fact, the conditions mirror the process of determining state-feedback robust control or optimal control for nonlinear uncertain systems (Isidori, 1995 & 1999; Khalil, 1996; Krstic, Kanellakppoulos, & Kokotovic, 1995; Qu, 1998). Case study and design examples are used to illustrate observer designs, including recursive designs. In particular, the conditions provide a natural way to search for global Lyapunov function (whether being quadratic or not) and to find the feedback function in the nonlinear observer. These features make the proposed method general, constructive, and promising. As an application, robust observer-based control is designed for a plant in which uncertainties are generated by an exogenous system. It is shown that the concept of CSM and the corresponding observer design can be directly applied to achieve global stability and convergence. This extends the results in Qu (2002) and Qu and Jin (2001) to non-affine nonlinear uncertain systems.

2. State estimation using CSM

In this section, the concept of CSM is introduced for the purpose of feedback state estimation of nonlinear uncertain systems. It will then be used in robust observer and control designs in the subsequent sections.

Consider the following system:

$$\dot{z} = F(t,z,r) + \Delta F(t,z,r,\eta),\ \ z(t_0) = z_0,$$
$$y = H(t,z,r),\ \ y_m = y + \Delta H(t,z,r,\eta_m), \tag{1}$$

where $z \in \Re^n$ is the state, $r \in \Re^m$ is the command, $\eta \in \Re^l$ is uncertainty, $y \in \Re^p$ is the output, y_m is measurement of y, and $\eta_m \in \Re^{l_m}$ is measurement noise.

The following assumptions are introduced, and most of their requirements are standard while partial differentiability of $F(\cdot)$ and $H(\cdot)$ enables us to define shortly the so-called Jacobian system.

Assumption 1. *Known dynamics of $F(t,z,r)$ and $H(t,z,r)$ are uniformly bounded with respect to t, locally uniformly bounded with respect to z and r, and differentiable with respect to z. Command $r(t)$ is uniformly bounded as $\|r(t)\| \leqslant \bar{c}_r$ and $\|\dot{r}(t)\| \leqslant \bar{c}_{\dot{r}}(t) \leqslant \bar{c}'_{\dot{r}}$.*

Assumption 2. *Unknown state z is uniformly bounded, i.e., $\|z(t)\| \leqslant \bar{c}_z$ and $\|\dot{z}(t)\| \leqslant \bar{c}_{\dot{z}}(t) \leqslant \bar{c}'_{\dot{z}}$.*

Assumption 3. *Uncertainty η belongs to a bounded set such that $\|\Delta F(t,z,r,\eta)\| \leqslant \bar{c}_\eta(t) \leqslant \bar{c}'_\eta$.*

Assumption 4. *Measurement noise η_m is bounded as $\|\Delta H(t,z,r,\eta_m)\| \leqslant \bar{c}_{\eta_m}(t) \leqslant \bar{c}'_{\eta_m}$.*

In general, $\Delta F(\cdot)$ would be bounded in size by a function of $\|z\|$ (or $\|y\|$) as $\|\Delta F(t,z,r,\eta)\| \leqslant \bar{c}_z(t,\|z\|)$. In light of Assumptions 2 and 3 can always be established, and hence Theorem 1 is little changed. If Assumption 2 is to be established (as will be in Theorem 2), the proof of Theorem 2 can be extended to the general case of $\bar{c}_z(t,\|z\|)$ by applying Lemma 2. The same can be said about Assumption 4.

2.1. System mappings and robust observer

System (1) can be viewed as an *input-to-state/output mapping*: $[r^{\mathrm{T}}\ \eta^{\mathrm{T}}\ \eta_m^{\mathrm{T}}]^{\mathrm{T}} \to z/y$. If there is neither uncertainty nor measurement noise, the system reduces to the so-called CSM: $r \to z/y$. For state estimation, the CSM is of the main concern as system uncertainty and noise are simply unknown. To estimate the state of system (1), consider the following general class of nonlinear observers:

$$\dot{\hat{z}} = F(t,\hat{z},r) - [G(t,e_{y_m},y_m,r) - G(t,e_{y_m},\hat{y},r)],$$
$$\hat{y} = H(t,\hat{z},r),\quad \hat{z}(t_0) = \hat{z}_0, \tag{2}$$

where $\hat{z}$ is the estimate of z, $e_{y_m} = y_m - \hat{y}$, and $G(\cdot)$ is the feedback function to be designed. Its CSM is $r \to \hat{z}/\hat{y}$. Thus, robust estimation is to make the CSM of an uncertain system practically converge to that of an uncertainty-free and noise-free CSM (of the observer), or vice versa. In other words, the two CSMs are asymptotically and practically convergent to each other if

$$\|z-\hat{z}\| \leqslant \alpha_1(c_{0,r}, t-t_0) + \alpha_2(\overline{c}'_\eta) + \alpha_3(\overline{c}'_{\eta_m}), \tag{3}$$

where $c_{0,r} \stackrel{\triangle}{=} \max\{\|z_0\|, \|\hat{z}_0\|, \overline{c}_r\}$, $\alpha_1(\cdot)$ is a class-$\mathscr{K}L$ function, and $\alpha_2(\cdot)$, $\alpha_3(\cdot)$ are class-$\mathscr{K}$ functions.[1]

2.2. Error dynamics

Let state and output estimation errors be $e = z - \hat{z}$ and $e_y = y - \hat{y}$, respectively. It follows from (1) and (2) that the error dynamics are

$$\dot{e} = \mathscr{F}_\delta(t, e_y, z, z-e, r) + \Delta F + \Delta F_m, \tag{4}$$

where ΔF and ΔH are defined in (1), $e_{y_m} = e_y + \Delta H$,

$$\mathscr{F}(t, e_y, z, r) \stackrel{\triangle}{=} F(t, z, r) + G(t, e_y, H(t, z, r), r),$$

$$\mathscr{F}_\delta(t, e_y, z, z-e, r) \stackrel{\triangle}{=} \mathscr{F}(t, e_y, z, r) - \mathscr{F}(t, e_y, z-e, r),$$

and

$$\Delta F_m(t, y_m, y, \hat{y}, r) \stackrel{\triangle}{=} G(t, e_{y_m}, y_m, r) - G(t, e_{y_m}, \hat{y}, r) - [G(t, e_y, y, r) - G(t, e_y, \hat{y}, r)].$$

It will be shown later that convergent CSMs do not imply Lyapunov stability or input-to-state stability of system (1) or (2). Instead, the concept of convergent CSM is on stability of error dynamics. Specifically, we know from (3) that the nominal estimation error system

$$\dot{e} = \mathscr{F}_\delta(t, e_y, z, z-e, r) \tag{5}$$

is asymptotically stable and that error system (4) is input-to-state stable with respect to both η and η_m.

Convergence of CSMs is closely related to incremental stability in Angeli (2002). Their difference is that the relationship $r \to z$ is investigated as the CSM for either a given r or a class of commands while command r is treated in incremental stability as the "disturbance." By focusing upon specific choice(s) of r, the CSM can be used to develop stability and convergence conditions explicitly in terms of r and z (as will be shown in Theorems 1 and 2), which is not only useful for all the systems but also critical to those systems (such as non-holonomic systems in the chained form) whose observability is command-dependent.

2.3. Jacobian system and its global equivalence

In order to facilitate observer design and its associated search of Lyapunov function, a nonlinear *Jacobian system* is defined. For nominal error system (5), the so-called Jacobian system is defined by: for some $0<\delta<1$,

$$\dot{e} = [\triangledown_w \mathscr{F}(t, e_y, w, r)|_{w=z-\delta e}]e \stackrel{\triangle}{=} \overline{A}(t, e_y, \delta, e, z, r)e, \tag{6}$$

where $\triangledown_w \mathscr{F}(\cdot) \stackrel{\triangle}{=} \partial\mathscr{F}(\cdot)/\partial w$. It should be noted that locally-defined Jacobian systems, especially linearized systems at the origin by a first-order Taylor expansion, have been widely used. In the context of observer design, observer synthesis using a locally-defined Jacobian together with constant Lyapunov function matrix P has been pursued in Lohmiller and Slotine (1998), Tsinias (1990), Praly (2001). The following lemma shows that, although the mean value theorem does not hold in general for vector functions, Lyapunov stability of error system (5) and Jacobian system (6) are closely linked. In particular, there is an equivalence in Lyapunov stability argument between the two systems, not only locally but also *globally*.

