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Distributed estimation of algebraic connectivity of directed networks

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ABSTRACT

In directed network, algebraic connectivity is defined as the second smallest eigenvalue of graph Laplacian, and it captures the most conservative estimate of convergence rate and synchronicity of networked systems. In this paper, distributed estimation of algebraic connectivity of directed and connected graphs is studied using a decentralized power iteration scheme. Specifically, the proposed scheme is introduced in discrete time domain in order to take advantage of the discretized nature of information flow among networked systems and it shows that, with the knowledge of the first left eigenvector associated with trivial eigenvalue of graph Laplacian, distributed estimation of algebraic connectivity becomes possible. Moreover, it is revealed that the proposed estimation scheme still performs in estimating the complex eigenvalues. Simulation results demonstrate the effectiveness of the proposed scheme.

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1. Introduction

Because of its emerging potential in networked control systems, distributed estimation of network connectivity became an attractive topic in the past decade [1]. Common challenge in this venue is that each system estimates the global criteria of the overall network using only the local measurements and information received from its connected peers. Such global criteria could be eigenvalues and its associated eigenvectors, or most recently the first left eigenvector (corresponding to eigenvalue 0 of graph Laplacian). Among them, the most prominent and essential one is the second smallest eigenvalue of graph Laplacian, also known as algebraic connectivity, which captures the synchronicity and convergence rate of networked systems [2,3]. Hence, it shall be both theoretical interesting and practically useful if algebraic connectivity can be known explicitly and locally in the absence of global information.

In the event that network is either undirected or directed but balanced, algebraic connectivity is defined as the second smallest eigenvalue of graph Laplacian (i.e., Fiedler value [4]). Since analytical solution to Fiedler value is known [3,5], numerical solution can thus be readily applied to estimate or improve its value. For instance, a decentralized orthogonal iteration approach is proposed in [6] to estimate the leading k eigenvectors, but this approach is not scalable and also requires a centralized initialization, similar work on this topic can be found in [7]; another breakthrough worth noting is using the Fast Fourier Transform (FFT) [8] by constructing distributed oscillators whose states oscillate at frequencies corresponding to the eigenvalues of graph Laplacian, however FFT is not appropriate for real-time implementation nor for handling switching topologies. The arguably most effective schemes in estimating Fiedler value are [9] and [10]. Specifically, a decentralized power iteration approach is introduced in [9] to estimate components of the Fiedler eigenvector and then the Fiedler value in a continuous-time fashion, but the proposed estimation scheme requires global initialization and cannot handle switching topologies. In [10], power of adjacency matrix are used to calculate the upper and lower boundary of algebraic connectivity. In terms of improving Fiedler value or algebraic connectivity, a centralized semi-definite programming (SDP) solver is proposed in [11] to maximize Fiedler value directly; and a similar approach can be found in [12], where relay locations are selected to optimize the connectivity. As an extension, a decentralized supergradient algorithm is proposed in [13], but this requires the a priori knowledge of Fiedler eigenvector and communication overhead during each iteration. Components of distance-dependent graph Laplacian are optimized in [14] by driving mobile robots to appropriate locations, and algebraic connectivity has been maximized as a result.

However, it should be pointed out that all of the aforementioned results on distributed estimation and control are restricted to undirected or directed but balanced network. For directed networks, the most notable work is focused on distributed estimation of the first left eigenvector and its applications in improving network convergence [15–17]. To be more precise, a supervisory node with global information of network topology is

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introduced in [15] to improve network convergence by making time derivatives of cooperative control Lyapunov functions more negative; extensions of this approach can be found in [16,17], where network performance is enhanced distributively with local estimation of the first left eigenvector, consensus vector, and consequently the cooperative control Lyapunov function. To the best of our knowledge, little is available on distributed estimation of algebraic connectivity of directed network.

In this paper, algebraic connectivity of directed graph is estimated using a decentralized power iteration scheme, whose effectiveness has already been verified in calculating eigenstructure [9,18]. It is shown that, with local knowledge of the first left eigenvector and affine transformation, algebraic connectivity of directed network can be estimated distributively. It is demonstrated that the proposed scheme overcomes the inherent shortcomings associated with power iteration. That is, despite outputs of power iteration always being real, it is applicable in estimating complex eigenvalues. This paper is organized as follows. In Section 2, preliminary results of graph theory and the relevant mathematical results on the first left eigenvector are summarized for directed and switching networks. In Section 3, estimation of algebraic connectivity of a digraph is formulated and the main findings of this paper are presented. Section 4 focuses upon the proof of the main theorem. More specifically, in Section 4.1, properties of the affine transformation are studied and its relation with respect to algebraic connectivity is explicitly found. Then, in Section 4.2, the proof of main theorem is carried out for digraphs with simple and complex eigenvalues, and effectiveness of the proposed scheme is verified by numerical examples. In Section 5, conclusion of the underlying problem is reached.

2. Preliminaries on algebraic graph theory

In this paper, we consider a digraph $\mathcal{D} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, n\}$ and $\mathcal{E}$ denote the sets of vertices/nodes and directed edges, respectively. Unless otherwise specified, vertex j is said to be adjacent to vertex i if there exists a directed edge $(j, i) \in \mathcal{E}$ with i being tail of the edge and j being the head. Analogously, neighborhood set $\mathcal{N}_i \subseteq \mathcal{V}$ of vertex i is $\{k \in \mathcal{V} \mid (k, i) \in \mathcal{E}\}$, the set of all vertices that are adjacent to vertex i. Hence, cardinality of $\mathcal{N}_i$ represents the numbers of connected neighbor(s) of vertex i. If $j \in \mathcal{N}_i$, $i \in \mathcal{N}_j$ holds for undirected graph, but is not necessarily valid for a digraph.

Without loss of any generality, the adjacency matrix $\mathcal{A}(\mathcal{D})$ considered in this paper is weighted and normalized as

$$[\mathcal{A}(\mathcal{D})]_{ik} = \begin{cases} a_{ik} > 0 & \text{if } (k, i) \in E \\ 1 - \sum_{k \neq i} a_{ik} & \text{if } k = i \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

That is, $\mathcal{A}(\mathcal{D})$ is designed to be nonnegative and row-stochastic, and we assume that all the non-zero and hence positive weighting factors are both uniformly lower and upper bounded, i.e., $a_{ij} \in [\underline{a}\ 1]$, that is $0 < \underline{a} \leq a_{ij} \leq 1$, $\forall j \in \mathcal{N}_i$. As such, the degree matrix for the underlying graph becomes trivial (i.e., identity), and its corresponding weighted Laplacian is

$$\mathcal{L}(\mathcal{D}) \triangleq I_n - \mathcal{A}(\mathcal{D}). \tag{2}$$

Moreover, digraph $\mathcal{D}$ is said to be connected if it has one globally reachable node, or equivalently $\mathcal{A}(\mathcal{D})$ is lower triangularly complete; digraph $\mathcal{D}$ is called strongly connected if it contains a directed path (could be multi-hop) from i to k and a directed path from k to i for every pair of vertices i, k or equivalently $\mathcal{A}(\mathcal{D})$ is irreducible [19,20].

