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Cascaded feedback linearization and its application to stabilization of nonholonomic systems

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Abstract

In this paper, a new cascaded feedback linearization problem is formulated and a set of conditions on the cascaded feedback linearizability are established for a class of two-input affine nonlinear systems. The proposed cascaded feedback linearization method enlarges the classes of nonlinear systems which can be dealt with using the feedback linearization technique. In particular, the proposed design can be applied to address the feedback stabilization problem for a few classes of nonlinear systems which have uncontrollable linearization and do not satisfy the standard feedback linearization conditions. As an illustrative application, the proposed cascade feedback linearization concept is used to solve the feedback stabilization problem of nonholonomic systems within the framework of continuously differentiable state feedback control. Simulation results are provided to illustrate the proposed method.
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1. Introduction

Feedback linearization technique has been one of the most important tools in the study of nonlinear control systems. Different from the Jacobian linearization of a nonlinear system, the purpose of feedback linearization is to transform a given nonlinear system into a linear system via feedback control and states transformations. The exact state feedback linearization problem was pioneered and elegantly solved in [13,9,8,6], and sufficient and necessary conditions for exact feedback linearization of large classes of affine nonlinear systems were established and documented in texts [7,17]. To enlarge the class of nonlinear systems which can be handled using the differential geometric approach, the dynamic feedback linearization problem was initiated and addressed in [4] by introducing dynamic compensators and searching for the corresponding state and control transformations in the augmented state spaces. Sufficient conditions for dynamic feedback linearization were given in [5] and necessary conditions were established in [26]. Partial feedback linearization problem was formulated and studied in [15,25] by identifying the largest feedback linearizable subsystems, where conditions were given to transform a portion of the nonlinear system into a linear part. When the relative degree of the considered nonlinear system is less than system dimension, feedback linearization based nonlinear control can also render the transformed system consisting of a nonlinear zero dynamics plus a linear controllable system (the so-called normal form) [7]. The difference between the normal form and the partial feedback linearizable form is that the nonlinear zero dynamics in the normal form is only driven by the states of the linear controllable system while the nonlinear part in partial feedback linearizable system can contain control inputs. More recently, nonregular feedback linearization problem was defined in [27], where the purpose is to transform the nonlinear system into the linear controllable form with reduced control input dimensions.

On the other hand, to enhance the robustness and take advantage of the beneficial nonlinearities in the design of

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nonlinear controls, the Lyapunov direct method has also been extensively explored for nonlinear control [11], such as recursive backstepping design for nonlinear systems in the strict-feedback form [14], and nonlinear robust design for nonlinear systems with unmatched and/or generalized matched uncertainties [20,21,23,19,22]. While Lyapunov direct method has been proven to be effective for solving nonlinear control problem, the difficulty usually comes from the construction of a suitable Lyapunov function. In particular, for some classes of inherently nonlinear systems, such as nonlinear systems with uncontrollable linearization [2] and with topological obstructions to smooth (even continuous) state feedback stabilization [3], neither standard feedback linearization technique nor Lyapunov direct method can be straightforwardly applied for control design. In such a case, the discontinuous and/or time-varying feedback controls have to be sought. A typical such class of nonlinear systems are nonholonomic systems [12], which are not feedback linearizable and their feedback stabilizing control design is challenging due to Brockett's necessary condition [3].

In this paper, to deal with the feedback control problem for a class of more general nonlinear systems which are not exact or dynamic feedback linearizable, we propose a new cascaded feedback linearization approach by adopting the merits of both feedback linearization technique and Lyapunov direct method. The novelty of the proposed cascaded feedback linearization concept lies in that by properly introducing exogenous dynamics and defining the state and input transformations, the nonlinear systems can be transformed into the cascaded linear controllable subsystems with nonlinear coupling terms. Then, upon the appropriate design of the controls for the cascaded linear nominal systems, the Lyapunov direct method can be invoked to conclude the asymptotic stability of the overall closed-loop system. It is shown that under certain conditions on the choice of exogenous dynamics, the feedback linearization technique reduce to the standard (dynamic) feedback linearization or is equivalent to partial feedback linearization. In addition, the flexibility in the choice of exogenous dynamics provides extra benefits for the control design using the cascaded feedback linearization technique particularly for nonlinear systems which do not satisfy Brockett's necessary condition. In this paper, as an illustrative application, the cascaded feedback linearization method is applied in the feedback stabilization of nonholonomic chained systems [16]. In contrast to the existing control designs for nonholonomic systems [1,10,24,18], the proposed cascaded feedback linearization control design is simple and also renders smooth control.

The paper is organized as follows. In Section 2, the cascaded feedback linearization problem is formulated. Section 3 provides a set of conditions for solvability of the cascaded feedback linearization problem of a class of two-input nonlinear systems, and its stabilizing control design is explored in Section 4. In Section 5, the choice of exogenous dynamics is discussed and illustrated through examples. Section 6 presents an application to the stabilization of nonholonomic systems, and simulation results are given in Section 7. Section 8 concludes the paper.

2. Problem formulation

Consider the class of multi-input affine nonlinear systems given by

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i = f(x) + G(x)u, \tag{1}$$

where $x \in \Re^n$ is the state, $u \in \Re^m$ is the input, $f(0) = 0$, $m < n$, the entries of $f(x)$ and $G(x)$ are analytic functions of x, and rank $G(x) = m$ for all $x \in \Re^n$.

Let pair $\{A, B\}$ (or lower-dimensional sub-pairs $\{A_i, B_i\}$) denote the linear time-invariant controllable canonical form of proper dimension. Then, the standard feedback linearization problem is to find a state transformation $z = \phi(x) \in \Re^n$ and a control mapping $u = \alpha(x) + \beta(x)v$ with $v \in \Re^m$ such that the resulting transformed system is given by $\dot{z} = Az + Bv$. Conditions under which nonlinear system (1) is feedback linearizable can be found in texts such as [7,17]. For those systems that are not exact feedback linearizable, the problem of partial feedback linearization was studied in [15] to transform system into a partially linear controllable form as

$$\dot{z}_1 = Az_1 + Bv, \quad \dot{z}_2 = \gamma(z_1, z_2) + \beta(z_1, z_2)v,$$

where $z_1 \in \Re^p$ and $z_2 \in \Re^{n-p}$. As a more general extension to exact feedback linearization, the dynamic feedback linearization problem was studied in [5,26,7] by using the following dynamic compensator:

$$\dot{w} = a(x, w) + b(x, w)v, \quad u = \alpha(x, w) + \beta(x, w)v,$$

with $a(0, 0) = 0$ and $\alpha(0, 0) = 0$. Clearly, feedback linearization is closely related to controllability and, when applicable, renders simple solution to the stabilization problem.