Lemma 1. *Consider systems* (5) *and* (6). *Given any Lyapunov function* $V(t, e, z, r)$, *suppose that one of the following inequalities holds for some function* $\gamma(\cdot)$:

$$\triangledown_t V + \triangledown_e^{\mathrm{T}} V \mathscr{F}_\delta(t, e_y, z, z-e, r) \leqslant -\gamma(\|e\|), \tag{7}$$

$$\triangledown_t V + \triangledown_e^{\mathrm{T}} V \triangledown_w \mathscr{F}(t, e_y, w, r)|_{w=z-\delta e} e \leqslant -\gamma(\|e\|). \tag{8}$$

Then, inequality (7) *implies that inequality* (8) *holds for some constant* $0<\delta<1$. *Conversely, if inequality* (8) *holds for all choices of constant* $\delta \in (0,1)$ (*or, more restrictively, for all* w), *inequality* (7) *holds.*

Proof. Let function $\xi(\cdot): \Re^+ \to \Re$ be defined by

$$\xi(\delta) \stackrel{\triangle}{=} [\triangledown_e^{\mathrm{T}} V]\mathscr{F}(t, e_y, z-\delta e, r).$$

It follows that $\xi(1) - \xi(0) = -[\triangledown_e^{\mathrm{T}} V]\mathscr{F}_\delta(t, e_y, z, z-e, r)$. Applying the mean value theorem (Grossman, 1986) yields that $\xi(1) - \xi(0) = \triangledown_\delta \xi(\delta)|_{\delta=\delta^*}(1-0)$ holds for some $\delta^* \in (0,1)$. Direct computation yields

$$\triangledown_\delta \xi(\delta) = -[\triangledown_e^{\mathrm{T}} V][\triangledown_w \mathscr{F}(t, e_y, w, r)|_{w=z-\delta e}]e.$$

Therefore, there exists $\delta^* \in (0,1)$ such that

$$[\triangledown_e^{\mathrm{T}} V][\triangledown_w \mathscr{F}(t, e_y, w, r)|_{w=z-\delta^* e}]e = [\triangledown_e^{\mathrm{T}} V]\mathscr{F}_\delta(t, e_y, z, z-e, r),$$

from which the two statements can be concluded. □

2.4. Useful stability results

Lemma 2 combines several standard results from texts (Khalil, 1996; Qu, 1998) into a concise form conducive to the subsequent analysis, and its proof is omitted.

Lemma 2. *Suppose that a Lyapunov function* $V(t) \geqslant 0$ *satisfies the differential inequality*

$$\dot{V} \leqslant -\lambda_0 \gamma(V) + \lambda_1(t)\gamma(V) + \sum_{i=2}^{l_v} \lambda_i(t)\gamma^{\beta_i}(V), \tag{9}$$

[1] As defined in Khalil (1996), $\alpha(s): \Re^+ \to \Re^+$ is a class-$\mathscr{K}$ function if $\alpha(0)=0$ and it is strictly increasing; $\alpha(s): \Re^+ \to \Re^+$ is a class-$\mathscr{K}_\infty$ function if it is class-$\mathscr{K}$ and $\alpha(s) \to \infty$ as $s \to \infty$; $\alpha(s,t): \Re^+ \times \Re^+ \to \Re^+$ is a class-$\mathscr{K}L$ function if, for each fixed t, it is a class-$\mathscr{K}$ function of s and if, for each fixed s, it is decreasing with respect to t and $\alpha(s,t) \to 0$ as $t \to \infty$.

where $\lambda_0 > 0$, $l_v \geqslant 2$ *is an integer*, $0 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_{l_v}$, $0 \leqslant \lambda_i(t) \leqslant \overline{\lambda}_i \leqslant \overline{\lambda}$, *and* $\gamma(\cdot)$ *is a class-*$\mathscr{K}_\infty$ *function. Then,*

(i) *If* $\beta_{l_v} < 1$ *and if* $\lambda_0 > \overline{\lambda}_1$, $V(t)$ *is globally uniformly bounded and also ultimately bounded by a class-*$\mathscr{K}$ *function of* $\overline{\lambda}/(\lambda_0 - \overline{\lambda}_1)$. *Furthermore, if* $\lambda_i(t)$ *converge asymptotically to zero for all i, so does* $V(t)$.

(ii) *If* $\beta_2 > 1$ *and if* $\lambda_0 > \overline{\lambda}_1$, $V(t)$ *is locally asymptotically stable.*

(iii) *If* $\beta_2 < 1$ *and* $\beta_{l_v} > 1$, $V(t)$ *is locally uniformly bounded and locally ultimately bounded provided that polynomial equation of* $-(\lambda_0 - \overline{\lambda}_1)p^{1-\beta_2} + \sum_{i=2}^{l_v} \overline{\lambda}_i p^{\beta_i - \beta_2} = 0$ *has two* (*or more*) *positive solutions for p.*

Lemma 3 relaxes the condition imposed on λ_0 and $\lambda_1(t)$ in Lemma 2, and it can be extended to the case that $\gamma(V)$ has certain property but is of a generic expression.

Lemma 3. *Consider case* (i) *in Lemma* 2. *If inequality* (9) *holds with* $\gamma(V) = V^q$ *and* $0 < q \leqslant 1$, *condition* $\lambda_0 > \overline{\lambda}_1$ *in Lemma* 2 *can be relaxed to be*

$$\lambda_0 > \lim_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} \lambda_1(\tau)\,\mathrm{d}\tau. \tag{10}$$

Proof. It follows from (9) that, for any given constant $0 < c < 1$, there exists a constant M (which depends on c and $\overline{\lambda}_i$, $i = 1, \ldots, l_v$) such that, for all $V \geqslant M$,

$$\dot{V} \leqslant -c\lambda_0 V^q + \lambda_1(t) V^q.$$

Solving the above differential inequality yields that, for all $V \geqslant M$,

$$\begin{array}{ll} V^{1-q}(t) \leqslant V^{1-q}(0) - (1-q)t\theta(t) & \text{if } 0 < q < 1, \\ V(t) \leqslant V(0)\mathrm{e}^{-t\theta(t)} & \text{if } q = 1, \end{array}$$

where $\theta(t) = \lambda_0 - \int_{t_0}^{t} \lambda_1(\tau)\,\mathrm{d}\tau / t$. Although $\theta(t)$ may assume negative values during a finite interval, it becomes positive and remains so after some finite time instant, and hence stability claim in case (i) of Lemma 2 can be established under inequality (10). □

It is worth noting that (10) is much less restrictive than $\lambda_0 > \overline{\lambda}_1$. For instance, inequality (10) becomes trivial for any function $\lambda_1(t)$ belonging to L_1 space. Also note that (10) is insufficient for $\gamma(V) = V^q$ with $q > 1$ as $\theta(t)$ assuming positive values over a finite interval can induce singularity and possibly a finite escape time for solution $V(t)$.

2.5. Stability conditions of robust observer

It follows that Jacobian system of error dynamics (4) is:

$$\dot{e} = \overline{A}e + \Delta F + \overline{E}\Delta H, \tag{11}$$

where $\Delta F(t, z, r, \eta)$ is the uncertainty, $\Delta H(t, z, r, \eta_m)$ is the noise, $0 < \delta < 1$, $A(t, w, r) \triangleq \bigtriangledown_w F(t, w, r)$, $C(t, w, r) \triangleq \bigtriangledown_w H(t, w, r)$, $L(t, e_y, q, r) \triangleq \bigtriangledown_q G(t, e_y, q, r)$,

$$\overline{A}_w(t, e_y, w, r) \triangleq A + L(t, e_y, H(t, w, r), r)C, \tag{12}$$

$$\overline{A}(t, e_y, \delta, e, z, r) = \overline{A}_w(t, e_y, z - \delta e, r), \tag{13}$$

and matrix $\overline{E}(t, \delta, e_y, y, r, \eta_m)$ is defined by

$$\begin{aligned} \overline{E} = {} & L(t, e_y + \Delta H, y - \delta e_y + (1-\delta)\Delta H, r) \\ & - L(t, e_y, y - \delta e_y, r). \end{aligned} \tag{14}$$

Matrix $\overline{A}(\cdot)$ in (13) is the same as that in (6), while matrix $\overline{A}_w(\cdot)$ defined in (12) will be used later.

Theorem 1 provides an explicit set of conditions for CSMs to be asymptotically convergent and, as a result of Lemma 1, it applies to both error system (4) and Jacobian system (11).