According to (2), $\mathcal{L}(\mathcal{D})$ always has zero row sum, and if symmetric, is also positive semi-definite. It follows that $\lambda_1 = 0$ is the smallest eigenvalue of $\mathcal{L}$ with right eigenvector $\nu_1 \triangleq \frac{1}{\sqrt{n}}\mathbf{1}_n \triangleq \frac{[1,\ldots,1]^T}{\sqrt{n}}$ and left eigenvector γ_1, and all other eigenvalues have positive real parts if digraph is at least connected. Suppose the smallest non-zero eigenvalue of $\mathcal{L}$ is denoted by $\lambda_2(\mathcal{L})$, and [20,21]

- $\lambda_2(\mathcal{L})$ is of multiplicity 1 if $\mathcal{L}(\mathcal{D})$ is at least connected;
- $\lambda_2(\mathcal{L})$ is real if digraph $\mathcal{D}$ is rooted out-branching[1];
- if $\lambda_2(\mathcal{L})$ is complex, then there exists another eigenvalue $\lambda_3(\mathcal{L})$ be its conjugate.

In addition to λ_2, recent advances also prompt γ_1 to describe network connectivity of a digraph. Specifically, connectedness or network social standings of a digraph is preserved by γ_1 and its components as summarized in the following lemma, whose proof can be found in [16,17] and hence omitted here.

Lemma 1. *Consider digraph $\mathcal{D}$ and its adjacency matrix $\mathcal{A}(\mathcal{D})$, suppose γ_1 is the first left eigenvector of $\mathcal{A}(\mathcal{D})$ and $\gamma_{1,i}$ be its ith component. That is, $\mathcal{A}^T\gamma_1 = \gamma_1$. Then,*

- *γ_1 is unique and positive if digraph $\mathcal{D}$ is strongly connected;*
- *γ_1 is unique and nonnegative if digraph $\mathcal{D}$ is connected, and $\gamma_{1,i} > 0$ if system i belongs to the leader group,[2] and $\gamma_{1,i} = 0$ if system i is otherwise.* □

In this paper, we attempt to find a distributive scheme to estimate λ_2 for directed and switching networks with local knowledge of γ_1, and as implied by (2), this can be accomplished by estimating the second largest eigenvalue of adjacency matrix $\mathcal{A}$, and for the sake of notation brevity, λ_2 is used to denote this particular eigenvalue hereafter, unless otherwise specified.

3. Problem formulation

In this section, algebraic connectivity of digraph under switching topologies is formulated and its distributed estimation is motivated. Moreover, the main findings of this paper is presented and its proof will be carried out rigorously in the next section. In what follows, we define the time sequences $\{t_k : k \in \aleph\}$ for $\aleph = \{0, 1, \ldots, \infty\}$, and without loss of any generality, digraph $\mathcal{D}$ is assumed to be time invariant during each interval $[t_k,\ t_{k+1})$, and its corresponding $\mathcal{A}$ or $\mathcal{L}$ is piecewise-constant. That is, $\mathcal{D}(t_k) = \mathcal{D}(t^-_{k+1})$. However, $\mathcal{D}(t^-_{k+1}) = \mathcal{D}(t^+_{k+1})$ does not necessarily hold due to the existence of switching topologies. The following assumptions on connectivity and eigenstructure of digraph captured by the underlying adjacency matrix $\mathcal{A}(\mathcal{D})$ are assumed:

Assumption 1. The digraph is assumed to be strongly connected at each interval. Moreover, let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $\mathcal{A}(\mathcal{D})$, sorted in order of decreasing magnitude. That is, $1 = \lambda_1 > |\lambda_2| \geq \cdots \geq |\lambda_n|$. □

Assumption 2. Time sequence $\{t_k : k \in \aleph\}$ has the property that $(t_{k+1} - t_k) \geq T$ for some known constant $T > 0$.[3]

---

1 A digraph is rooted out-branching if it contains no directed cycle and there exists a vertex ι such that for any other vertex $\kappa \in E$, there is a directed path from ι to κ.

2 System i is said to be a leader if all edges initiated at system i are tails. In contrast, system i is said to be a follower if all edges are heads.

3 If $(t_{k+1} - t_k) \geq T$ does not hold most of the time, there is little chance for any online estimation scheme to work.

In addition, the unity[4] right and left eigenvectors associated with eigenvalue λ_i are defined as ν_i and γ_i, respectively. That is,

$$\mathcal{A}\nu_i = \lambda_i \nu_i, \quad \text{and} \quad \mathcal{A}^T\gamma_i = \lambda_i\gamma_i. \tag{3}$$

Assumption 1 implies that $\mathcal{A}(\mathcal{D})$ is irreducible and eigenvalue $\lambda_1 = 1$ is of multiplicity 1[5] at each interval, and γ_1 is unique and $\gamma_1 > 0$ [17], while Assumption 2 ensures switching frequency is bounded by $1/T$ such that distributed estimation of connectivity is plausible. However, it should be pointed out that, unlike undirected or directed but balanced networks, being strongly connected does not guarantee λ_2 be real in a digraph, rather it ensures the real part of λ_2 be positive, but no restrictions can be made on its complex part, which makes estimation of λ_2 even more challenging.

As is well established, $\lambda_2(\mathcal{L})$ is commonly known as Fiedler value or algebraic connectivity if the underlying graph is undirected [4], and it is the most cited and prominent factor when describing network connectivity. To be more precise, convergence rate as well as algebraic connectivity is determined by $\lambda_2(\mathcal{L})$ in this case [22], while, for digraphs, determination of convergence rate is rather involved, it depends on not only $\lambda_2(\mathcal{L})$, but also the current state and the state space to which it belongs [17]. Nonetheless, $\lambda_2(\mathcal{L})$ of a digraph captures the essence of network performance and represents the most conservative estimate of convergence rate, as well as synchronicity of the overall network [2]. In other words, though $\lambda_2(\mathcal{L})$ of digraph cannot be used directly to quantitatively characterize network performance, its value, if obtained, can be treated as a criterion to network performance and as an inspiration to motivate higher-level control, in order to enhance convergence rate of networked systems. Therefore, a distributive knowledge of λ_2 is not only necessary, but also intuitive in studying directed networks.