However, many systems of practical importance are not (exact or dynamic) feedback linearizable even they are partial feedback linearizable, for instance, nonholonomic systems. Nonetheless, for nonlinear systems in the partial feedback linearizable form (the conditions were obtained for the construction of the largest feedback lineariable subsystem [15]), it remains in general to be difficult to solve the control design problem from the obtained partial feedback linearizable system due to the complexity involved with nonlinear subsystem. In this paper, we introduce a new concept of cascaded feedback linearization as defined below, which can be viewed as a special case of general partial feedback linearization form but with the characteristic of the cascaded structure of linear controllable subsystems coupled by nonlinear dynamics. The advantage of cascaded feedback linearization is that, with the proposed cascaded structure, the feedback control design can be readily performed using Lyapunov direct method. It is shown in this paper that a few classes of these systems (including nonholonomic systems) are cascaded feedback linearizable, and in particular for a class of two-input nonlinear systems the conditions of cascaded feedback linearization are explicitly given.

Definition 1. Nonlinear system (1) is cascaded feedback linearizable if there exist

- exogenous dynamics $\dot{w}=q(w)\in\Re^p$ whose state $w(t)$ stays in a bounded set Ω_w for all $t\in[t_0,\infty)$,
- state transformation $z\overset{\triangle}{=}[z_1^{\mathrm{T}},\ldots,z_m^{\mathrm{T}}]^{\mathrm{T}}=[\phi_1^{\mathrm{T}}(x,w),\ldots,\phi_m^{\mathrm{T}}(x,w)]^{\mathrm{T}}\overset{\triangle}{=}\Phi(x,w)\in\Re^n$ (with $z_i\in\Re^{n_i}$),
- control mappings $u_i=\alpha_i(x,w,u_1,\ldots,u_{i-1})+\beta_i(x,w)v_i$ with $v_i=v_i(z_i)\in\Re$

such that, for all $x\in\Re^n$ and all $w\in\Omega_w$,

(i) entries of $\phi_i(\cdot)$, $\alpha_i(\cdot)$ and $\beta_i(\cdot)$ are analytic, and the transformation $z=\Phi(x,w):\Re^{n+p}\to\Re^n$ has an inverse $x=\Phi^{-1}(z,w)$ in the sense that $\Phi^{-1}(\Phi(x,w),w)=x$,

(ii) under the state transformation and control mapping, system (1) is mapped into

$$\dot{z}_1=A_1z_1+B_1v_1, \tag{2}$$

$$\dot{z}_i=A_iz_i+B_iv_i+\sum_{j<i}\mathscr{L}_{ij}(\overline{z}_{i-1},z_i,w)z_j,\quad 1<i\leqslant m, \tag{3}$$

where pairs $\{A_i,B_i\}$ are of Brunovsky controllable form, $\overline{z}_{i-1}\overset{\triangle}{=}[z_1^{\mathrm{T}},\ldots,z_{i-1}^{\mathrm{T}}]^{\mathrm{T}}$, and $\mathscr{L}_{ij}(\cdot)$ are locally uniformly bounded functions in terms of their arguments and satisfy the growth condition of $\|\mathscr{L}_{ij}(\overline{z}_{i-1},z_i,w)\|\leqslant\rho_{i1}+\rho_{i2}\|\overline{z}_{i-1}\|^{\rho_{i3}}+\rho_{i4}\|z_i\|$ for some constants $\rho_{i1},\rho_{i2},\rho_{i4}\geqslant 0$ and $\rho_{i3}>0$.

Clearly, if $w=0$ and if $\mathscr{L}_{ij}(\cdot)=0$ for all $1<i\leqslant m$, cascaded feedback linearizable systems reduce to (dynamic) feedback linearizable systems. If $w=0$ but $\mathscr{L}_{ij}(\cdot)\neq 0$, cascaded feedback linearizable systems are equivalent to the partial feedback linearizable systems with the linear part in (2) and the nonlinear part in (3). Generally speaking, the cascaded feedback linearization concept proposed in this paper is related to but different from the partial feedback linearization method studied in [15]. Instead of seeking a set of feedback invariant controllability indices for the largest feedback linearizable subsystem, we constructively introduce exogenous dynamics which provide some kind of flexibility in rendering the system into the cascaded structure, from which the globally stabilizing control in the original state space can be designed and it is possible that the continuous/smooth feedback control design can be done for nonlinear systems which fail to satisfy the Brockett's necessary condition. The method is also different from the dynamic feedback linearization technique because the introduced exogenous dynamics are not taken as additional new states and feedback linearization is still sought in the original state space instead of doing that in the augmented state space. Hence, the concept of cascaded feedback linearization enlarges the classes of systems that can be handled using the standard feedback linearization methodology. The following examples are used to illustrate the basic idea of cascaded feedback linearization. Specifically, Example 1 shows that the cascaded feedback linearization is possible without introducing control augmentation or exogenous dynamics, and Example 2 illustrates a more sophisticated case in which the cascaded feedback linearization is realized with the aid of exogenous dynamics.

Example 1. Consider the system:

$$\dot{x}_1=u_1,\quad \dot{x}_2=x_4+x_3u_1,\quad \dot{x}_3=x_3+x_4,\quad \dot{x}_4=u_2, \tag{4}$$

which is neither static feedback linearizable nor globally dynamic feedback linearizable. It is apparent that the system is cascaded feedback linearizable to

$$\dot{z}_1=v_1,$$

$$\dot{z}_2=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}z_2+\begin{bmatrix}0\\0\\1\end{bmatrix}v_2+\begin{bmatrix}0&-1&0\\0&0&0\\0&0&0\end{bmatrix}z_2v_1$$

under the following transformations:

$$z_1=x_1,\quad v_1=u_1,\quad z_2=[x_2-x_3,\ \ -x_3,\ \ -x_3-x_4]^{\mathrm{T}},$$

$$v_2=-u_2-x_3-x_4.$$

Example 2. Consider the system:

$$\dot{x}_1=u_1,\quad \dot{x}_2=x_3x_1+x_3u_1,\quad \dot{x}_3=u_2. \tag{5}$$

It follows that $f=[0,\ \ x_3x_1,\ \ 0]^{\mathrm{T}}$, $g_1=[1,\ \ x_3,\ \ 0]^{\mathrm{T}}$, and $g_2=[0,\ \ 0,\ \ 1]^{\mathrm{T}}$. By checking that $ad_{g_1}g_2=[0,\ \ -1,\ \ 0]^{\mathrm{T}}\notin$ span $\{g_1,g_2\}$, we know that system (5) is not feedback linearizable. One can also check that system (5) is not dynamic feedback linearizable. This can be seen by simply introducing new state $x_4=u_1$ and adding one integrator $\dot{x}_4=v_1$, and then checking the feedback linearization condition under the new vector fields given by

$$\overline{f}=[x_4,\ \ x_3x_1+x_3x_4,\ \ 0,\ \ 0]^{\mathrm{T}},\quad \overline{g}_1=[0,\ \ 0,\ \ 0,\ \ 1]^{\mathrm{T}},$$

$$\overline{g}_2=[0,\ \ 0,\ \ 1,\ \ 0]^{\mathrm{T}},$$

from which the constructed distribution $\{\overline{g}_1,\overline{g}_2,ad_{\overline{f}}\overline{g}_1,ad_{\overline{f}}\overline{g}_2\}$ does not have a constant rank. Nonetheless, system (5) is cascaded feedback linearizable. This can be seen by introducing exogenous dynamics $\dot{w}=-w/2$ with $w(0)>0$ as well as the following transformations[1]:

$$z_1=\phi_1(x,w)=\frac{x_1+w}{w},\quad v_1=\frac{2u_1-w}{2w},$$

$$z_2=[\,z_{21}\quad z_{22}\,]^{\mathrm{T}}=\phi_2(x,w)=\begin{bmatrix}\dfrac{x_2}{w} & x_3\end{bmatrix}^{\mathrm{T}},\quad v_2=u_2.$$