Theorem 1. *Consider system* (1) *under Assumptions* 1–4. *Then, inequality* (3) *holds if function* $G(\cdot)$ *and Lyapunov function* $V_e(t, e, z, r)$ *are found to meet the following inequalities*: *for any* $0 < \delta < 1$, *for some* $0 < \beta_1, \beta_2 < 1$ *and* $0 < \beta_3, \beta_4, q \leqslant 1$, *for constants* $c_z, c_r, c_d \geqslant 0$, $c_e, c_\gamma, c_q > 0$, *for some class-*$\mathscr{K}_\infty$ *functions* $\gamma_i(\cdot)$, *and for all* $s \geqslant 0$,

$$\gamma_1(\|e\|) \leqslant V_e \leqslant \gamma_2(\|e\|), \tag{15}$$

$$\bigtriangledown_t V_e + [\bigtriangledown_e^{\mathrm{T}} V_e]\overline{A}(t, e_y, \delta, e, z, r)e \leqslant -k_c\gamma_3(\|e\|), \tag{16}$$

$$\|\bigtriangledown_e V_e\| \leqslant c_e\gamma_3^{\beta_1}(\|e\|), \quad \|[\bigtriangledown_e^{\mathrm{T}} V_e]\overline{E}\| \leqslant c_d\gamma_3^{\beta_2}(\|e\|), \tag{17}$$

$$\|\bigtriangledown_z V_e\| \leqslant c_z\gamma_3^{\beta_3}(\|e\|), \quad \|\bigtriangledown_r V_e\| \leqslant c_r\gamma_3^{\beta_4}(\|e\|), \tag{18}$$

$$\gamma_3 \circ \gamma_1^{-1}(s) \leqslant c_\gamma \gamma_3 \circ \gamma_2^{-1}(s) = c_q s^q \triangleq c_\gamma \gamma(s), \tag{19}$$

where gain $k_c > 0$ *and constants* c_z, c_r *observe the conditions that*

$$k_c > \begin{cases} \delta_d(1-\beta_3)c_z c_\gamma \overline{c}'_{\dot{z}} \\ \quad + \delta_d(1-\beta_4)c_r c_\gamma \overline{c}'_{\dot{r}} & \text{if } q > 1, \\ \delta_d(1-\beta_3)c_z c_\gamma \overline{\rho}_{\dot{z}} \\ \quad + \delta_d(1-\beta_4)c_r c_\gamma \overline{\rho}_{\dot{r}} & \text{if } 0 < q \leqslant 1, \end{cases} \tag{20}$$

$$c_z \lim_{t\to\infty} \overline{c}_{\dot{z}}(t) = 0 \quad \text{if } \beta_3 < 1,$$
$$c_r \lim_{t\to\infty} \overline{c}_{\dot{r}}(t) = 0 \quad \text{if } \beta_4 < 1,$$

$\overline{\rho}_{\dot{z}} \triangleq \lim_{t\to\infty} (1/t)\int_{t_0}^{t} \overline{c}_{\dot{z}}(\tau)\,\mathrm{d}\tau$, $\overline{\rho}_{\dot{r}} \triangleq \lim_{t\to\infty} (1/t)\int_{t_0}^{t} \overline{c}_{\dot{r}}(\tau)\,\mathrm{d}\tau$, *and* $\delta_d(\cdot)$ *is the discrete impulse function.*

Proof. It follows that, for Jacobian system (11),

$$\begin{aligned} \dot{V}_e = {} & \bigtriangledown_t V_e + \bigtriangledown_e^{\mathrm{T}} V_e \dot{e} + \bigtriangledown_z^{\mathrm{T}} V_e \dot{z} + \bigtriangledown_r^{\mathrm{T}} V_e \dot{r} \\ \leqslant {} & \bigtriangledown_t V_e + \bigtriangledown_e^{\mathrm{T}} V_e \overline{A}e + \|\bigtriangledown_e^{\mathrm{T}} V_e \overline{E}\| \cdot \|\Delta H\| \\ & + \|\bigtriangledown_e V_e\| \cdot \|\Delta F\| + \|\bigtriangledown_z^{\mathrm{T}} V_e\| \cdot \|\dot{z}\| + \|\bigtriangledown_r^{\mathrm{T}} V_e\| \cdot \|\dot{r}\| \\ \leqslant {} & -k_c\gamma_3(\|e\|) + c_d\overline{c}'_{\eta_m}\gamma_3^{\beta_2}(\|e\|) + c_e\overline{c}'_{\eta}\gamma_3^{\beta_1}(\|e\|) \\ & + c_z\overline{c}_{\dot{z}}(t)\gamma_3^{\beta_3}(\|e\|) + c_r\overline{c}_{\dot{r}}(t)\gamma_3^{\beta_4}(\|e\|), \end{aligned}$$

in which (15)–(18) are used, and Assumptions 1–4 are invoked. It follows from (19) that

$$\dot{V}_e \leqslant -k_c\gamma(V_e) + c_d\overline{c}'_{\eta_m}c_\gamma\gamma^{\beta_2}(\|V_e\|) + c_e\overline{c}'_\eta c_\gamma\gamma^{\beta_1}(V_e) + c_z\overline{c}_{\dot{z}}(t)c_\gamma\gamma^{\beta_3}(V_e) + c_r\overline{c}_{\dot{r}}(t)c_\gamma\gamma^{\beta_4}(V_e). \tag{21}$$

Proof can be completed by invoking Lemmas 2 and 3. □

Several of the conditions in Theorem 1 are worth elaborating. First, condition (16) is equivalent to $\bigtriangledown_t V_e + [\bigtriangledown_e^{\mathrm{T}} V_e]\mathscr{F}_\delta \leqslant -k_c\gamma_3(\|e\|)$ for error system (4) and, if $\bigtriangledown_t V_e = 0$, it says that matrix $\overline{A}(\cdot)$ of the Jacobian system is asymptotically stable for all "frozen" values of command $r(t)$ and state $z(t)$. Second, gain $k_c > 0$ is usually the result of observer design, and the choices of feedback $G(\cdot)$ and its Lyapunov function $V_e(\cdot)$ are to achieve stability and to attenuate the effects of uncertainty and measurement noise. Third, the conditions are stated in simpler forms to expose the basic result of Theorem 1 and to simplify its proof, and they can be relaxed (by invoking Lemma 2 or an improved version of Lemma 3); for instance, β_2 in (17) does not have to be less than one, bounding function on $\|\bigtriangledown_e^{\mathrm{T}} V_e\overline{E}\|$ may have multiple terms, and function $\gamma(s)$ in (19) can be a function of different type or generic expression.

To achieve global convergence of estimation, condition (20) puts restrictions on the impacts of command $r(t)$ and state z by limiting their magnitudes or their rates of change in the limit. It follows from (18) and (17) that, for most choices of Lyapunov function $V_e(\cdot)$, both β_3 and β_4 are less than one. Thus, (20) becomes trivial in the following two situations. The first is that a Lyapunov function is independent of $\dot{r}$ and $\dot{z}$ (i.e., $V_e(t,e,z,r)=V_e(t,e)$) and hence $c_z=c_r=0$ in (20). The second is that, as $t\to\infty$, $r(t)\to r_{\mathrm{ss}}$ and $z(t)\to z_{\mathrm{ss}}$ for some steady states r_{ss} and z_{ss}. While command $r(t)$ is known, it is unlikely that $z(t)$ has a steady-state due to the presence of uncertainties. Thus, (20) is restrictive mainly for the case that Lyapunov function $V_e(\cdot)$ contains z while $\beta_3<1$. In this case, the magnitude of $c_{\dot{z}}(t)$ will depend upon $c_\eta(t)$, which is admissible for a convergent CSM. However, for a CSM to be convergent, the impacts of $z(0)$ and $r(t)$ on $c_{\dot{z}}(t)$ must also be limited by a class-$\mathscr{K}L$ function. Without pre-qualifying these impacts, Theorem 1 is established by imposing the condition $c_z \lim_{t\to\infty}\overline{c}_{\dot{z}}(t)=0$ whenever $\beta_3<1$. On the other hand, the impacts can be properly quantified if certain property of command-related dynamics and certain stability property of the nominal system are available. Such a result is provided by the following theorem.

Theorem 2. *Consider system* (1) *under Assumptions* 1, 3, *and* 4. *Then, bounds $\overline{c}_z(t)$ on $\|z\|$ and $\overline{c}_{\dot{z}}(t)$ on $\|\dot{z}\|$ are ultimately bounded by class-K_∞ functions of $c_\eta(t)$, respectively, if $A(t,z,r)\overset{\triangle}{=}\partial F(t,z,r)/\partial z$ is bounded as*

$$\|A(t,z,r)\| \leqslant \alpha_f(\|z\|), \tag{22}$$

and if the origin of $z=0$ is asymptotically attractive for the nominal system in the sense that

$$\lim_{t\to\infty}\|F(t,0,r)\|=0, \tag{23}$$

and that a Lyapunov function $V_z(t,z,r)$ exists to establish the following inequalities: for any $0<\delta<1$, for some $0<\beta_5<1$ and $0<\beta_6, q'\leqslant 1$, for constants $c'_z, c'_r, \geqslant 0$, $c'_\gamma, c'_q>0$, for gain $k_f>0$, and for some class-$\mathscr{K}_\infty$ functions $\gamma_i(\cdot)$,

$$\gamma_4(\|z\|)\leqslant V_z(t,z,r)\leqslant\gamma_5(\|z\|),$$

$$\bigtriangledown_t V_z + [\bigtriangledown_z^{\mathrm{T}} V_z]A(t,\delta z,r)z \leqslant -k_f\gamma_6(\|z\|), \tag{24}$$

$$\|\bigtriangledown_z V_z\|\leqslant c'_z\gamma_6^{\beta_5}(\|z\|),\quad \|\bigtriangledown_r V_z\|\leqslant c'_r\gamma_6^{\beta_6}(\|z\|),$$

$$\gamma_6\circ\gamma_4^{-1}(s)\leqslant c'_\gamma\gamma_6\circ\gamma_5^{-1}(s)=c'_q s^{q'}\overset{\triangle}{=}c'_\gamma\gamma_z(s), \tag{25}$$

and

$$k_f > \begin{cases} c'_r c'_\gamma \overline{c}'_{\dot{r}}, & \text{if } q'>1 \text{ and } \beta_6=1,\\ c'_r c'_\gamma \overline{\rho}_{\dot{r}}, & \text{if } 0<q'\leqslant 1 \text{ and } \beta_6=1,\end{cases}$$
$$c'_r\lim_{t\to\infty}\overline{c}_{\dot{r}}(t)=0 \quad \text{if } \beta_6<1, \tag{26}$$

where $\overline{\rho}_{\dot{r}}=\lim_{t\to\infty}\int_{t_0}^{t}\overline{c}_{\dot{r}}(\tau)\,\mathrm{d}\tau/t$.