While the expression of algebraic connectivity is unique for undirected networks, however, its corresponding form in digraph varies due to the complexity of its eigenstructure. In what follows, definition of algebraic connectivity in undirected network is generalized to a digraph, and [2]:

Definition 1. For a digraph with adjacency matrix $\mathcal{A}(\mathcal{D})$, let λ_2 is its second largest eigenvalue. Then, algebraic connectivity α of digraph $\mathcal{D}$ is the real number defined by

$$\alpha = \sqrt{(1 - |\lambda_2|\cos\theta_2)^2 + |\lambda_2|^2 \sin^2\theta_2} \tag{4}$$

where $|.|$ is the absolute value operator, and θ_2 is the phase angle of $\lambda_2(\mathcal{A})$.

The above definition is reduced to Fiedler's version of algebraic connectivity when restricted to undirected and connected graphs. As is well known [18], power iteration is an effective and widely applied approach in calculating dominant (in terms of magnitude) eigenvalue of any particular matrix. Specifically, power iteration estimates eigenvector associated with the dominant eigenvalue at first, then the underlying eigenvalue can be calculated consequently as a result. Indeed, it does not require performing matrix decomposition at each system nor is it necessary, thus making its distributive implementation possible. For instance, suppose λ' and ν' are dominant eigenvalues (with strictly greater magnitude compared to other eigenvalues) and its associated eigenvectors of matrix D. It follows that, ν' can be estimated by power iteration and at the kth step,

$$\hat{\nu}'(k) = \frac{D\hat{\nu}'(k-1)}{\|D\hat{\nu}'(k-1)\|} \tag{5}$$

where $\hat{\nu}'(k)$ is estimate of ν' at the kth iteration, and $\hat{\nu}'(0)$ is the initial vector with $\hat{\nu}'(0)\cdot\nu' \neq 0$.

Apparently that power iteration is a recursive procedure with a strict requirement on initial conditions, and a normalization (i.e., deflation) is inevitable at each step in order to make sure $\hat{\nu}'$ converge to a finite vector. Therefore, what prevents its distributive implementation is the normalization, and as will be shown later, this can be resolved with proper design. Another fact worth noting is that its outputs are always real, which precludes its application in estimating the complex eigenvector/eigenvalue. In principle, should $\mathcal{L}(\mathcal{D})$ or $\mathcal{A}(\mathcal{D})$ be symmetric, λ_2 is real and positive if the underlying undirected graph is connected, it follows that power iteration fits intuitively in estimating λ_2 of undirected graph successful examples can be found in [9,10].

In this paper, we focus on application of power iteration (5) to estimate $\lambda_2(\mathcal{A})$ of a digraph. It should be pointed out that such application is not trivial because of possible complex eigenstructure of $\mathcal{A}(\mathcal{D})$ and admittance of switching topologies. Once successful, convergence rate of overall network becomes available at each system, such that correlative behavior can be carried out distributively to improve network performance. Moreover, rather than designing a continuous-time observer, estimation in this paper will be carried out in the discretized-time domain by taking advantage of the discrete nature of information flow among networked systems as well as the knowledge of the first left eigenvector γ_1. The following lemma summarizes existing results on distributed estimation of γ_1, its proof is omitted here since it combines results in [17,20].

Lemma 2. *Consider adjacency matrix defined in (1) and its first left eigenvector $\gamma_1 \in \Re^n$,[6] satisfying $\mathcal{A}^T\gamma_1 = \gamma_1$ and $\gamma_1^T\mathbf{1}_n = 1$. Then,*

- *If $\mathcal{A}$ is non-irreducible, and there exists a permutation matrix P, such that*

  $$P\mathcal{A}P^T = \begin{bmatrix} E_{11} & 0 \\ E_{21} & E_{22} \end{bmatrix}$$

  where E_{11} is irreducible, systems of block E_{11} form a leader group, and if $E_{21} \neq 0$ and E_{21} is irreducible (otherwise, the same argument can recursively be applied to block E_{22}), systems corresponding to block E_{22} form a follower group.
- *γ_1 can be estimated distributively at system i by*

  $$\hat{\gamma}_1^{(i)}(\ell+1) = \sum_{k=1}^{n} a_{ik}\hat{\gamma}_1^{(k)}(\ell), \quad \hat{\gamma}_1^{(i)}(0) = e_i \tag{6}$$

  where $\hat{\gamma}_1^{(i)}$ is estimated γ_1 at system i, $e_i \in \Re^n$ is a vector of zeros except its ith entry being 1, and $\hat{\gamma}_1^{(i)}(\ell)$ is updated according to, at interval $[t_k,\ t_{k+1})$,

  $$\hat{\gamma}_1^{(i)}(\ell) = \hat{\gamma}_1^{(i)}(t_k + \ell T_s^*), \quad \ell = 0, 1, \ldots, N(\ell) \tag{7}$$

where $0 < T_s^* \ll 1$ is the sample period of observer (6), and $N(\ell)$ will be specified later. □

[4] In this paper, a vector $x \in \Re^n$ is said to be unity if $\|x\| = 1$ and unit-sum if $x^T\mathbf{1}_n = 1$, where $\|.\|$ represents Euclidean norm.

[5] In the case that multiplicity of λ_1 is η and $\eta > 1$, network is consisted of η subgraphs, the proposed distributed estimation scheme can be readily applied in each subgraph provided it is strongly connected.

[6] In this paper, the unit-sum first left eigenvector and unity first left eigenvector are assumed to be interchangeable for the sake of notation brevity.

In essence, observer (6) allows each system have local knowledge of γ_1, which, as will be revealed later, is critically important in distributed estimation of λ_2 in a digraph. Without loss of any generality, γ_1 and its various forms are assumed to be known locally hereafter.

In what follows, λ_2 and ν_2 are defined of the form

$$\lambda_2 \triangleq |\lambda_2| (\cos\theta_2 + j\sin\theta_2) \tag{8}$$

$$\nu_2 \triangleq \begin{bmatrix} \nu_{2,1} \\ \vdots \\ \nu_{2,n} \end{bmatrix} \triangleq \begin{bmatrix} |\nu_{2,1}|(\cos\phi_1 + j\sin\phi_1) \\ \vdots \\ |\nu_{2,n}|(\cos\phi_n + j\sin\phi_n) \end{bmatrix} \tag{9}$$

where ϕ_i is phase angle of $\nu_{2,i}$, and $j = \sqrt{-1}$ is the imaginary unit, apparently that λ_2 and ν_2 are real if $\theta_2 = \phi_i = 0$.