It is obvious that bounded set for $w(t)$ is $\Omega_w=(0,w(0)]$ in which the above transformations are analytical and have

[1] In the case that there are round-off errors or measurement noises in the values of x_i and w, the calculation of the transformations need to be properly saturated in a small neighborhood around the origin in order to ensure boundedness of z, in which case the state x can be made to be uniformly ultimately bounded with respect to a small neighborhood around the origin. Such a stability result is the best achievable under any control in the presence of noises and/or round-off errors.

inverse as

$$x_1 = \phi_1^{-1}(z, w) = wz_1 - w, \quad u_1 = wv_1 - 0.5w,$$

$$[\, x_2 \quad x_3 \,]^{\mathrm{T}} = \phi_2^{-1}(z, w) = [\, wz_{21} \quad z_{22} \,]^{\mathrm{T}}, \quad u_2 = v_2.$$

It is straightforward to verify that, under the state transformation and control mapping, the system becomes

$$\dot{z}_1 = \frac{z_1}{2} + v_1,$$

$$\dot{z}_2 = \begin{bmatrix} 1/2 & -1/2 \\ 0 & 0 \end{bmatrix} z_2 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_2 + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z_2(z_1 + v_1),$$

which are in the form of (2) and (3).

It should be noted that inverse state transformation has the property of $\phi^{-1}(0, 0) = 0$ and that, even though $w \to 0$ as $t \to \infty$, the existence of transformation $\phi(x, w)$ at $w = 0$ (i.e., either $\phi(x, 0)$ or $\phi(0, 0)$) is not required in Definition 1. This is because $0 \notin \Omega_w$ (thus no singularity problem for $t \in [t_0, \infty)$) and, under an appropriately designed control, $z \to 0$ as $t \to \infty$. In other words, the existence of $\phi(0, 0)$ is achieved by control design in the sense that x is forced to converge faster than that of w.

In the next section, the conditions of cascaded feedback linearization for a class of two-input nonlinear systems are developed. Upon satisfying these conditions, control design can be proceeded in a manner similar to that for feedback linearizable systems.

3. Conditions of cascaded feedback linearization

To avoid unnecessary complexity, let us consider the following class of two-input nonlinear systems:

$$\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2. \tag{6}$$

The following theorem presents a set of conditions on cascaded feedback linearizability of a two-input nonlinear system given by (6). In the proof of the theorem, the following generalized Lie derivative with respect to a sub-vector is used for notational convenience: given a scalar function $\gamma_1(\xi)$ and a vector field $\gamma_2(\xi) \in \Re^{n'}$ with $\xi' \in \Re^{n'}$ and $\xi' \subseteq \xi$,

$$\mathscr{L}^0_{\xi', \gamma_2} \gamma_1 \triangleq \gamma_1,$$

$$\mathscr{L}_{\xi', \gamma_2} \gamma_1 \triangleq \frac{\partial \gamma_1}{\partial \xi'} \gamma_2, \ldots, \mathscr{L}^k_{\xi', \gamma_2} \gamma_1 \triangleq \frac{\partial \mathscr{L}^{k-1}_{\xi', \gamma_2} \gamma_1}{\partial \xi'} \gamma_2, \quad k \geqslant 1.$$

Theorem 1. *For two-input affine nonlinear system* (6), *suppose that there exist an integer* n_1 *and exogenous dynamics* $\dot{w} = q(w)$ *with* $w \in \Omega_w$ *for* $t \in [t_0, \infty)$ *such that*: *for all* $x \in \Re^n$ *and for all* $w \in \Omega_w$,

(i) *the matrix* $[\mathscr{G}_{10}, \mathscr{G}_{11}, \ldots, \mathscr{G}_{1,n_1-1}]$ *has rank* n_1;
(ii) *the distribution* $\mathscr{D}(x, w) = \mathrm{span}\{\mathscr{G}_{10}, \mathscr{G}_{11}, \ldots, \mathscr{G}_{1,n_1-2}, \mathscr{G}_{20}, \mathscr{G}_{21}, \ldots, \mathscr{G}_{2,n_1-1}\}$ *is involutive*,

where

$$\mathscr{G}_{10}(x, w) \triangleq g_1,$$

$$\mathscr{G}_{1k}(x, w) \triangleq \frac{\partial \mathscr{G}_{1,k-1}}{\partial x} f - \frac{\partial f}{\partial x} \mathscr{G}_{1,k-1} + \frac{\partial \mathscr{G}_{1,k-1}}{\partial w} q, \quad k \geqslant 1,$$

$$\mathscr{G}_{20}(x, w) \triangleq g_2,$$

$$\mathscr{G}_{2k}(x, w) \triangleq \frac{\partial \mathscr{G}_{2,k-1}}{\partial x} f - \frac{\partial f}{\partial x} \mathscr{G}_{2,k-1} + \frac{\partial \mathscr{G}_{2,k-1}}{\partial w} q, \quad k \geqslant 1.$$

Then, *system* (6) *is cascaded feedback linearizable if and only if there exist diffeomorphic state transformations* $\eta_1 = \eta_1(x, w) \in \Re^{n_1}$ *and* $\eta_2 = \eta_2(x, w) \in \Re^{n_2}$ *with* $n_1 + n_2 = n$ *such that the following conditions are satisfied*: *for all* $x \in \Re^n$ *and for all* $w \in \Omega_w$,

(iii) *the matrix* $[\mathscr{G}'_{20}, \mathscr{G}'_{21}, \ldots, \mathscr{G}'_{2,n_2-1}]$ *has rank* n_2;
(iv) *the distribution* $\mathscr{D}'(\eta_1, \eta_2, w) = \mathrm{span}\{\mathscr{G}'_{20}, \mathscr{G}'_{21}, \ldots, \mathscr{G}'_{2,n_2-2}\}$ *is involutive*,

where

$$\mathscr{G}'_{20} \triangleq \frac{\partial \eta_2}{\partial x} g_2,$$

$$\mathscr{G}'_{2k} \triangleq \frac{\partial \mathscr{G}'_{2,k-1}}{\partial \eta_2} f'_2 - \frac{\partial f'_2}{\partial \eta_2} \mathscr{G}'_{2,k-1} + \frac{\partial \mathscr{G}'_{2,k-1}}{\partial w} q, \quad k \geqslant 1,$$

where $f'_2 = (\partial \eta_2 / \partial x) f + (\partial \eta_2 / \partial w) q - (\partial \eta_2 / \partial x) g_1 (\mathscr{L}^{n_1}_{\overline{x}, \overline{f}} h_1 / \mathscr{L}_{x, g_1} \mathscr{L}^{n_1-1}_{\overline{x}, \overline{f}} h_1)$ *with* $\overline{x} = [x^{\mathrm{T}}, w^{\mathrm{T}}]^{\mathrm{T}}$ *and* $\overline{f} = [f^{\mathrm{T}}, q^{\mathrm{T}}]^{\mathrm{T}}$.