Proof. It follows from (1) and Lemma 1 that stability of system

$$\dot{z}=[F(t,z,r)-F(t,0,r)]+[F(t,0,r)+\Delta F(t,z,r,\eta)]$$

can be studied by analyzing its Jacobian system

$$\dot{z}=A(t,\delta z,r)z+[F(t,0,r)+\Delta F(t,z,r,\eta)], \tag{27}$$

where $0<\delta<1$. It follows from (24) and (25) that, along all the trajectories of (27), the time derivative of Lyapunov function $V_z(\cdot)$ is bounded from above as

$$\dot{V}_z\leqslant -k_f\gamma_z(V_z)+c'_z c'_\gamma c_\eta\gamma_z^{\beta_5}(V_z)+c'_r c'_\gamma\overline{c}_{\dot{r}}\gamma_z^{\beta_6}(V_z). \tag{28}$$

Thus, we know from (26) and Lemma 3 that V_z, in turn $\|z\|$, and its bound $\overline{c}'_z$ are all uniformly ultimately bounded by a class-K_∞ function of c_η. It then follows from (27), (23), and (22) that

$$\lim_{t\to\infty}\sup_{t\geqslant s}\overline{c}_{\dot{z}}(s)\leqslant\lim_{t\to\infty}\sup_{t\geqslant s}[\alpha_f(\overline{c}_z(s))\overline{c}_z(s)+c_\eta(s)],$$

at which the proof is completed. □

Theorem 2 has three implications. First, Theorem 2 establishes the requirements on $z(t)$ and $\dot{z}(t)$ that are required in Assumption 2. Second, Corollary 1 stated below can be proven by invoking Theorem 2 and then by using inequality (21) and Lemma 2. Third, the existence of Lyapunov function $V_z(t,z,r)$ facilitates the search for $V_e(t,e,z,r)$ as $G(\cdot)\equiv 0$ yields $\overline{A}(t,e_y,\delta,e,z,r)=A(t,z-\delta e,r)$.

Corollary 1. *If Theorem* 2 *holds, Theorem* 1 *can still be applied after removing from* (20) *the condition of* $\lim_{t\to\infty}\overline{c}_{\dot{z}}(t)=0$ *whenever* $\beta_3<1$ *and* $c_z\neq 0$.

There is no need to explain the conditions in Theorem 2 as they are parallel to those in Theorem 1. Together, Theorems 1 and 2 provide a set of explicit conditions to check stability, convergence, and robustness of a nonlinear observer. The subject of the next section is to show how the two theorems can be applied to synthesize nonlinear observers. Such a constructive design involves the search for feedback $G(\cdot)$ and its corresponding Lyapunov function $V_e(\cdot)$. It will be shown that, through the use of matrix representation in the Jacobian system, the proposed design leads naturally to algebraic (state-dependent) matrix equations which could be solved in many cases and in the way comparable to a linear time-invariant design.

We conclude this section by comparing the concept CSM and standard stability concepts. In Theorem 1, state z is assumed to be uniformly bounded in the presence of uncertainty. In other words, fictitious system $\dot{z} = F(t, z, r) + u$ is bounded-input bounded-output with respect to "input" u. However, this does not mean that the fictitious system is input-to-state stable (Sontag & Wang, 1995), since input-to-state stability implies $z \to 0$ as $u \to 0$. Also note that the convergence of CSM is defined in terms of either one specific command $r(t)$ or one class of commands. As such, it is not necessary for system $\dot{z} = F(t, z, 0) - F(t, 0, 0)$ to be asymptotically stable. For example, consider the second-order system

$$\dot{z}_1 = -z_1 + z_2 + \Delta F_1, \quad \dot{z}_2 = -r^2(t) z_2 + \Delta F_2.$$

For these reasons, convergent CSM is proposed in the paper, and results on asymptotic stability or input-to-state stability are not invoked. Should command $r(t)$ be arbitrary, system $\dot{z} = F(t, z, r) - F(t, 0, 0) + u$ would have to be input-to-state stable with respect to both r and u in order for system (1) to remain bounded.

3. Nonlinear observer design using CSM

Jacobian system (11) makes it possible to express error dynamics globally into a standard matrix form. If the corresponding search of Lyapunov function is narrowed to the class of time-independent functions $V_e(e, z, r)$, a matrix representation can also be used. That is, matrix $P(e, z, r)$ is said to be *Lyapunov-integrable* if the partial differential equation

$$\bigtriangledown_e V_e(e, z, r) = e^{\mathrm{T}} P(e, z, r) \tag{29}$$

is integrable and the resulting scalar function $V_e(e, z, r)$ is positive definite with respect to e (as specified by (15)). Using the matrix representations, the conditions in Theorem 1 lead naturally to Lyapunov-based criteria for nonlinear observer designs, as evidenced by the following corollary. Its proof is obvious from (16) and (13).

Corollary 2. *Consider system* (1) *under Assumptions* 1, 3, *and* 4. *Nonlinear observer* (2) *can be found from the following steps*:

(i) *Function $G(\cdot)$ in* (2) *is chosen such that, given matrix $\overline{A}(t, e_y, \delta, e, z, r)$ in* (13), *state-dependent Lyapunov matrix equation*

$$P(e, z, r)\overline{A} + \overline{A}^{\mathrm{T}} P^{\mathrm{T}}(e, z, r) = -Q(t, e_y, \delta, e, z, r), \tag{30}$$

admits a positive definite matrix $Q(\cdot)$ and a Lyapunov-integrable matrix $P(\cdot)$ for all $\delta \in (0, 1)$, for r, and for all (t, e_y, e, z).

(ii) *Determine gain k_c and function $\gamma_3(\|e\|)$ defined in* (16) *by evaluating inequality $0 < k_c \gamma_3(\|e\|) \leqslant e^{\mathrm{T}} Q(t, e_y, \delta, e, z, r) e$.*

(iii) *Calculate the partial derivatives in* (18) *and* (17), *and find the set of constants $c_d, c_e, c_z, c_r, c_\gamma, c_q, \beta_i$.*

(iv) *Conclude stability, convergence and robustness of the observer by checking conditions* (19) *and* (20). *Whenever applicable*, *Theorem* 2 *and Corollary* 1 *should be invoked*.

The key step of the observer design process in Corollary 2 is the algebraic Lyapunov matrix (30). The following corollary provides further simplification.

Corollary 3. *Let $G(t, e_y, y, r) = G(t, 0, y, r)$. Then, Corollary* 2 *holds if Lyapunov matrix equation* (30) *is replaced by, for all $\chi = [e^{\mathrm{T}}\ z^{\mathrm{T}}]^{\mathrm{T}}$, for all $w \in \Re^n$ satisfying $|w_i| \leqslant |e_i| + |z_i|$, and for the given r,*

$$P(\chi, r)\overline{A}_w + \overline{A}_w^{\mathrm{T}} P^{\mathrm{T}}(\chi, r) = -Q(t, w, \chi, r), \tag{31}$$

where $A_w(t, 0, w, r)$ is the matrix defined in (12).

Three points are worth making here regarding nonlinear algebraic Lyapunov matrix (30) or (31). First, state-dependent (30) or (31) implies nonlinear observability, which includes standard linear results as special cases. For state estimation only, reachability is not required. However, unlike the case of linear systems, one cannot simply solve Eq. (31) for any positive definite choice of $Q(\cdot)$. This is because the resulting matrix $P(\cdot)$ must also be Lyapunov-integrable as required by (29). If Eq. (30) or (31) does admit a constant and symmetric solution P, integrability is guaranteed and the resulting Lyapunov function becomes quadratic in e.

Second, $\dot{w} = \overline{A}_w(t, 0, w, r) w$ is the Jacobian system corresponding to system $\dot{z} = F(t, z, r) + G(t, 0, H(t, z, r), r)$, and hence $\overline{A}_w(\cdot)$ can be viewed as its nonlinear system matrix. While pointwise linearization and the local Jacobian system around the origin have been commonly used, the proposed design makes their application global in the state space. Consequently, Eq. (31) enables the designer to find Lyapunov matrix $P(e, z, r)$ that depends upon e, or z, or r, or their combinations. Thus, Lyapunov function $V_e(e, z, r)$ is not restricted to be either quadratic in e or independent of z and r, which makes the proposed design more general than the existing design methods.