Moreover, suppose $\hat{\nu}_2(0)$ is the initial condition for power iteration, and it can be decomposed along the eigenspace [23]

$$\hat{\nu}_2(0) = c_1\nu_1 + c_2\nu_2 + \cdots + c_n\nu_n \tag{10}$$

where c_i are gains to be specified, and if ν_2 is complex and of the form (9), then

$$c_2 = |c| (\cos\theta_c + j\sin\theta_c) \tag{11}$$

where $|c|$ and θ_c are absolute value and phase angle of c_2, respectively.

In this paper, a consensus observer as well as a decentralized power iteration scheme will be introduced to estimate λ_2 and consequently α at each system. The main results are summarized into the following theorem, its proof will be carried out successively in the next section.

Theorem 1. *Consider the adjacency matrix defined in (1) satisfying Assumptions 1 and 2, and λ_2 is the second largest eigenvalue of $\mathcal{A}(\mathcal{D})$ defined in (8). Then, algebraic connectivity of the underlying digraph can be estimated distributively at each system. Specifically, at system i*

$$\alpha^{(i)} = \sqrt{\left[1 - |\lambda_2^{(i)}(k)|\cos\hat{\theta}_2(k)\right]^2 + |\lambda_2^{(i)}(k)|^2 \sin^2\hat{\theta}_2(k)} \tag{12}$$

with

$$|\lambda_2^{(i)}(k)| = \frac{\cos\left[(k-1)\hat{\theta}_2 + \hat{\phi}_i + \hat{\theta}_c\right]}{\cos(k\hat{\theta}_2 + \hat{\phi}_i + \hat{\theta}_c)} \frac{\hat{\nu}_{2,i}(k+1)}{|\hat{\nu}_{2,i}(k)|} \|\Upsilon^{-1}\hat{\omega}_i(k)\| \tag{13}$$

and

$$\hat{\nu}_{2,i}(k+1) = \frac{1}{\|\Upsilon^{-1}\hat{\omega}_i(k)\|}\left[\sum_{k=1}^{n} a_{ik}(k)\hat{\nu}_{2,k}(k) - \gamma_{1,i}(k)\hat{\omega}_i^T(k)\mathbf{1}_n\right] \tag{14}$$

and $\hat{\nu}_{2,i}(k)$ is updated according to, at interval $[t_k,\ t_{k+1})$

$$\hat{\nu}_{2,i}(k) = \hat{\nu}_{2,i}(t_k + kT_s), \quad k = 0, 1, \ldots, M(k)$$

where T_s is the sample period of observer (14), $M(k) = (t_{k+1} - t_k)/T_s - \mathrm{mod}((t_{k+1} - t_k)/T_s)$, and

- *$\alpha^{(i)}$ is the estimated α at system i*
- *$\lambda_2^{(i)}$ is the estimated λ_2 at system i,*
- *$\hat{\theta}_2$ is the estimated phase angle of λ_2,*
- *$\hat{\nu}_{2,i}$ is the estimated ith entry of $\hat{\nu}_2$,*
- *$\hat{\phi}_i$ is the estimated phase angle of $\nu_{2,i}$,*
- *$\hat{\theta}_c$ is the estimated phase angle of c_2,*
- $\Upsilon \triangleq \mathrm{diag}[\gamma_1]$
- *$\hat{\omega}_i \in \Re^n$ is an observer established at system i, and*

$$\hat{\omega}_i(\ell + 1) = \sum_{k=1}^{n} a_{ik}(\ell)\hat{\omega}_k(\ell) \tag{15}$$

and $\hat{\omega}_i(\ell)$ is updated according to, at interval $t \in [t_k + kT_s\ t_k + (k+1)T_s]$

$$\hat{\omega}_i(\ell) = \hat{\omega}_i(t_k + kT_s + \ell T_s^*), \quad \text{and} \quad \hat{\omega}_i(0) = \hat{\nu}_{2,i}(k)e_i \tag{16}$$

where e_i is the unity vector defined in (6), and $\ell = 0, 1, \ldots, N(\ell)$ with $N(\ell) = T_s/T_s^* - \mathrm{mod}(T_s/T_s^*)$. □

It follows that, in order to estimate algebraic connectivity distributively, each system requires local estimation of not only the magnitude of λ_2, but also its phase angle as well as phase angles of ν_2. Note that, phase angle calculation is numerical and will be introduced in the next section. As a special case, should λ_2 be real, Theorem 1 is reduced to the following corollary:

Corollary 1. *Consider the same set up as of Theorem 1 and under the assumption that λ_2 is real. Then, algebraic connectivity can be estimated distributively at each system. That is, at system i*

$$\alpha^{(i)} = 1 - \|\Upsilon^{-1}\hat{\omega}_i(k)\| \tag{17}$$

where $\hat{\omega}_i$ is the same observer as defined in (15).

In principle, the proposed scheme consists of two observers, observer (14) is essentially a decentralized and discrete power iteration scheme, dedicated to approximating eigenvector ν_2, while system (15) is a consensus observer, designed to propagate current estimates of $\hat{\nu}_2$ over the entire network. Both observers utilize the same topological protocol, yet evolve at a different pace. More specifically, convergence of system (15) should be achieved within a time period of T_s such that estimates of (14) can be broadcast accurately to all the systems, provided that digraph is strongly connected. That is, $T_s \gg T_s^*$. To be more precise, a conservative estimate of how small T_s^* should be to satisfy the underlying restriction will be provided in the next section, as well as performance of both observers.

4. Proof of the main theorem

In this section, proof of Theorem 1 is carried out. Special attention will be paid to decentralized implementation of power iteration and its application in estimating complex eigenvalues. As indicated in (5), classical power iteration dictates that a normalization/deflation should be performed at each step with explicit knowledge of the current estimates. However, in networked control systems, such recursive deflations can not be executed locally nor is it possible. On the other hand, for undirected network, due to the fact that its first left eigenvector is trivial (i.e., $\gamma_1 = \mathbf{1}/\sqrt{n}$), deflation of power iteration can be fulfilled with a continuous observer [9], however, this solution is no longer applicable in directed network simply because γ_1 is no longer trivial. In this section, we propose a modified power iteration approach, whose deflation is performed without the current estimates, rather estimates of the last step is used instead. As will be shown later, this one-step-back power iteration scheme is permissible and its distributive implementation is possible with the propose consensus-based observer (15). Note that, (15) is designed to propagate information over the entire network, and it could be considered as an auxiliary observer operated distributively under digraph $\mathcal{D}$, does not require any physical motion, thus its convergence can be made arbitrarily fast. Performances of consensus observer (15) are summarized into the following lemma; its proof follows immediately.