Proof. It follows from conditions (i) and (ii) and from Frobenius theorem that there exists a sufficiently smooth scalar function $h_1(x, w)$ such that

$$\mathscr{L}_{x, \mathscr{G}_{10}} h_1 = \mathscr{L}_{x, \mathscr{G}_{11}} h_1 = \cdots = \mathscr{L}_{x, \mathscr{G}_{1,n_1-2}} h_1 = 0$$

and

$$\mathscr{L}_{x, \mathscr{G}_{20}} h_1 = \mathscr{L}_{x, \mathscr{G}_{21}} h_1 = \cdots = \mathscr{L}_{x, \mathscr{G}_{2,n_1-1}} h_1 = 0.$$

Taking time derivative of $h_1(x, w)$ along the trajectory of system (6) yields

$$\frac{\partial h_1(x, w)}{\partial x} [\mathscr{G}_{10}, \mathscr{G}_{11}, \ldots, \mathscr{G}_{1,n_1-1}] = [0, \ldots, 0, \mathscr{L}_{x, \mathscr{G}_{1,n_1-1}} h_1].$$

Since rank of the matrix $[\mathscr{G}_{10} \ \mathscr{G}_{11} \ \cdots \ \mathscr{G}_{1,n_1-1}]$ is n_1 and since $\partial h_1(x, w)/\partial x \neq 0$ (otherwise, function $h_1(\cdot)$ is independent of x and thus trivial), we have $\mathscr{L}_{x, \mathscr{G}_{1,n_1-1}} h_1 \neq 0$.

On the other hand, using the following extended version of Jacobi identity:

$$\mathscr{L}_{x, \mathscr{G}_{i1}} h_1 = \mathscr{L}_{x, f} \mathscr{L}_{x, \mathscr{G}_{i0}} h_1 + \mathscr{L}_{w, q} \mathscr{L}_{x, \mathscr{G}_{i0}} h_1 - \mathscr{L}_{x, \mathscr{G}_{i0}} \mathscr{L}_{x, f} h_1 - \mathscr{L}_{x, \mathscr{G}_{i0}} \mathscr{L}_{w, q} h_1, \quad i = 1, 2,$$

it is easy to verify that

$$\mathscr{L}_{x,g_1}h_1 = \mathscr{L}_{x,g_1}\mathscr{L}_{\overline{x},\overline{f}}h_1 = \cdots = \mathscr{L}_{x,g_1}\mathscr{L}^{n_1-2}_{\overline{x},\overline{f}}h_1 = 0,$$

$$\mathscr{L}_{x,g_1}\mathscr{L}^{n_1-1}_{\overline{x},\overline{f}}h_1 \neq 0$$

and that

$$\mathscr{L}_{x,g_2}h_1 = \mathscr{L}_{x,g_2}\mathscr{L}_{\overline{x},\overline{f}}h_1 = \cdots = \mathscr{L}_{x,g_2}\mathscr{L}^{n_1-1}_{\overline{x},\overline{f}}h_1 = 0.$$

Thus, a partial state transformation $\eta_1 = [\eta_{11}, \eta_{12}, \ldots, \eta_{1n_1}]^{\mathrm{T}} \in \Re^{n_1}$ with

$$\eta_{11} = h_1, \quad \eta_{12} = \mathscr{L}_{\overline{x},\overline{f}}h_1, \ldots, \eta_{1n_1} = \mathscr{L}^{n_1-1}_{\overline{x},\overline{f}}h_1$$

and control mapping

$$v_1 = \mathscr{L}^{n_1}_{\overline{x},\overline{f}}h_1 + \mathscr{L}_{x,g_1}\mathscr{L}^{n_1-1}_{\overline{x},\overline{f}}h_1 u_1$$

can be calculated, and their dynamics are described by

$$\dot{\eta}_1 = A_1\eta_1 + B_1 v_1, \tag{7}$$

where the pair $\{A_1, B_1\}$ is of Brunovsky canonical form. Upon having η_1, it is always possible to find the remaining part of the state transformation, η_2, such that there is a diffeomorphism between x and $[\eta_1^{\mathrm{T}}, \eta_2^{\mathrm{T}}]^{\mathrm{T}}$ and that the remaining dynamics become

$$\dot{\eta}_2 = f_2'(\eta_1, \eta_2, w) + g_{21}'(\eta_1, \eta_2, w)v_1 + \mathscr{G}_{20}'(\eta_1, \eta_2, w)u_2, \tag{8}$$

where $g_{21}' = (1/\mathscr{L}_{x,g_1}\mathscr{L}^{n_1-1}_{\overline{x},\overline{f}}h_1)(\partial\eta_2/\partial x)g_1$.

In what follows, cascaded feedback linearizability of system (6) is investigated. Note that subsystem (7) is of the form (2) by setting $z_1 = \eta_1$. Subsystem (8) is feedback linearizable into (3) if and only if a sufficiently smooth function $h_2(\eta_1, \eta_2, w)$ exists such that, under coordinate transformation $z_2(\eta_1, \eta_2, w) = [z_{21}, \ldots, z_{2n_2}]^{\mathrm{T}} \in \Re^{n_2}$,

$$z_{21} = h_2, z_{2i} = \mathscr{L}_{\overline{\eta}_2,\overline{f}_2'}z_{2,i-1}, i = 2, \ldots, n_2,$$

$$\begin{aligned}\dot{z}_{2i} &= \frac{\partial z_{2i}}{\partial\eta_2}\dot{\eta}_2 + \frac{\partial z_{2i}}{\partial w}\dot{w} + \frac{\partial z_{2i}}{\partial\eta_1}\dot{\eta}_1\\ &= \mathscr{L}_{\overline{\eta}_2,\overline{f}_2'}z_{2i} + \mathscr{L}_{\eta_2,\mathscr{G}_{20}'}z_{2i}u_2 + \mathscr{L}_{\eta_2,g_{21}'}z_{2i}v_1\\ &\quad + \frac{\partial z_{2i}}{\partial\eta_1}(A_1\eta_1 + B_1v_1)\\ &= z_{2,i+1} + \mathscr{L}_{\eta_2,g_{21}'}z_{2i}v_1 + \frac{\partial z_{2i}}{\partial\eta_1}(A_1\eta_1 + B_1v_1),\\ &\quad i = 1, \ldots, n_2 - 1,\end{aligned}$$