Third, the proposed observer design is closely connected to nonlinear control design. Specifically, since Lyapunov matrix equations (30) and (31) are both pointwise and global, the process of solving for solutions $P(\cdot)$ and feedback $G(\cdot)$ (for some $Q(\cdot) > 0$) is parallel to that in the problem of nonlinear optimal control over an infinite horizon. Also, output equation $y = H(t, z, r)$ generally implies that matrix $\partial H(t, w, r)/\partial w$ is a flat matrix, which makes the choice of $G(\cdot)$ be a dual problem to robust control designs under the matching conditions (Corless & Leitmann, 1981), or equivalently matched dynamics/uncertainties (Qu, 1992), or the generalized matching conditions (Qu, 1993).

In what follows, a 2-by-2 block design and three examples are presented to illustrate the process of choosing $Q(\cdot)$ and $\partial G(t, 0, y, r)/\partial y$ and then solving algebraic Lyapunov matrix (31) pointwise.

3.1. 2-by-2 block design

Consider uncertain system (1) and let the state be partitioned by the output as

$$H(t, z, r) = z_1 \in \Re^{n_1}, \implies y = [I \ \ 0]\, z = Cz, \tag{32}$$

where $z = [z_1^{\mathrm{T}} \ z_2^{\mathrm{T}}] \in \Re^n$. Accordingly, partition system matrix $A(t, w, r) \triangleq \nabla_w F(t, w, r)$ and Lyapunov matrix $P(\chi, r)$ in (29) and (31) as, with $\chi = [e^{\mathrm{T}} \ z^{\mathrm{T}}]^{\mathrm{T}}$,

$$A(t, w, r) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad P(\chi, r) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix},$$

respectively, where $P_{11}, A_{11} \in \Re^{n_1 \times n_1}$. On the other hand, one can choose

$$\begin{aligned} &G(t, 0, y, r) \triangleq R(t, r)\mathscr{G}(y, r), \quad R \triangleq \begin{bmatrix} R_1(t, r) \\ R_2(t, r) \end{bmatrix}, \\ &\nabla_y \mathscr{G}(y, r) = S(y, r), \end{aligned} \tag{33}$$

where $R_1, S \in \Re^{n_1 \times n_1}$ and $R_2 \in \Re^{n_2 \times n_1}$ are matrices to be chosen, and matrix $S(\cdot)$ needs to be integrable.

It follows from (31) that

$$Q(t, w, \chi, r) = \begin{bmatrix} Q_{11}(t, w, \chi, r) & Q_{12}(t, w, \chi, r) \\ Q_{12}^{\mathrm{T}}(t, w, \chi, r) & Q_{22}(t, w, \chi, r) \end{bmatrix},$$

where

$$\begin{aligned} -Q_{11} &= P_{11}A_{11} + A_{11}^{\mathrm{T}}P_{11}^{\mathrm{T}} + P_{12}A_{21} + A_{21}^{\mathrm{T}}P_{12}^{\mathrm{T}} \\ &\quad + P_{11}R_1S + S^{\mathrm{T}}R_1^{\mathrm{T}}P_{11}^{\mathrm{T}} + P_{12}R_2S + S^{\mathrm{T}}R_2^{\mathrm{T}}P_{12}^{\mathrm{T}}, \\ -Q_{12} &= P_{11}A_{12} + A_{11}^{\mathrm{T}}P_{21}^{\mathrm{T}} + P_{12}A_{22} + A_{21}^{\mathrm{T}}P_{22}^{\mathrm{T}} \\ &\quad + S^{\mathrm{T}}R_1^{\mathrm{T}}P_{21}^{\mathrm{T}} + S^{\mathrm{T}}R_2^{\mathrm{T}}P_{22}^{\mathrm{T}}, \end{aligned}$$

and

$$-Q_{22}(t, w, \chi, r) = P_{21}A_{12} + A_{12}^{\mathrm{T}}P_{21}^{\mathrm{T}} + P_{22}A_{22} + A_{22}^{\mathrm{T}}P_{22}^{\mathrm{T}}.$$

Therefore, observer (2) can be constructively designed if matrices R_i and S and matrix blocks $P_{ij}(\cdot)$ are chosen such that matrix $P(\chi, r)$ is Lyapunov-integrable as defined in (29), that $S(y, r)$ is integrable as defined in (33), and that, for all χ, for all w with $|w_i| \leqslant |e_i| + |z_i|$, and for r,

$$\begin{aligned} &\lambda_{\min}(Q_{11}) > 0, \quad \lambda_{\min}(Q_{22}) > 0, \\ &4\lambda_{\min}(Q_{11})\lambda_{\min}(Q_{22}) > \|Q_{12}\|^2, \end{aligned} \tag{34}$$

where $\lambda_{\min}(\cdot)$ denotes the operation of finding the minimum pointwise eigenvalue of its matrix argument.

Summarizing the above derivations under block partition (32), we know that Corollary 3 and observer design reduce to the following: Given the matrix pair $\{A(t, z, r), \ C\}$, choose a state-independent matrix $R(t, r)$ and an integrable gain matrix $S(Cz, r)$ such that algebraic Lyapunov matrix equation (31) with $\overline{A} = A + RSC$ yields a Lyapunov-integrable matrix $P(\chi, r)$ as its pointwise (yet global) solution, all for some $Q(\cdot) > 0$ (specified by (34)). Once gain, this result is a dual to the nonlinear control problem. For system whose Jacobian matrices have triangular structures, the block design can be recursively applied to synthesize an observer, which is parallel to that in control designs and will be shown by Example 3.

In many cases, making matrix $P(\cdot)$ symmetrical helps to ensure its Lyapunov-integrability. For instance, let matrix $P(\chi, r)$ be

$$P(\chi, r) = \begin{bmatrix} P_{11}(e_1, z, r) & P_{12} \\ P_{12}^{\mathrm{T}} & P_{22}(e_2, z, r) \end{bmatrix}, \tag{35}$$

then $P(\cdot)$ is Lyapunov-integrable if $P_{ii}(e_i)$ are integrable and if

$$\begin{aligned} &\int e_i^{\mathrm{T}} P_{ii}(e_i, z, r)\, \mathrm{d}e_i > 0, \quad i = 1, 2, \\ &\int e_2^{\mathrm{T}} P_{11}\, \mathrm{d}e_2 + \int e_2^{\mathrm{T}} P_{22}\, \mathrm{d}e_2 > 2e_1^{\mathrm{T}} P_{12} e_2. \end{aligned} \tag{36}$$

Condition (36) can be used to select non-quadratic Lyapunov function, as will be in Example 1.

Having a symmetrical $P(\chi, r)$ also facilitates the choice of matrix $Q(\cdot)$. It follows from the expression of Q_{22} that $\lambda_{\min}(Q_{22}) > 0$ implies that, for r and for all $\{(t, w, \chi) : |w_i| \leqslant |e_i| + |z_i|\}$,

$$P_{22}A'_{22} + [A'_{22}]^{\mathrm{T}} P_{22} < 0, \tag{37}$$

where $A'_{22} \triangleq A_{22} + P_{22}^{-1}P_{12}^{\mathrm{T}}A_{12}$. In other words, matrix A'_{22} (through the choices of P_{22} and P_{12}) has "uniformly stable pointwise eigenvalues," i.e., eigenvalues are stable both pointwise and uniformly for r and for all (t, w, χ) with $|w_i| \leqslant |e_i| + |z_i|$. Similarly, $\lambda_{\min}(Q_{11}) > 0$ implies

$$P_{11}A'_{11} + [A'_{11}]^{\mathrm{T}} P_{11} < 0, \tag{38}$$

where matrix $A'_{11} \triangleq A_{11} + P_{11}^{-1}P_{12}A_{21} + (R_1 + P_{11}^{-1}P_{12}R_2)S$ must be uniformly pointwise stable. One possibility to achieve uniform pointwise stability for matrix A'_{11} is to make $[R_1 + P_{11}^{-1}P_{12}R_2]S$ uniformly pointwise stable (through choosing R_1 and R_2) and to make $\|S\|$ large if, as a sufficient condition and through the choice of $P_{11}(\cdot)$, $A_{11} + P_{11}^{-1}P_{12}A_{21}$ is uniformly bounded in norm by a class-$\mathscr{K}$ function of r. Once Q_{11} and Q_{22} are made to be positive definite, all the available choices should be explored to meet the last condition in (34). As will be illustrated in Examples 1 and 2, choices should be finalized based on dynamics of the Jacobian system.

3.2. Design examples

The following example shows that Lyapunov matrix (31) often leads to non-quadratic Lyapunov functions.