Lemma 3. *Consider nonnegative and row-stochastic adjacency matrix $\mathcal{A}(\mathcal{D})$ defined in (1) satisfying Assumption 1, and suppose $\hat{\omega}_i$ is the observer established at system i and it evolves according to (15) and is updated by (16). Then, at interval $t \in [t_k,\ t_{k+1})$*

$$\lim_{\ell\to\infty} \hat{\omega}_i(\ell) = \Upsilon \hat{\nu}_2(\ell)$$

can be ensured at system i. Moreover, convergence of (15) can be made arbitrarily fast with sufficiently small sampling period T_s^*, and a conservative estimate of T_s^* can be chosen by

$$T_s^* \le \frac{\underline{a}^{(n-1)}}{4(n-1)} T_s \tag{18}$$

where $\underline{a}$ is defined in (1).

Proof. According to (15), the closed-loop system for observer $\hat{\omega}_i$ is

$$\underbrace{\begin{bmatrix} \hat{\omega}_1 \\ \vdots \\ \hat{\omega}_n \end{bmatrix}}_{\hat{\omega}} (\ell+1) = \mathcal{A}(k) \otimes I_n \begin{bmatrix} \hat{\omega}_1 \\ \vdots \\ \hat{\omega}_n \end{bmatrix} (\ell)$$

where $\otimes$ denotes Kronecker product.

Since [17,20],

$$\lim_{\ell\to\infty} \mathcal{A}^\ell = \mathbf{1}_n \otimes \gamma_1^T.$$

Hence

$$\lim_{\ell\to\infty} \hat{\omega}(\ell) = \left[\left(\mathbf{1}_n \otimes \gamma_1^T\right) \otimes I_n\right] \hat{\omega}(0). \tag{19}$$

Since $\hat{\omega}_i(0) = \hat{\nu}_{2,i}(k) e_i$ for any i, yields

$$\lim_{\ell\to\infty} \hat{\omega}_i(\ell) = \underbrace{\begin{bmatrix} \gamma_{1,1} & & \\ & \ddots & \\ & & \gamma_{1,n} \end{bmatrix}}_{\Upsilon} \hat{\nu}_2(k).$$

It follows from Assumption 1 that $\gamma_1 > 0$, as such $\Upsilon^{-1}\hat{\omega}_i = \hat{\nu}_2(k)$. Moreover, in order for observer (15) to converge, $N(\ell)$ should satisfy [17]

$$N(\ell) \ge \frac{4}{\left(1 - \max|\lambda_j(\mathcal{A})|^{\frac{1}{n-1}}\right)}, \quad \forall j \ne 1$$

and since [24,25]

$$|\lambda_j(\mathcal{A})| \le \frac{1-\underline{a}}{1+\underline{a}}$$

for any eigenvalue $\lambda_j \ne 1$. Hence, one conservative estimate of T_s^* can be chosen by (18), which concludes the proof of Lemma 3. □

The proposed consensus observer works intuitively to broadcast last step estimate $\hat{\nu}_{2,i}$ to the entire network. Shall network be strongly connected at each interval, each system has explicit knowledge of last estimates $\hat{\nu}_2$ such that $\|\hat{\nu}_2(k)\|$ can be performed and deflation of power iteration becomes possible. Moreover, with help of (18), convergence of observers (15) and (14) is effectively separated. Consequently, the closed-loop system of observer (14) becomes

$$\hat{\nu}_2(k+1) = \frac{1}{\|\hat{\nu}_2(k)\|} \left[\mathcal{A}(k) - \hat{\gamma}_1(k)\hat{\gamma}_1^T(k)\right] \hat{\nu}_2(k). \tag{20}$$

It is obvious that system (20) is a discretized power iteration approach, dedicated to estimating eigenvector associated with the dominant eigenvalue of matrix $\left[\mathcal{A}(k) - \gamma_1(k)\gamma_1^T(k)\right]$, which, as will be shown later, preserves the eigenstructure of matrix $\mathcal{A}(k)$, thus makes estimation of λ_2 possible.

Table 1
Eigenstructures after affine transformations.

$\mathcal{A}$			$\mathcal{A}_{L_1}$			$\mathcal{A}_{R_1}$		
λ	Left	Right	λ	Left	Right	λ	Left	Right
1	γ_1	$\mathbf{1}/\sqrt{n}$	0	γ_1	ν_1'	0	γ_1'	$\mathbf{1}/\sqrt{n}$
λ_2	γ_2	ν_2	λ_2	γ_2'	ν_2	λ_2	γ_2	ν_2'
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
λ_n	γ_n	ν_n	λ_n	γ_n'	ν_n	λ_n	γ_n	ν_n'

4.1. Invariance of eigenstructure under affine transformation

In this section, affine transformation of adjacency matrix $\mathcal{A}$ is studied under both right and left eigenvectors. It is revealed that eigenstructure of adjacency matrix $\mathcal{A}$ is preserved if affine transformation is performed with left eigenvector and such property will sever as the basis of distributed estimation of algebraic connectivity.

The results of affine transformation are summarized in the following lemma:

Lemma 4. *Consider adjacency matrix $\mathcal{A} \in \Re^{n\times n}$ defined in (1) satisfying Assumption 1. Define the affine transformations:*

$$\mathcal{A}_{L_1} = \mathcal{A} - \gamma_1\gamma_1^T, \qquad \mathcal{A}_{R_1} = \mathcal{A} - \nu_1\nu_1^T. \tag{21}$$

Then, matrices $\mathcal{A}_{L_1}$ and $\mathcal{A}_{R_1}$ have the properties of eigenstructure invariance as shown in Table 1, where left *and* right *are specifications of the eigenvector.*

Proof. It follows from $\mathcal{A}^T\gamma_1 = 0$ and $\|\gamma_1\| = 1$ that

$$\mathcal{A}_{L_1}^T\gamma_1 = \gamma_1 - \gamma_1^T\gamma_1\gamma_1 = 0,$$

which implies that the pair of 0 and γ_1 is one of the eigenvalues and left-eigenvector pairs for $\mathcal{A}_{L_1}$.

It follows from (3), that

$$\gamma_1\gamma_1^T\nu_i = \frac{1}{\lambda_i}\gamma_1\gamma_1^T\mathcal{A}\nu_i = \frac{1}{\lambda_i}\gamma_1\gamma_1^T\nu_i,$$

or equivalently,

$$\left[1 - \frac{1}{\lambda_i}\right]\gamma_1\gamma_1^T\nu_i = 0.$$

Since $|\lambda_i| > 0$, we know that, for $i = 2, \ldots, n$,

$$\gamma_1\gamma_1^T\nu_i = 0. \tag{22}$$

Consequently,

$$\mathcal{A}_{L_1}\nu_i = \mathcal{A}\lambda_i = \lambda_i\nu_i, \quad i = 2, \ldots, n,$$

which implies that λ_i and ν_i are eigenvalue and right-eigenvector, respectively, for $\mathcal{A}_{L_1}$.