$$\begin{aligned}\dot{z}_{2n_2} &= \mathscr{L}_{\overline{\eta}_2,\overline{f}_2'}z_{2n_2} + \mathscr{L}_{\eta_2,\mathscr{G}_{20}'}z_{2n_2}u_2 + \mathscr{L}_{\eta_2,g_{21}'}z_{2n}v_1\\ &\quad + \frac{\partial z_{2n_2}}{\partial\eta_1}(A_1\eta_1 + B_1v_1),\end{aligned} \tag{9}$$

that is, if and only if the following set of conditions are met:

$$\begin{aligned}&\mathscr{L}_{\eta_2,\mathscr{G}_{20}'}h_2 = 0,\\ &\mathscr{L}_{\eta_2,\mathscr{G}_{20}'}\mathscr{L}_{\overline{\eta}_2,\overline{f}_2'}h_2 = 0, \ldots, \mathscr{L}_{\eta_2,\mathscr{G}_{20}'}\mathscr{L}^{n_2-2}_{\overline{\eta}_2,\overline{f}_2'}h_2 = 0,\\ &\mathscr{L}_{\eta_2,\mathscr{G}_{20}'}\mathscr{L}^{n_2-1}_{\overline{\eta}_2,\overline{f}_2'}h_2 \neq 0,\end{aligned} \tag{10}$$

where $\overline{\eta}_2 = [\eta_2^{\mathrm{T}}, w^{\mathrm{T}}]^{\mathrm{T}}$ and $\overline{f}_2' = [(f_2')^{\mathrm{T}}, q^{\mathrm{T}}]^{\mathrm{T}}$. Therefore, we only need to show that existence of function $h_2(\cdot)$ satisfying (10) is equivalent to conditions (iii) and (iv).

Sufficiency: Suppose that conditions (iii) and (iv) are satisfied. By Frobenius theorem, there exists function $h_2(\cdot)$ such that

$$\mathscr{L}_{\eta_2,\mathscr{G}_{20}'}h_2 = \mathscr{L}_{\eta_2,\mathscr{G}_{21}'}h_2 = \cdots = \mathscr{L}_{\eta_2,\mathscr{G}_{2,n_2-2}'}h_2 = 0.$$

Furthermore, it follows that

$$\frac{\partial h_2}{\partial\eta_2}[\mathscr{G}_{20}', \mathscr{G}_{21}', \ldots, \mathscr{G}_{2,n_2-1}'] = [0, \ldots, 0, \mathscr{L}_{\eta_2,\mathscr{G}_{2,n_2-1}'}h_2].$$

Since rank of the matrix $[\mathscr{G}_{20}' \ \mathscr{G}_{21}' \ \cdots \ \mathscr{G}_{2,n_2-1}']$ is n_2 and since $\partial h_2/\partial\eta_2 \neq 0$, we have $\mathscr{L}_{\eta_2,\mathscr{G}_{2,n_2-1}}h_2 \neq 0$. Thus, using the following extended version of Jacobi identity

$$\begin{aligned}\mathscr{L}_{\eta_2,\mathscr{G}_{21}'}h_2 &= \mathscr{L}_{\eta_2,f_2'}\mathscr{L}_{\eta_2,\mathscr{G}_{20}'}h_2 + \mathscr{L}_{w,q}\mathscr{L}_{\eta_2,\mathscr{G}_{20}'}h_2\\ &\quad - \mathscr{L}_{\eta_2,\mathscr{G}_{20}'}\mathscr{L}_{\eta_2,f_2'}h_2 - \mathscr{L}_{\eta_2,\mathscr{G}_{20}'}\mathscr{L}_{w,q}h_2,\end{aligned}$$

all the expressions in (10) can be verified.

Necessity: Suppose that function $h_2(\cdot)$ satisfies the expressions in (10). We first prove the following result by an induction in terms of j:

$$\mathscr{L}_{\eta_2,\mathscr{G}_{2j}'}\mathscr{L}^k_{\overline{\eta}_2,\overline{f}_2'}h_2 = \begin{cases}0 & 0\leqslant j+k < n_2 - 1,\\ (-1)^j\mathscr{L}_{\eta_2,\mathscr{G}_{20}'}\mathscr{L}^{n_2-1}_{\overline{\eta}_2,\overline{f}_2'}h_2 \neq 0 & j+k = n_2 - 1.\end{cases} \tag{11}$$

Note first that, at $j = 0$ and for $k = 1, \ldots, n_2 - 1$, the result in (11) is implied by the expressions in (10). Next, assume that (11) hold for some $j \geqslant 0$ in order to show (11) also holds for $j+1$. At $j+1$, it follows from the Jacobi identity that, for any real-valued function γ and integer $j \geqslant 0$,

$$\begin{aligned}\mathscr{L}_{\eta_2,\mathscr{G}_{2,j+1}'}\gamma &= \mathscr{L}_{\eta_2,f_2'}\mathscr{L}_{\eta_2,\mathscr{G}_{2j}'}\gamma + \mathscr{L}_{w,q}\mathscr{L}_{\eta_2,\mathscr{G}_{2j}'}\gamma\\ &\quad - \mathscr{L}_{\eta_2,\mathscr{G}_{2j}'}\mathscr{L}_{\eta_2,f_2'}\gamma - \mathscr{L}_{\eta_2,\mathscr{G}_{2j}'}\mathscr{L}_{w,q}\gamma\end{aligned}$$

and

$$\begin{aligned}\mathscr{L}_{\eta_2,\mathscr{G}_{2,j+1}'}\mathscr{L}^k_{\overline{\eta}_2,\overline{f}_2'}h_2 &= \mathscr{L}_{\eta_2,f_2'}\mathscr{L}_{\eta_2,\mathscr{G}_{2j}'}\mathscr{L}^k_{\overline{\eta}_2,\overline{f}_2'}h_2\\ &\quad + \mathscr{L}_{w,q}\mathscr{L}_{\eta_2,\mathscr{G}_{2j}'}\mathscr{L}^k_{\overline{\eta}_2,\overline{f}_2'}h_2\\ &\quad - \mathscr{L}_{\eta_2,\mathscr{G}_{2j}'}\mathscr{L}_{\eta_2,f_2'}\mathscr{L}^k_{\overline{\eta}_2,\overline{f}_2'}h_2\\ &\quad - \mathscr{L}_{\eta_2,\mathscr{G}_{2j}'}\mathscr{L}_{w,q}\mathscr{L}^k_{\overline{\eta}_2,\overline{f}_2'}h_2.\end{aligned} \tag{12}$$

Since result (11) is assumed to hold at j, it follows from (12) that

$$\begin{aligned}\mathscr{L}_{\eta_2,\mathscr{G}'_{2,j+1}}\mathscr{L}^k_{\overline{\eta}_2,\overline{f}'_2}h_2 &= -\mathscr{L}_{\eta_2,\mathscr{G}'_{2j}}\mathscr{L}_{\eta_2,f'_2}\mathscr{L}^k_{\overline{\eta}_2,\overline{f}'_2}h_2 - \mathscr{L}_{\eta_2,\mathscr{G}'_{2j}}\mathscr{L}_{w,q}\mathscr{L}^k_{\overline{\eta}_2,\overline{f}'_2}h_2\\ &= -\mathscr{L}_{\eta_2,\mathscr{G}'_{2j}}\mathscr{L}^{k+1}_{\overline{\eta}_2,\overline{f}'_2}h_2\\ &= \begin{cases}0, & 0\leqslant j+k+1<n_2-1,\\ (-1)^{j+1}\mathscr{L}_{\eta_2,\mathscr{G}'_{20}}\mathscr{L}^{n_2-1}_{\overline{\eta}_2,\overline{f}'_2}h_2\neq 0, & j+k+1=n_2-1,\end{cases}\end{aligned} \tag{13}$$

which is the same result (11) but at $j+1$.