Example 1. Consider system (1) with

$$F(t,z,r)=\begin{bmatrix}-z_1+0.5rz_2^2\\ -z_2+r\end{bmatrix},\quad H(t,z,r)=z_1.$$

It is straightforward to show (using Lyapunov function $L=z_1^2+cz_2^4$) that the uncertain system is globally uniformly bounded. Also, it follows that

$$A\triangleq\nabla_w F(t,w,r)=\begin{bmatrix}-1 & rw_2\\ 0 & -1\end{bmatrix},$$

which is globally and uniformly asymptotically stable (with $w=\delta z$ for all $\delta\in(0,1)$ and using the Lyapunov function L again). Thus, Theorem 2 applies if $\lim_{t\to\infty} r=0$.

It is clear that, for any choice of constant matrix P, Lyapunov matrix equation (31) does not yield a globally positive definite matrix Q. Hence, a non-quadratic Lyapunov function is needed for designing a global observer. Now, choose matrix $P(\cdot)$ to be that in (35) with $P_{12}=1$. In light of (37), we know that $A'_{22}=-1+rw_2/P_{22}(e_2,z,r)<-0.5$ is ensured for all $\{w_2:|w_2|\leqslant|e_2|+|z_2|\}$ by choosing

$$P_{22}(e_2,z,r)=2(2+\bar{c}_r^2e_2^2+\bar{c}_r^2z_2^2).$$

It follows from (38) that $A_{11}+P_{11}^{-1}P_{12}A_{21}+(R_1+P_{11}^{-1}P_{12}R_2)S=-1+(R_1+P_{11}^{-1}R_2)S\leqslant-1$ holds for all $P_{11},S>0$ and $R_1,R_2\leqslant 0$.

Next, choose $R_1=-1$, $R_2=0$, $P_{11}=1$, and $S\in(0,2+4\sqrt{2})$. Under these choices, $Q_{22}\geqslant 2+\bar{c}_r^2e_2^2+\bar{c}_r^2z_2^2$, $Q_{11}=2(1+S)$, $Q_{12}=2+S-rw_2$, and hence the last inequality in (34) is met. It follows from (35) and (36) that the resulting Lyapunov matrix P is Lyapunov-integrable and that $V_e(e,z,r)=\frac{1}{2}e_1^2+e_1e_2+(2+\bar{c}_r^2z_2^2)e_2^2+\frac{1}{2}\bar{c}_r^2e_2^4$. Thus, the resulting robust observer is given by (2) with $G(t,e_y,y,r)=[-Sy\ \ 0]^{\mathrm{T}}$.

Dynamics of the following example contain high-order terms of z_2 that cannot be eliminated in the design, either directly or through the operating of being bounded. As a result, existing results cannot be applied here.

Example 2. Consider system (1) with $y=z_1$ and

$$F=\begin{bmatrix}r^2z_1-\frac{1}{2l+1}z_1^{2l+1}-2(1+r)z_2+\frac{c_1}{3}z_2^3+r^2\\ -c_1z_1z_2^2-c_1z_1^2z_2-r^2z_2-c_1z_2^3\end{bmatrix},$$

where $l>1$ and $c_1\geqslant 0$ are system parameters. It is straightforward to show (using Lyapunov function $L=z_1^2+z_2^2$) that the system is not Lyapunov stable around the origin and that, since $l>1$, the system is robustly uniformly bounded unless $c_1=0$ and $r\to 0$. It follows that $A(t,w,r)\triangleq\nabla_w F(t,w,r)$ where

$$A=\begin{bmatrix}r^2-w_1^{2l} & -2(1+r)+c_1w_2^2\\ -c_1w_2^2-2c_1w_1w_2 & -r^2-c_1(w_1+w_2)^2-2c_1w_2^2\end{bmatrix}.$$

Note that the Jacobian matrix A has several useful properties: diagonal elements have negative terms that are of highest orders in w_1 and w_2 (among all the terms in the matrix), and both of them are either negative or negative-definite except for the term $+r^2$; a pair of off-diagonal elements are skew symmetric, and the rest off-diagram terms are a cross product term of w_1w_2 and a term of $2(1+r)$. Thus, there is little difficulty to choose constant sub-blocks P_{ii} to satisfy the inequalities in (37) and (38). For example, let matrix $P(\cdot)$ be that in (35) with $P_{11}=4$ and $P_{12}=P_{22}=1$ (hence P is positive definite and integrable). Also, choose $R_1=1$, and $R_2=-1$. It follows that

$$Q_{11}=8\left[-r^2+w_1^{2l}+\frac{1}{4}(c_1w_2^2+2c_1w_1w_2)-\frac{3}{4}S\right],$$

$$Q_{12}=8(1+r)+w_1^{2l}+4c_1w_1w_2+c_1w_1^2$$

and

$$Q_{22}=1+(1+r)^2+c_1w_1^2+2c_1w_1w_2+2c_1w_2^2.$$

Applying inequality $a^2+b^2\geqslant 2ab$ yields

$$|Q_{12}|\leqslant 8(1+r)+w_1^{2l}+5c_1w_1^2+c_1w_2^2,$$

$$Q_{22}\geqslant 1+(1+r)^2+c_1w_2^2,$$

and

$$Q_{11}\geqslant 16k_1(r)+\frac{k_2(r)}{4}(w_1^{2l}+5c_1w_1^2)^2+c_1w_2^2+8q_g(w_1),$$

where

$$q_g(w_1)\triangleq-2k_1(r)-\frac{k_2(r)}{32}(w_1^{2l}+5c_1w_1^2)^2-r^2+w_1^{2l}-\frac{1}{2}c_1w_1^2-\frac{3}{4}S.$$

Therefore, inequalities in (34) holds if $k_1(r)>(1+r)^2/[1+(1+r)^2]$ and $k_2(r)\geqslant 1/[1+(1+r)^2]$ are chosen and if $q_g(w_1)>0$. The requirement of $q_g(w_1)>0$ can be guaranteed by choosing S, i.e., by setting function $G(\cdot)$ as

$$G(y,r)=\begin{bmatrix}1\\ -1\end{bmatrix}\left\{-\frac{8}{3}k_1(r)y-\frac{k_2(r)}{24(4l+1)}y^{4l+1}-\frac{5c_1}{12(2l+3)}y^{2l+3}-\frac{5}{24}c_1y^5-\frac{4}{3}r^2y+\frac{4}{3(2l+1)}y^{2l+1}-\frac{2}{9}c_1y^3\right\}.$$

Then, the resulting robust observer is given by (2).

The following example deals with the cases that the Jacobian system has a triangular structure under which observer design can be carried out recursively.

Example 3. Consider system (1) with output (32). Two cases of Jacobian matrix $A(\cdot)$ being triangular are investigated.

Suppose that $A(\cdot)$ is block lower triangular and hence $A_{12}=0$. Then, inequality (37) reduces to

$$-Q_{22}=P_{22}A_{22}+A_{22}^{\mathrm{T}}P_{22}<0,$$

and it holds if and only if matrix A_{22} is pointwise stable. The choice of P_{22} can help in making pointwise stability become uniform pointwise stability. Once Q_{22} is positive definite, choose $R_2=0$ and symmetrical $P=\mathrm{diag}\{P_{11},\ P_{22}\}$. It follows that

$$-Q_{11}=P_{11}(A_{11}+R_1S)+(A_{11}+R_1S)^{\mathrm{T}}P_{11},$$

and $-Q_{12}=A_{21}^{\mathrm{T}}P_{22}$. Therefore, Q_{11} can be made positive definite through the choice of R_1S as long as the portion of A_{11} that is not pointwise stable is uniformly bounded with respect to w_2, and condition $4\lambda_{\min}(Q_{11})\lambda_{\min}(Q_{22})>\|Q_{12}\|^2$ can always be guaranteed provided that the dependence of $\|A_{21}\|$ on w_2 is restricted by the square root of that of $\|Q_{22}\|$.

If $A(\cdot)$ is block upper triangular, $A_{21}=0$. Choose P_{11} and P_{22} to be symmetrical. In this case,

$$\begin{aligned}-Q_{11}&=P_{11}[A_{11}+(R_1+P_{11}^{-1}P_{12}R_2)S]\\&\quad+[A_{11}+(R_1+P_{11}^{-1}P_{12}R_2)S]P_{11}^{\mathrm{T}}<0,\end{aligned}$$

which holds if and only if matrix $[A_{11}+(R_1+P_{11}^{-1}P_{12}R_2)S]$ is uniformly pointwise stable. Given the choices of R_i, S, P_{11} and P_{12}, it is not difficult to see how to make $Q_{11}>0$. On the other hand, it follows from (37) that making $Q_{22}>0$ is also straightforward. On the other hand, we have

$$\begin{aligned}-Q_{12}&=P_{11}A_{12}+A_{11}^{\mathrm{T}}P_{21}^{\mathrm{T}}+P_{12}A_{22}+S^{\mathrm{T}}R_1^{\mathrm{T}}P_{21}^{\mathrm{T}}\\&\quad+S^{\mathrm{T}}R_2^{\mathrm{T}}P_{22},\end{aligned}$$

and the remaining goal is to satisfy the last inequality in (34). Could $P_{21}=0$ and $R_2=0$ be made, $-Q_{12}=P_{11}A_{12}$, and it is simple to check whether $4\lambda_{\min}(Q_{11})\lambda_{\min}(Q_{22})>\|Q_{12}\|^2$ can be ensured.