The proof of Lemma 3 is completed by applying the same argument to $\mathcal{A}_{R_1}$. □

Clearly that λ_2 is the dominant eigenvalue of $\mathcal{A}_{L_1}$ and $\mathcal{A}_{R_1}$. Hence, with the local knowledge of γ_1, $\mathcal{A}_{L_1}$ fits intuitively in the underlying problem. Moreover, the proposed affine transformation can be extended further with the inclusion of eigenvalues λ_j, $\forall j > 1$. In particular λ_{k+1} is the dominant eigenvalue of matrix $\mathcal{A}_{L_k} = [\mathcal{A} - \sum_{i=1}^k \lambda_i\gamma_i\gamma_i^T]$ and $\mathcal{A}_{R_k} = [\mathcal{A} - \sum_{i=1}^k \lambda_i\nu_i\nu_i^T]$. Therefore, estimation of eigenvalue $\lambda_i(\mathcal{A})$ can be accomplished provided left eigenvectors up to γ_{i-1} are known explicitly. In other words, estimation of left eigenvector(s) should be performed before any decentralized power iteration approach can be applied to estimate any eigenvalue.

4.2. Proof of Theorem 1

We start the proof by noticing that, under Assumption 1, $\mathcal{A}_{L_1}$ can be decomposed into the following form

$$\mathcal{A}_{L_1} = \underbrace{\begin{bmatrix} \nu_1' \; \nu_2 \dots \nu_n \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} 0 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}_{J} \underbrace{\begin{bmatrix} \nu_1' \; \nu_2 \dots \nu_n \end{bmatrix}^{-1}}_{F^{-1}}. \quad (23)$$

Therefore, system (20) can be rewritten to

$$\hat{\nu}_2(k+1) = \frac{\mathcal{A}_{L_1}^k \hat{\nu}_2(0)}{\|\mathcal{A}_{L_1}^{k-1} \hat{\nu}_2(0)\|} \quad (24)$$

where $\hat{\nu}_2(0)$ is the initial condition defined in (10) with ν_1 replaced by ν_1'.

Consequently, due to the fact that $\mathcal{A}_{L_1}^k = FJ^kF^{-1}$, yields

$$\hat{\nu}_2(k+1) = \frac{(FJ^kF^{-1})(c_1\nu_1' + c_2\nu_2 + \cdots + c_n\nu_n)}{\|(FJ^{k-1}F^{-1})(c_1\nu_1' + c_2\nu_2 + \cdots + c_n\nu_n)\|}. \quad (25)$$

Since λ_2 is defined in (8), it follows that λ_3 should be its conjugate eigenvalue. In addition, c_2 and c_3 in (10) should be conjugate to each other as well. Hence, according to (8) and (9), results

$$\lambda_3 = |\lambda_2|(\cos\theta_2 - j\sin\theta_2),$$
$$\nu_3 = \begin{bmatrix} |\nu_{2,1}|(\cos\phi_1 - j\sin\phi_1) \\ \vdots \\ |\nu_{2,n}|(\cos\phi_n - j\sin\phi_n) \end{bmatrix} \quad (26)$$

and since c_2 is defined by (11), yields

$$c_3 = |c|\left(\cos\theta_c - j\sin\theta_c\right).$$

As such, the power of λ_2 or λ_3 can be written in the form

$$\lambda_2^k = |\lambda_2|^k[\cos(k\theta_2) + j\sin(k\theta_2)]$$
$$\lambda_3^k = |\lambda_2|^k[\cos(k\theta_2) - j\sin(k\theta_2)].$$

Consequently,

$$\lim_{k\to\infty} \frac{1}{\lambda_2^k} FJ^kF^{-1}\nu_i = \begin{cases} \mathbf{0}_3 & \text{if } i \neq 2, 3 \\ \nu_2 & \text{if } i = 2 \\ \left(\dfrac{\lambda_3}{\lambda_2}\right)^k \nu_3 & \text{if } i = 3 \end{cases} \quad (27)$$

where $\mathbf{0}_3$ is a vector of zeros.

As a result, (25) becomes

$$\lim_{k\to\infty} \hat{\nu}_2(k+1) = \frac{c_2\lambda_2^k\nu_2 + c_3\lambda_3^k\nu_3}{\|c_2\lambda_2^{k-1}\nu_2 + c_3\lambda_3^{k-1}\nu_3\|}. \quad (28)$$

In addition, let $\varphi_i = \theta_c + \phi_i$, yielding

$$c_2\lambda_2^k\nu_2 + c_3\lambda_3^k\nu_3 = 2|c||\lambda_2|^k \begin{bmatrix} |\nu_{2,1}|\cos(k\theta_2 + \varphi_1) \\ \vdots \\ |\nu_{2,n}|\cos(k\theta_2 + \varphi_n) \end{bmatrix}.$$

It is again verified that, albeit the expected value, output of power iteration (i.e., $\hat{\nu}_2(k)$) is always real and its absolute value is time varying and changes periodically according to the phase angles. Before proceeding further, consider the special case where λ_2 is real and matrix $\mathcal{A}(k)$ satisfies Assumption 1. As such, (27) becomes

$$\lim_{k\to\infty} \frac{1}{\lambda_2^k} FJ^kF^{-1}\nu_i = \begin{cases} \mathbf{0}_3 & \text{if } i \neq 2 \\ \nu_2 & \text{if } i = 2 \end{cases} \quad (29)$$

and (28) is reduced to

$$\lim_{k\to\infty} \hat{\nu}_2(k+1) = \lambda_2 \left(\frac{\lambda_2}{|\lambda_2|}\right)^k \frac{c_2}{|c_2|} \frac{\nu_2}{\|\nu_2\|}. \quad (30)$$

That is,

$$\lim_{k\to\infty} \hat{\nu}_2(k+1) = \pm\lambda_2\nu_2. \quad (31)$$

Consequently,

$$\lim_{k\to\infty} \|\hat{\nu}_2(k)\| = |\lambda_2|. \quad (32)$$

Therefore, under the proposed conjecture, Euclidean norm of output of power iteration is $|\lambda_2|$ instead of unity. With local knowledge of $\hat{\omega}_i(\ell)$, $\alpha^{(i)}$ can be calculated by (17), which verifies the Corollary 1. In this case, an alternative approach to calculate λ_2 is, if $\hat{\nu}_{2,i}(k) \neq 0$

$$\lambda_2^{(i)}(k) = \frac{\sum_{k=1}^{n} a_{ik}\hat{\nu}_{2,k}(k)}{\hat{\nu}_{2,i}(k)}.$$