To show the necessity, it follows that

$$\begin{aligned}&\begin{bmatrix}\dfrac{\partial h_2}{\partial \eta_2}\\ \vdots\\ \dfrac{\partial \mathscr{L}^{n_2-1}_{\overline{\eta}_2,\overline{f}'_2}h_2}{\partial\eta_2}\end{bmatrix}[\mathscr{G}'_{20}\ \cdots\ \mathscr{G}'_{2,n_2-1}]\\ &= \begin{bmatrix}\mathscr{L}_{\eta_2,\mathscr{G}'_{20}}h_2 & \mathscr{L}_{\eta_2,\mathscr{G}'_{21}}h_2 & \cdots & \cdots & \mathscr{L}_{\eta_2,\mathscr{G}'_{2,n_2-1}}h_2\\ \mathscr{L}_{\eta_2,\mathscr{G}'_{20}}\mathscr{L}_{\overline{\eta}_2,\overline{f}'_2}h_2 & & & \mathscr{L}_{\eta_2,\mathscr{G}'_{2,n_2-2}}\mathscr{L}_{\overline{\eta}_2,\overline{f}'_2}h_2 & *\\ \vdots & & & & \vdots\\ \mathscr{L}_{\eta_2,\mathscr{G}'_{20}}\mathscr{L}^{n_2-1}_{\overline{\eta}_2,\overline{f}'_2}h_2 & * & \cdots & & *\end{bmatrix}\\ &= \begin{bmatrix}0 & \cdots & \cdots & 0 & \Delta_0\\ 0 & & & \Delta_1 & *\\ \vdots & & & & \vdots\\ 0 & \Delta_{n_2-2} & & & *\\ \Delta_{n_2-1} & * & \cdots & & *\end{bmatrix},\end{aligned} \tag{14}$$

in which the entries

$$\Delta_i = \mathscr{L}_{\eta_2,\mathscr{G}'_{2,(n_2-1-i)}}\mathscr{L}^i_{\overline{\eta}_2,\overline{f}'_2}h_2$$

are all nonzero according to (11). Hence, matrix $[\mathscr{G}'_{20},\mathscr{G}'_{21},\ldots,\mathscr{G}'_{2,n_2-1}]$ has rank n_2, which implies that distribution $\mathscr{D}'$ is nonsingular and has dimension n_2-1. On the other hand, it follows that

$$\frac{\partial h_2}{\partial\eta_2}[\mathscr{G}'_{20},\mathscr{G}'_{21},\ldots,\mathscr{G}'_{2,n_2-2}]=0,$$

which implies that $\mathscr{D}'$ is completely integrable and, by Frobenius theorem, $\mathscr{D}'$ is involutive. □

Theorem 1 states the conditions of cascaded feedback linearizability for the class of two-input affine nonlinear systems, and it can be extended to multi-input systems. As shown in the proof, the key steps are to solve for scalar functions $h_1(\cdot)$ and $h_2(\cdot)$ from a set of first-order partial differential equations, which is generally a difficult task. Fortunately, for many practical physical systems, it is often straightforward to find functions $h_1(\cdot)$ and $h_2(\cdot)$. Moreover, the introduction of exogenous dynamics w adds flexibility to finding the solutions, which will be discussed in Section 5.

Next, we consider the case that system (6) is already of the cascaded structure as given by

$$\dot{x}_1 = f_1(x_1)+g_{11}(x_1)u_1, \tag{15}$$

$$\dot{x}_2 = f_2(x)+g_{21}(x)u_1+g_{22}(x)u_2, \tag{16}$$

where $x_1\in\Re^{n_1}$ and $x_2\in\Re^{n_2}$. It follows that $x=[x_1^{\mathrm{T}},x_2^{\mathrm{T}}]^{\mathrm{T}}\in\Re^n$, $f=[f_1^{\mathrm{T}},f_2^{\mathrm{T}}]^{\mathrm{T}}$, $g_1=[g_{11}^{\mathrm{T}},g_{21}^{\mathrm{T}}]^{\mathrm{T}}$, and $g_2=[0,g_{22}^{\mathrm{T}}]^{\mathrm{T}}$. Should its cascaded structure render feedback linearization of two individual subsystems, the following theorem can be concluded. Unlike those in Theorem 1, the conditions in Theorem 2 are sufficient but may not be necessary since the state transformations in this case are limited to $z_1=\Phi_1(x_1,w)$ and $z_2=\Phi_2(x_2,w)$ (instead of the more general expressions of $z_1=\Phi_1(x,w)$ and $z_2=\Phi_2(x,w)$ used in Theorem 1). Proof of Theorem 2 is analogous to that of Theorem 1.

Theorem 2. *Consider the two-input affine nonlinear system in* (15) *and* (16) *and suppose that exogenous dynamics* $\dot{w}=q(w)$ *with* $w\in\Omega_w$ *for* $t\in[t_0,\infty)$ *such that the following conditions are satisfied: for all* $x\in\Re^n$ *and for all* $w\in\Omega_w$,

(i) *the matrix* $[G_{10},G_{11},\ldots,G_{1,n_1-1}]$ *has rank* n_1;
(ii) *the distribution* $\mathscr{D}_1(x_1,w)=\mathrm{span}\{G_{10},G_{11},\ldots,G_{1,n_1-2}\}$ *is involutive*;
(iii) *the matrix* $[G_{20},G_{21},\ldots,G_{2,n_2-1}]$ *has rank* n_2;