As long as structural properties of the Jacobian system hold for its sub-blocks, the above argument can be repeated in a recursive way.

4. Robust control by estimating uncertainty

Consider an uncertain system of form

$$\dot{x}=f(t,x,\upsilon,u), \tag{39}$$

where $f(\cdot)$ has a known functional expression, $x\in\Re^n$ is the state, $u\in\Re^m$ is the control, and $\upsilon\in\Re^l$ is an uncertainty generated by exogenous system

$$\dot{\upsilon}=g(t,\upsilon,x)+\Delta g(t,\upsilon,x). \tag{40}$$

The control problem is robust stabilization by estimating the uncertainties, the proposed observer design using CSM is applied, and the difference in observer design is that an "output equation" based not on υ but on state x from differential equation (39) needs to be constructed.

The following assumptions are made for the plant and the exogenous system. Assumption 5 guarantees controllability. Assumption 6 is parallel to Assumption 3. Assumption 7 provides a way to assess and quantify the impact of estimation error on control design and closed-loop stability.

Assumption 5. *A perfect-knowledge control* $u=-U(t,x,\upsilon)$ *and the corresponding Lyapunov function* $V_n(t,x)$ *can be found for system* (39) *such that*, $\forall\upsilon$,

$$f(t,x,\upsilon,-U(t,x,\upsilon))=f_n(t,x), \tag{41}$$

$$c_{n_1}\gamma_n^{\beta_1'}(\|x\|)\leqslant V_n\leqslant c_{n_2}\gamma_n^{\beta_2'}(\|x\|),\quad \|\nabla_x V_n\|\leqslant c_{n_3}V_n^{\beta_3'},$$

and

$$\nabla_t V_n+[\nabla_x^{\mathrm{T}}V_n]f_n(t,x)\leqslant -k_nV_n^{\beta_n}(t,x), \tag{42}$$

where $\gamma_n(\cdot)$ *is a class-*$\mathscr{K}_\infty$ *function, and* $c_{n_i},\beta_i',k_n,\beta_n>0$ *are constants.*

Assumption 6. *If* x *remains in a compact set, state* υ *of exogenous system* (40) *is uniformly bounded, and* $\|\Delta g(t,\upsilon,x)\|\leqslant c_gV_n^{\beta_4'}(t,x)$ *for constants* $c_g,\beta_4'\geqslant 0$.

Assumption 7. *For a given class of "output" functions* $h(x):\Re^n\to\Re^l$, *there exist constants* $c_f,c_\upsilon\geqslant 0$ *and* $\beta_5',\beta_6'\geqslant 0$ *such that*, $\forall\upsilon$ *and* $\forall\delta\in(0,1)$,

$$\begin{aligned}&\|\nabla_\xi f(t,x,\upsilon,-U(t,x,\xi))|_{\xi=\upsilon-\delta(\tilde{\upsilon}+h(\tilde{x}))}\|\\&\quad\leqslant c_fV_n^{\beta_5'}(t,x)+c_\upsilon\|\tilde{\upsilon}+h(\tilde{x})\|^{\beta_6'}.\end{aligned} \tag{43}$$

Note that a completely unknown exogenous model is admissible as $g(t,\upsilon,x)\equiv 0$ and that inequality (43) becomes independent of choices of $h(\cdot)$ and hence $c_\upsilon=0$ if $f(t,x,\upsilon,u)$ (or equivalently $f(t,x,0,-U(t,x,\upsilon))$) is affine in υ.

Consider the following observer-based robust control

$$u=-U(t,x,\hat{\upsilon}-h(\tilde{x})), \tag{44}$$

where $\tilde{x}=x-\hat{x}$, function $h(\cdot):\Re^n\to\Re^l$ is a chosen feedback of "estimation output" (which belongs to the class in Assumption 7 and meets the conditions to be stated in Theorem 3), $\hat{x}$ and $\hat{\upsilon}$ are defined by

$$\dot{\hat{x}}=k_o(x,\tilde{x})\tilde{x}+f_n(t,x), \tag{45}$$

$$\dot{\hat{\upsilon}}=g(t,\hat{\upsilon}-h(\tilde{x}),x)-k_o(x,\tilde{x})[\nabla_{\tilde{x}}h(\tilde{x})]\tilde{x}, \tag{46}$$

where $k_o(\cdot)>0$ is a scalar gain to be chosen. Clearly, control (44) is not one satisfying the separation principle.

It follows from (41) and from observer (45) and (46) that dynamics of estimation error under control (44) are

$$\dot{\tilde{x}} = -k_o(x,\tilde{x})\tilde{x} + f(t,x,\upsilon,-U(t,x,\hat{\upsilon}-h(\tilde{x}))) - f(t,x,\upsilon,-U(t,x,\upsilon)), \quad (47)$$

$$\dot{\tilde{\upsilon}} = g(t,\upsilon,x) - g(t,\hat{\upsilon}-h(\tilde{x}),x) + \Delta g(t,\upsilon,x) + k_o(x,\tilde{x})[\nabla_{\tilde{x}} h(\tilde{x})]\tilde{x}. \quad (48)$$

It follows from Lemma 1 that, for stability analysis, Eqs. (47), (48), and (39) are equivalent to the following system:

$$\dot{\tilde{z}} = \mathscr{A}(t,\tilde{z},\upsilon)\tilde{z} + \Delta\mathscr{A}(t,\tilde{z},\upsilon), \quad (49)$$

where $0<\delta<1$,

$$\tilde{z} = \begin{bmatrix}\tilde{z}_1\\ \tilde{z}_2\\ \tilde{z}_3\end{bmatrix} \triangleq \begin{bmatrix} x\\ \tilde{x}\\ \tilde{\upsilon}+h(\tilde{x})\end{bmatrix},$$

$$\mathscr{A} = \begin{bmatrix}\mathscr{A}_{11} & 0 & \mathscr{A}_{13}\\ 0 & \mathscr{A}_{22} & \mathscr{A}_{23}\\ 0 & 0 & \mathscr{A}_{33}\end{bmatrix},$$

$$\Delta\mathscr{A}(t,\tilde{z},\upsilon) = \begin{bmatrix}0\\ 0\\ \Delta g(t,\upsilon,x)\end{bmatrix},$$

$$\mathscr{A}_{11} = \nabla_\xi f_n(t,\xi)|_{\xi=\delta x},$$

$$\mathscr{A}_{13} = -\nabla_\xi f(t,x,\upsilon,-U(t,x,\xi))|_{\xi=\upsilon-\delta(\tilde{\upsilon}+h(\tilde{x}))},$$

$$\mathscr{A}_{22} = -k_o I_{n\times n}, \quad \mathscr{A}_{23} = \mathscr{A}_{13}, \quad \text{and}$$

$$\mathscr{A}_{33} = [\nabla_{\tilde{x}} h(\tilde{x})]\mathscr{A}_{13} + \nabla_\xi g(t,\xi,x)|_{\xi=\upsilon-\delta(\tilde{\upsilon}+h(\tilde{x}))}.$$

Comparing (49) and (11), we see that υ can be viewed as an unknown "command signal" and robust control design is to ensure the convergence of $\tilde{z}$ in the presence of the "command" and by attenuating uncertainty vector $\Delta\mathscr{A}(t,\tilde{z},\upsilon)$. It follows from Theorem 1 that the key to establish stability for system (49) (without any further information on υ) is to make its Lyapunov function be a function of $\tilde{z}$ but independent of υ. The structure of matrix $\mathscr{A}(t,\tilde{z},\upsilon)$ makes the search of such a Lyapunov function possible, and such a result is given by the following theorem. Its proof shows how stability conditions can be derived, but detailed expressions of the conditions are omitted due to space limitation.

Theorem 3. *Consider system* (39) *under Assumptions* 5–7. *Suppose that* "*output*" *function* $h(\tilde{x})$ *and Lyapunov function* $V_h(t,\tilde{z}_3)$ *can be chosen such that, for any* $0<\delta<1$ *and for some class-*$\mathscr{K}_\infty$ *function* $\gamma_h(\cdot)$,

$$\|\tilde{z}_3\| \leqslant c_{h_1} V_h^{\beta_7'}, \quad V_h \leqslant c_{h_2}\gamma_h^{\beta_8'}(\|\tilde{z}_3\|), \quad \|\nabla_{\tilde{z}_3} V_h\| \leqslant c_{h_3} V_h^{\beta_9'},$$

$$\nabla_t V_h + \left[\nabla_{\tilde{z}_3}^{\mathrm{T}} V_h\right]\mathscr{A}_{33}\tilde{z}_3 \leqslant -k_h V_h^{\beta_h}(t,\tilde{z}_3), \quad (50)$$

where $\beta_7', \beta_8', \beta_9', \beta_h, c_{h_1}, c_{h_2}, c_{h_3}, k_h > 0$. *Then, stability conditions can be found in terms of gains* k_n, k_h, *gain function* $k_o(x,\tilde{x})$, *and relevant constants such that augmented state* $\tilde{z}$ *is either globally asymptotically stable or locally asymptotically stable or globally uniformly ultimately bounded* (*and hence so are* x *and* $\tilde{\upsilon}$).