As implied by (32), the proposed scheme still performs when $\hat{\nu}_2^T(0)\nu_2 = 0$, which implies that performance of discretized power iteration is independent of initial conditions, and such independence is attributed to the nature of one-step-back power iteration. If λ_2 be complex, substituting (9) and (26) into (28), yields

$$\lim_{k\to\infty} \hat{\nu}_2(k+1) = \frac{|\lambda_2|}{\sqrt{\sum_{\ell=1}^{n} \nu_{2,\ell}^2 \cos^2\left[(k-1)\theta_2 + \varphi_\ell\right]}} \times \begin{bmatrix} |\nu_{2,1}|\cos(k\theta_2 + \varphi_1) \\ \vdots \\ |\nu_{2,n}|\cos(k\theta_2 + \varphi_n) \end{bmatrix}. \quad (33)$$

That is,

$$\frac{\hat{\nu}_{2,i}(k+1)}{\hat{\nu}_{2,\ell}(k+1)} = \left|\frac{\nu_{2,i}}{\nu_{2,\ell}}\right| \frac{\cos(k\theta_2 + \varphi_i)}{\cos(k\theta_2 + \varphi_\ell)}. \quad (34)$$

Consequently,

$$\frac{\hat{\nu}_{2,i}(k+1)\hat{\nu}_{2,\ell}(k)}{\hat{\nu}_{2,\ell}(k+1)\hat{\nu}_{2,i}(k)} = \frac{\cos(k\theta_2 + \varphi_i)\cos\left[\cos(k-1)\theta_2 + \varphi_\ell\right]}{\cos(k\theta_2 + \varphi_\ell)\cos\left[(k-1)\theta_2 + \varphi_i\right]}. \quad (35)$$

Moreover, (33) can be rewritten as, at system i

$$\lim_{k\to\infty} \hat{\nu}_2(k+1) = \frac{|\lambda_2|\cos(k\theta_2 + \varphi_i)}{\cos\left[(k-1)\theta_2 + \varphi_i\right]\sqrt{\sum_{\ell=1}^{n} \left|\frac{\nu_{2,\ell}}{\nu_{2,i}}\right|^2 \frac{\cos^2[(k-1)\theta_2 + \varphi_\ell]}{\cos^2[(k-1)\theta_2 + \varphi_i]}}} \times \begin{bmatrix} \left|\dfrac{\nu_{2,1}}{\nu_{2,i}}\right| \dfrac{\cos(k\theta_2 + \varphi_1)}{\cos(k\theta_2 + \varphi_i)} \\ \vdots \\ \left|\dfrac{\nu_{2,n}}{\nu_{2,i}}\right| \dfrac{\cos(k\theta_2 + \varphi_n)}{\cos(k\theta_2 + \varphi_i)} \end{bmatrix}. \quad (36)$$

Therefore, $|\lambda_2|$ can be calculated at system i by

$$|\lambda_2^{(i)}(k)| = \frac{\cos\left[(k-1)\theta_2 + \varphi_i\right]\hat{\nu}_{2,i}(k+1)}{\cos(k\theta_2 + \varphi_i)} \times \sqrt{\sum_{j=1}^{n} \left|\frac{\nu_{2,j}}{\nu_{2,i}}\right|^2 \frac{\cos^2\left[(k-1)\theta_2 + \varphi_j\right]}{\cos^2\left[(k-1)\theta_2 + \varphi_i\right]}}. \quad (37)$$

Substituting (34) into the above relation, yields

$$|\lambda_2^{(i)}(k)| = \frac{\cos[(k-1)\theta_2+\varphi_i]\,\hat{\nu}_{2,i}(k+1)}{\cos(k\theta_2+\varphi_i)} \frac{\sqrt{\sum_{\ell=1}^{n} \hat{\nu}_{2,\ell}^2(k)}}{|\hat{\nu}_{2,i}(k)|}. \tag{38}$$

Hence, after invoking Lemma 2, (38) can be further simplified to (13). In addition, if λ_2 and ν_2 are real (i.e., $\theta_2 = \varphi_i = 0$) and as $k \to \infty$, (13) can be further reduced to (31). This demonstrates the consistency of the proposed scheme in estimating both real and complex eigenvalues, and that Corollary 1 is a special case of Theorem 1.

Therefore, additional calculation of phase angle is not only necessary but also imperative in order to estimate the complex eigenvalue. It follows from (35) that the value as well as tendency of phase angles are preserved by estimates of (20) at each step. Hence, should each system have memories of previous estimates, a numerical approach can be proposed to solve the following equations: suppose $\ell \in \mathcal{N}_i$

$$\begin{cases} f_1(\theta_2,\varphi_i,\varphi_\ell) = \dfrac{\hat{\nu}_{2,i}(k+1)\hat{\nu}_{2,\ell}(k)}{\hat{\nu}_{2,j}(k+1)\hat{\nu}_{2,i}(k)} \\ \qquad - \dfrac{\cos(k\theta_2+\varphi_i)\cos[(k-1)\theta_2+\varphi_\ell]}{\cos(k\theta_2+\varphi_\ell)\cos[(k-1)\theta_2+\varphi_i]} \\ f_2(\theta_2,\varphi_i,\varphi_\ell) = \dfrac{\hat{\nu}_{2,i}(k)\hat{\nu}_{2,\ell}(k-1)}{\hat{\nu}_{2,\ell}(k)\hat{\nu}_{2,i}(k-1)} \\ \qquad - \dfrac{\cos[(k-1)\theta_2+\varphi_i]\cos[(k-2)\theta_2+\varphi_\ell]}{\cos[(k-1)\theta_2+\varphi_\ell]\cos[(k-2)\theta_2+\varphi_i]} \\ f_3(\theta_2,\varphi_i,\varphi_\ell) = \dfrac{\hat{\nu}_{2,i}(k-1)\hat{\nu}_{2,\ell}(k-2)}{\hat{\nu}_{2,\ell}(k-1)\hat{\nu}_{2,i}(k-2)} \\ \qquad - \dfrac{\cos[(k-2)\theta_2+\varphi_i]\cos[(k-3)\theta_2+\varphi_\ell]}{\cos[(k-2)\theta_2+\varphi_\ell]\cos[(k-3)\theta_2+\varphi_i]}. \end{cases} \tag{39}$$

As such, θ_2, φ_i and φ_ℓ can be calculated numerically by solving $f_1 = f_2 = f_3 = 0$. It should be pointed out that the existence and uniqueness of a solution to (39) can be proved trivially by noting that all functions are smooth and its corresponding Jacobian matrix is not singular [23]. Note that choice of system ℓ should satisfy that $\Delta \triangleq \hat{\nu}_{2,\ell}(k)\hat{\nu}_{2,\ell}(k-1)\hat{\nu}_{2,\ell}(k-2) \neq 0$. Should $\Delta = 0$ occur for any $\ell \in \mathcal{N}_i$, calculation of (39) will be halted and estimates of previous iterations will be applied until next initiation.