(iv) *the distribution* $\mathscr{D}_2(z_1, x_2, w) = \mathrm{span}\{G_{20}, G_{21}, \ldots, G_{2,n_2-2}\}$ *is involutive*,

then, there exist smooth scalar functions $h_1(x_1, w)$ *and* $h_2(z_1, x_1, w)$ *such that* (15) *and* (16) *are cascaded feedback linearizable with the following transformations*: $z_1 = [z_{11}, \ldots, z_{1n_1}]^{\mathrm{T}} \in \Re^{n_1}$, $z_2 = [z_{21}, \ldots, z_{2n_2}]^{\mathrm{T}} \in \Re^{n_2}$,

$$z_{11} = h_1(x_1, w), \quad z_{1i} = \mathscr{L}^{i-1}_{\overline{x}_1, \overline{f}_1} h_1, \quad i = 2, \ldots, n_1 - 1,$$

$$u_1 = \frac{v_1 - \mathscr{L}^{n_1}_{\overline{x}_1, \overline{f}_1} h_1}{\mathscr{L}_{x_1, g_{11}} \mathscr{L}^{n_1-1}_{\overline{x}_1, \overline{f}_1} h_1}$$

$$z_{21} = h_2(z_1, x_2, w), \quad z_{2i} = \mathscr{L}^{i-1}_{\overline{x}_2, \overline{f}_2} h_2, \quad i = 2, \ldots, n_2 - 1,$$

$$u_2 = \frac{v_2 - \mathscr{L}^{n_2}_{\overline{x}_2, \overline{f}_2} h_2}{\mathscr{L}_{x_2, g_{22}} \mathscr{L}^{n_2-1}_{\overline{x}_2, \overline{f}_2} h_2},$$

where

$$G_{10}(\overline{x}_1) \stackrel{\triangle}{=} g_{11},$$

$$G_{1k}(\overline{x}_1) \stackrel{\triangle}{=} \frac{\partial G_{1,k-1}}{\partial x_1} f_1 - \frac{\partial f_1}{\partial x_1} G_{1,k-1} + \frac{\partial G_{1,k-1}}{\partial w} q, \quad k \geqslant 1,$$

$$G_{20}(z_1, \overline{x}_2) \stackrel{\triangle}{=} g_{22},$$

$$G_{2k}(z_1, \overline{x}_2) \stackrel{\triangle}{=} \frac{\partial G_{2,k-1}}{\partial x_2} f_2' - \frac{\partial f_2'}{\partial x_2} G_{2,k-1} + \frac{\partial G_{2,k-1}}{\partial w} q, \quad k \geqslant 1,$$

with $\overline{x}_1 = [x_1^{\mathrm{T}}, w^{\mathrm{T}}]^{\mathrm{T}}$, $\overline{x}_2 = [x_2^{\mathrm{T}}, w^{\mathrm{T}}]^{\mathrm{T}}$, $\overline{f}_1 = [f_1^{\mathrm{T}}, q^{\mathrm{T}}]^{\mathrm{T}}$, $\overline{f}_2 \stackrel{\triangle}{=} [(f_2')^{\mathrm{T}}, q^{\mathrm{T}}]^{\mathrm{T}}$, *and* $f_2' = f_2 - g_{21} \frac{\mathscr{L}_{\overline{x}_1, \overline{f}_1} z_{1n_1}}{\mathscr{L}_{x_1, g_{11}} z_{1n_1}}$.

4. Stabilization of cascaded feedback linearizable systems

It is shown in this section that, under the proposed controls, cascaded feedback linearized systems are exponentially convergent and so are their original systems.

Theorem 3. *Suppose that system* (6) *is cascaded feedback linearizable, that asymptotically stable exogenous dynamics* $\dot{w} = q(w)$ *with* $w \in \Omega_w$ *for* $t \in [t_0, \infty)$ *have been found, and that, under transformations* $z = \Phi(x, w)$ *and* $u = \alpha(x, w) + \beta(x, w)v$, *the system has been transformed into the following canonical form*: *for all* $x \in \Re^n$ *and for all* $w \in \Omega_w$,

$$\dot{z}_1 = A_1 z_1 + B_1 v_1, \tag{17}$$

$$\dot{z}_2 = A_2 z_2 + B_2 v_2 + \mathscr{Z}(z_1, z_2, w) z_1, \tag{18}$$

where nonlinear term $\mathscr{Z}(z_1, z_2, w)$ *is bounded as*

$$\|\mathscr{Z}(z_1, z_2, w)\| \leqslant \rho_1 + \rho_2 \|z_1\|^{\rho_3} + \rho_4 \|z_2\| \tag{19}$$

for constants $\rho_1, \rho_2, \rho_4 \geqslant 0$ *and* $\rho_3 > 0$. *Then, under the control*

$$v_1 = -r_1^{-1} B_1^{\mathrm{T}} P_1 z_1, \quad v_2 = -r_2^{-1} B_2^{\mathrm{T}} P_2 z_2, \tag{20}$$

where $r_1 > 0$ *and* $r_2 > 0$ *are design parameters, and* $P_i > 0$ *are the solutions to algebraic Riccati equations* $P_i A_i + A_i^{\mathrm{T}} P_i - P_i B_i r_i^{-1} B_i^{\mathrm{T}} P_i + Q_i = 0$ *for any choices of* Q_i *satisfying* $0 < \underline{q} I \leqslant Q_i \leqslant \overline{q} I$, *the transformed state* z *is exponentially convergent. If the inverse transformation* $x = \Phi^{-1}(x, w)$ *has the property that* $\Phi^{-1}(0, 0) = 0$, *the original state* $x(t)$ *also convergence to the origin asymptotically*.

Proof. Consider Lyapunov function $V = V_1 + V_2$ where $V_i = z_i^{\mathrm{T}} P_i z_i$. It follows that, along any trajectory of subsystem (17) and under control (20),

$$\dot{V}_1 = -z_1^{\mathrm{T}} Q_1 z_1 \leqslant -\frac{\underline{q}}{\lambda_{\max}(P_1)} V_1 \stackrel{\triangle}{=} -\lambda V_1,$$

from which exponential stability of $\|z_1\|$ can be concluded as

$$\|z_1(t)\| \leqslant \frac{\lambda_{\max}(P_1)}{\lambda_{\min}(P_1)} \|z_1(t_0)\| \mathrm{e}^{-\lambda(t-t_0)}. \tag{21}$$

Now, consider subsystem (18) under control (20). It follows that

$$\begin{aligned}\dot{V}_2 &\leqslant -z_2^{\mathrm{T}} Q_2 z_2 + 2 z_2^{\mathrm{T}} P_2 \mathscr{Z} z_1 \\ &\leqslant -\underline{q}\|z_2\|^2 + 2\|z_2\| \|P_2\| (\rho_1 + \rho_2 \|z_1\|^{\rho_3} + \rho_4 \|z_2\|) \|z_1\| \\ &\leqslant [-2\beta_0 + 2\beta_2 \lambda \mathrm{e}^{-\lambda(t-t_0)}] V + 2\beta_1 \sqrt{V}\, \mathrm{e}^{-\lambda(t-t_0)},\end{aligned} \tag{22}$$

where

$$c_0 = \frac{\lambda_{\max}(P_1)}{\lambda_{\min}(P_1)} \|z_1(t_0)\|, \quad \beta_0 = \frac{\underline{q}}{2\lambda_{\max}(P_2)},$$

$$\beta_1 = \frac{\lambda_{\max}(P_2)}{\sqrt{\lambda_{\min}(P_2)}} (\rho_1 + \rho_2 c_0^{\rho_3}) c_0, \quad \beta_2 = \frac{c_0}{\lambda} \frac{\lambda_{\max}(P_2)}{\lambda_{\min}(P_2)}.$$