Proof. Choose Lyapunov function to be

$$L(t,\tilde{z}) = \frac{c_1}{1+c_2}V_n^{1+c_2}(t,x) + \frac{1}{2+c_3}\|\tilde{x}\|^{2+c_3} + V_h(t,\tilde{z}_3),$$

where $c_1>0$ and $c_2, c_3 \geqslant 0$ are constants. It follows from (49) and (50) and from Assumptions 5–7 that

$$\begin{aligned}\dot{L} \leqslant & -c_1 k_n V_n^{c_2+\beta_n} + c_1 V_n^{c_2}[\nabla_x^{\mathrm{T}} V_n]\mathscr{A}_{13}\tilde{z}_3 - k_o\|\tilde{x}\|^{2+c_3} \\ & + \|\tilde{x}\|^{c_3}\tilde{x}^{\mathrm{T}}\mathscr{A}_{23}\tilde{z}_3 - k_h V_h^{\beta_h} + [\nabla_{\tilde{z}_3}^{\mathrm{T}} V_h]\Delta g \\ \leqslant & -c_1 k_n V_n^{c_2+\beta_n} + c_1 c_{n_3} c_f c_{h_1} V_n^{c_2+\beta_3'+\beta_5'} V_h^{\beta_7'} \\ & + c_1 c_{n_3} c_\upsilon c_{h_1}^{\beta_6'} V_n^{c_2+\beta_3'} V_h^{\beta_6'\beta_7'} - k_o\|\tilde{x}\|^{2+c_3} \\ & + c_f c_{h_1}\|\tilde{x}\|^{1+c_3} V_n^{\beta_5'} V_h^{\beta_7'} + c_\upsilon c_{h_1}^{\beta_6'+1}\|\tilde{x}\|^{1+c_3} V_h^{(\beta_6'+1)\beta_7'} \\ & - k_h V_h^{\beta_h} + c_{h_3} c_g V_n^{\beta_4'} V_h^{\beta_9'}.\end{aligned}$$

Let $\wp : \Re^+ \times \Re^+ \to \Re$ be a polynomial of two variables as

$$\wp(p,q) = -ap^{l_1} + bp^{m_1}q^{m_2} - cq^{l_2}, \quad a,b,c,l_i,m_i>0.$$

It follows from Holder's inequality that

$$\wp(p,q) \leqslant -(l_1+l_2)\left(\frac{a}{l_1}\right)^{l_1/(l_1+l_2)}\left(\frac{c}{l_2}\right)^{l_2/(l_1+l_2)} \times p^{l_1^2/(l_1+l_2)}q^{l_2^2/(l_1+l_2)} + bp^{m_1}q^{m_2}.$$

Thus, we know that, if $l_i^2 > (l_1+l_2)m_i$ for $i=1,2$,

$\exists\ M_1>0$ such that $p,q \geqslant M_1$ implies $\wp(p,q)<0$,

that, if $l_i^2 < (l_1+l_2)m_i$ for $i=1,2$,

$\exists\ M_2>0$ such that $0<p,q\leqslant M_2$ implies $\wp(p,q)<0$,

and that, if $l_i^2 = (l_1+l_2)m_i$ for $i=1,2$ and if $(a/l_1)^{l_1/(l_1+l_2)}(c/l_2)^{l_2/(l_1+l_2)} > b/(l_1+l_2)$,

$\wp(p,q)<0 \quad \forall p,q>0.$

The argument can be repeated to establish three cases of $\wp'(p,q,s)<0$ for polynomial

$$\wp'(p,q,s) = -ap^{l_1} + bp^{m_1}q^{m_2}s^{m_3} - cq^{l_2} - ds^{l_3}$$

with $a,b,c,d,l_i,m_i>0$.

It follows from Lemma 2 (or Theorem 2.15 in Qu (1998)) that the three cases mentioned above but established for $\dot{L}$ correspond to the three types of stability in the statement of theorem, respectively. Thus, stability conditions can be concluded by grouping the last expression of $\dot{L}$ into several polynomials (each of which contains one cross-product term and a fraction of the negative definite terms of V_n, $\|\tilde{x}\|$ and V_h), by finding the conditions of three cases for each polynomial, and by combining all the conditions. □

Function $h(\cdot)$ in control (44) should be chosen primarily to satisfy (50), and the process of selecting $h(\cdot)$ is essentially the same as that of $G(\cdot)$ in the observer design. Thus, all the discussions in preceding sections are applicable here.

5. Conclusions

In this paper, the robust estimation problem is studied by introducing the concept of convergent command-to-state mappings and by developing a Jacobian system equivalent to error dynamics *everywhere* in the state space and for both analysis and design. Conditions are found on designing any nonlinear observer (from a general class of candidates) for nonlinear uncertain systems with measurement noise. These conditions are then restated as stabilization conditions in terms of an algebraic Lyapunov matrix equation. It is shown that the conditions can be used to search for an appropriate Lyapunov function (whether it is quadratic or not) and to constructively design a (globally convergent) observer in ways parallel to those used in designing robust and/or optimal state-feedback controls. It is also shown that, for systems whose uncertainties are generated by an exogenous systems, robust control can be designed by designing an observer to estimate the uncertainties.

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References

Angeli, D. (2002). A Lyapunov approach to incremental stability properties. *IEEE Transactions on Automatic Control*, *47*, 410–421.

Corless, M. J., & Leitmann, G. (1981). Continuous state feedback guaranteeing uniform ultimate boundedness for uncertain dynamic systems. *IEEE Transactions on Automatic Control*, *26*, 1139–1144.

Dawson, D. M., Qu, Z., & Carroll, J. (1992). On the state observation and output feedback problems for nonlinear uncertain dynamic systems. *Systems & Control Letters*, *18*(2), 217–222.

Gauthier, J., Hammouri, H., & Othman, S. (1992). A simple observer for nonlinear systems: Applications to bioreactors. *IEEE Transactions on Automatic Control*, *37*, 875–880.

Grossman, S. (1986). *Multivariable calculus linear algebra and differential equations* (2nd ed.) New York: Academic Press.

Isidori, A. (1995 & 1999). *Nonlinear control systems I & II*. Berlin: Springer.

Khalil, H. (1996). *Nonlinear systems* (2nd ed.) Englewood Cliffs, NJ: Prentice-Hall.

Khalil, H. K. (1996). Adaptive output feedback control of nonlinear systems represented by input-output models. *IEEE Transactions on Automatic Control*, *41*(2), 177–188.

Khalil, K. H., & Esfandiari, F. (1993). Semiglobal stabilization of a class of nonlinear systems using Output feedback. *IEEE Transactions on Automatic Control*, *38*(9), 1412–1415.

Krstic, M., Kanellakppoulos, I., & Kokotovic, P. V. (1995). *Nonlinear and adaptive control design*. New York: Wiley.

Lohmiller, W., & Slotine, J. J. (1998). On contraction analysis for nonlinear systems. *Automatica*, *34*, 683–696.

Praly L. (2001). On observers with state independent error Lyapunov function. *NOLCOS* 1425–1430.

Qu, Z. (1992). Global stabilization of nonlinear systems with a class of unmatched uncertainties. *Systems & Control Letter*, *18*, 301–307.

Qu, Z. (1993). Robust control of nonlinear uncertain systems under generalized matching conditions. *Automatica*, *29*, 985–998.

Qu, Z. (1998). *Robust control of nonlinear uncertain systems*. New York: Wiley.

Qu, Z. (2002). Robust control of nonlinear systems by estimating time variant uncertainties. *IEEE Transactions on Automatic Control*, *47*(1), 115–121.

Qu, Z., & Jin, Y. (2001). Robust control of nonlinear systems in the presence of unknown exogenous dynamics. *2001 IEEE conference on decision and control* (pp. 2784–2790). Orlando, Florida.

Shim, H., & Seo, J. H. (2003). Recursive nonlinear observer design: Beyond the uniform observability. *IEEE Transactions on Automatic Control*, *48*(2), 294–298.

Sontag, E. D., & Wang, Y. (1995). On characterizations of the input-to-state stability property. *Systems & Control Letters*, *24*, 351–359.

Tsinias, J. (1989). Observer design for nonlinear systems. *Systems & Control Letters*, *13*, 135–142.

Tsinias, J. (1990). Further results on the observer design problem. *Systems & Control Letters*, *14*, 411–418.

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