In the event that $\hat{\nu}_2^T(0)(\nu_2 + \nu_3) = 0$ or equivalently $\hat{\nu}_2^T(0)\mathrm{Re}(\nu_2) = 0$ (i.e., $\varphi_i = \pi/2$, $\forall i$), (34) is rendered to

$$\frac{\hat{\nu}_{2,i}(k+1)}{\hat{\nu}_{2,\ell}(k+1)} = \frac{|\nu_{2,i}|}{|\nu_{2,\ell}|}\frac{\sin(k\theta_2)}{\sin(k\theta_2)}. \tag{40}$$

In this case, (39) can still be applied in finding θ_2, with which $|\lambda_2^{(i)}|$ can be calculated by (13), which again implies that the proposed estimation scheme works for any initial condition. Hence, with local knowledge of $|\lambda_2^{(i)}|$ and $\hat{\theta}_2$, algebraic connectivity can be known explicitly and locally at each system. The proof of Theorem 1 is completed by noticing the above argument. □

Specifically, implementation of the proposed estimation scheme is fully decentralized, and it requires an exchange of a composite vector $x_i = [\hat{\omega}_i\ \hat{\nu}_{2,i}(k)\ \hat{\nu}_{2,i}(k-1)\ \hat{\nu}_{2,i}(k+1)]^T$ of dimension $(n+3)$ at each system, which indicates that the proposed scheme performs at the expense of communication overheads. Also, accurate estimation demands distributive calculation of θ_2 and φ_i; this can be accommodated with onboard DSP chip. In other words, the proposed estimation scheme captures a tradeoff between communication and possible computational overhead and connectivity self-awareness. If γ_1 and $|\lambda_2|$ are known locally, each system will not only have explicit knowledge of topological structure or social standing of the entire network, but also the criterion of convergence rate of overall network, which potentially enables and motivates high-level corrective control strategies carried out at each system, to improve network performance.

Remark 1. In the case that λ_2 is not unique, or in other words, the multiplicity of λ_2 is larger than 1, it can be easily verified that (28) still applies, consequently it is safe to conclude that the proposed scheme still performs in this case.

Remark 2. As stated in Assumption 1, the proposed scheme only works if $\mathcal{A}$ is irreducible, any less restrictive topology will preclude the convergence of consensus observer (15). However, the proposed estimation scheme is still applicable if matrix $\mathcal{A}$ is non-irreducible. To be more precise, as suggested in Lemma 1, algebraic connectivity of irreducible blocks E_{11} and E_{22} are independent when $\mathcal{A}$ is lower triangularly incomplete. As such, the proposed estimation scheme can be applied straightforwardly at each irreducible block. Moreover, if $\mathcal{A}$ is non-irreducible and lower triangularly complete or the corresponding digraph has a spinning tree, each system can distributively identify its own social standing as well as its connected neighbors' based on γ_1. Then, systems of both leader block E_{11} and follower block E_{21} can perform the same estimation procedure such that algebraic connectivity of each respective block is known locally. In addition, $|\lambda_2(E_{11})|$ can be broadcast to systems of E_{22} such that systems in block E_{22} know explicitly the convergence rate of the overall network.

The performance of the proposed scheme is illustrated in the following simple example:

Example 1. Given the following adjacency matrices

$$\mathcal{A}_1 = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0.2 & 0.3 \end{bmatrix}, \qquad \mathcal{A}_2 = \begin{bmatrix} 0.3 & 0 & 0.7 \\ 0.5 & 0.5 & 0 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}$$

Note that $\lambda_2(\mathcal{A}_1) = 0.5,\ \lambda_2(\mathcal{A}_2) = 0.2 + 0.2236j$. Therefore,

$$\alpha(\mathcal{A}_1) = 0.5, \qquad \alpha(\mathcal{A}_2) = 0.83.$$

Simulation is initiated with $\hat{\nu}_2(0) = [0.1\ 0.8\ 1.5]^T$, and the sample period is chosen as $T_s = 0.1$ s and $T_s^* = 0.01$ s, respectively. And, Eq. (39) is solved by "fsolve" in Matlab/Simulink. Estimation of α of $\mathcal{A}_i$, $\forall i$ are provided in Fig. 1. It is clear that all estimations converge to the expected value with acceptable accuracy, albeit the initial deviation, and the convergence is prompt.

5. Conclusion

This paper investigates distributed estimation of algebraic connectivity of a digraph. The proposed scheme is based on a decentralized power iteration and affine transformation of the first left eigenvector. It is shown that, with knowledge of the first left eigenvector and the proposed consensus observer, distributed estimation of algebraic connectivity is possible, even when eigenstructure is complex. Indeed, the proposed scheme, together with numerical calculation, is also applicable when the eigenvalue (to be estimated) is complex.

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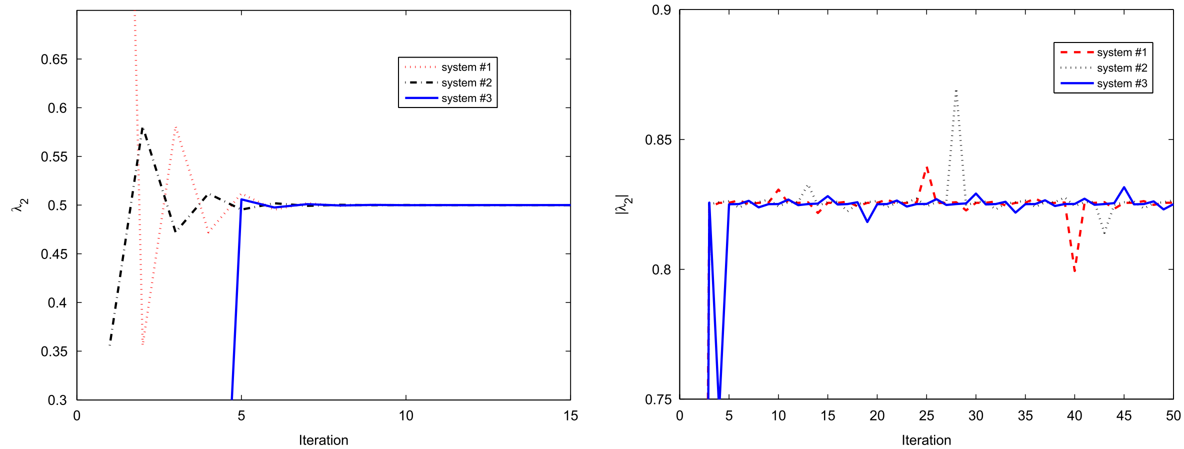

(a) A_1. (b) A_2.

Fig. 1. Estimation of algebraic connectivity.

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