The solution to inequality (22) is

$$\begin{aligned}\sqrt{V(t)} &\leqslant \sqrt{V(t_0)} \mathrm{e}^{\int_{t_0}^t (-\beta_0 + \beta_2 \lambda \mathrm{e}^{-\lambda(\tau - t_0)})\, \mathrm{d}\tau} \\ &\quad + \int_{t_0}^t \mathrm{e}^{\int_s^t (-\beta_0 + \beta_2 \lambda \mathrm{e}^{-\lambda(\tau - s)})\, \mathrm{d}\tau} \beta_1 \mathrm{e}^{-\lambda(s-t_0)}\, \mathrm{d}s \\ &\leqslant \sqrt{V(t_0)} \mathrm{e}^{\beta_2} \mathrm{e}^{-\beta_0(t-t_0)} \\ &\quad + \frac{\beta_1 \mathrm{e}^{\beta_2}}{\beta_0 - \lambda} [\mathrm{e}^{-\lambda(t-t_0)} - \mathrm{e}^{-\beta_0(t-t_0)}],\end{aligned}$$

from which exponential convergence of $\|z_2\|$ is obvious. Asymptotic convergence of $\|x\|$ can be concluded by noting the convergence of z and w and using the property of $\Phi^{-1}(0, 0) = 0$. □

5. Roles of exogenous dynamics

In this section, the roles of exogenous dynamics are studied for the proposed cascaded feedback linearization methodology and its associated problems of stabilization and control design. Generally, whether to introduce exogenous dynamics and what are their choices depend upon structural properties of the nonlinear system under investigation. Nonetheless, it is possible to point out the benefits of introducing exogenous dynamics. Specifically, exogenous dynamics are introduced to alleviate the difficulties in the following aspects:

(a) Rank conditions on distribution of vector fields.
(b) Singularity encountered in a backstepping control design.
(c) Brockett's necessary condition on the existence of smooth feedback control.

Item (a) is most relevant to feedback linearization. It has been shown in Example 2, exogenous dynamics can be introduced to meet the rank conditions required for cascaded feedback linearization. As will be shown in the next section, exogenous dynamics facilitate cascaded feedback linearization of classes of nonholonomic systems.

With or without carrying out feedback linearization, exogenous dynamics can be introduced to avoid a possible singularity problem that may be encountered in a backstepping control design. Example 3 is provided below to illustrate item (b). For the problem of stabilization, it is preferred that exogenous dynamics be chosen to be asymptotically/exponentially stable.

Besides the singularity problem, the control design problem also involves the choice of controllers. It is well known that, for nonlinear system $\dot{x}=f(x,u)$ with $f(0,0)=0$, the existence of smooth feedback control requires the so-called Brockett necessary condition [3]. That is, there is no smooth feedback control if the algebraic mapping $(x,u)\mapsto f(x,u)$ is not onto a neighborhood around the origin. In the case that the system can be stabilized but the mapping $(x,u)\mapsto f(x,u)$ is not onto, one has to search for a state feedback discontinuous control or a smooth time-varying state feedback control, and those designs are often less systematic. Example 4 is included below to illustrate item (c). That is, the Brockett necessary condition is met by introducing exogenous dynamics and hence a smooth control depending upon exogenous dynamics (or simply upon time) can be proceeded with. As a result, the same control design procedure can be applied without regard to the Brockett necessary condition so long as the condition, if not met originally, can be satisfied by the introduction of exogenous dynamics.

Example 3. Consider the stabilization problem of the following cascaded nonlinear system:

$$\dot{x}_1=u_1, \tag{23}$$

$$\dot{x}_2=x_3x_1,\quad \dot{x}_3=u_2. \tag{24}$$

It is apparent that, due to the singularity when $x_1=0$, the backstepping method is not directly applicable to design control u_2. To alleviate this problem, we introduce the following exogenous dynamics: for some $w(t_0)>0$,

$$\dot{w}=-w.$$

It follows that, for $t\in[t_0,\infty)$, $w\in(0,w(t_0)]$. Thus, the following transformations can be defined for $t\in[t_0,\infty)$:

$$z_1=\frac{x_1-w}{w},\quad v_1=\frac{u_1+x_1}{w}, \tag{25}$$

under which the system becomes

$$\dot{z}_1=v_1, \tag{26}$$

$$\dot{x}_2=x_3w+x_3wz_1,\quad \dot{x}_3=u_2. \tag{27}$$

Since $w(t)\neq 0$ for $t\in[t_0,\infty)$, the backstepping design can be applied to (27), that is, we choose for $t\in[t_0,\infty)$

$$z_2=x_2,\quad z_3=x_3+\frac{x_2}{w},\quad v_2=u_2+z_3+x_3z_1, \tag{28}$$

by which the equations in (27) can be rewritten as

$$\dot{z}_2=-z_2+[z_3w+(z_3w-z_2)z_1],\quad \dot{z}_3=v_2. \tag{29}$$

It is clear that controls $v_1=-z_1$ and $v_2=-z_3$ stabilize the transformed system of z_i, as a result, transformations (25) and (28) are well defined for all t and original state variables x_i asymptotically converge to zero.

Example 4. Consider again the chained system:

$$\dot{x}_1=u_1,\quad \dot{x}_2=x_3u_1,\quad \dot{x}_3=u_2.$$

The system fails to satisfy the Brockett's necessary condition. For any exogenous dynamics $\dot{w}=q(w)$, the following transformations can be applied:

$$x_1'=x_1-w,\quad x_2'=x_2,\quad x_3'=x_3,\quad u_1'=u_1-q(w),$$
$$u_2'=u_2,$$

under which the system becomes

$$\dot{x}_1'=u_1',\quad \dot{x}_2'=x_3'q(w)+x_3'u_1',\quad \dot{x}_3'=u_2'.$$

Hence, the mapping $(x_1',x_2',x_3',u_1',u_2')\mapsto f'=[u_1',x_3'q(w)+x_3'u_1',u_2']^{\mathrm{T}}$ is onto a small neighborhood of $x'=[x_1',x_2',x_3']^{\mathrm{T}}=0$, that is,

$$f'=\xi=[\xi_1,\xi_2,\xi_3]^{\mathrm{T}}\quad \text{with } 0<\|\xi\|\ll 1$$

provided that

$$u_1'=\xi_1,\quad u_2'=\xi_3,\quad x_3'=\frac{\xi_2}{q(w)+\xi_1}.$$

Clearly, the above solution exists and hence the Brockett's necessary condition is made valid with the aid of w if exogenous dynamics are chosen such that $q(w)+\xi_1$ is a lower-order infinitesimal than ξ_2.

6. Application to the stabilization of nonholonomic systems

The proposed cascaded feedback linearization method can be directly applied to the stabilization problem of nonholonomic systems. Consider the class of chained systems: $\dot{x}=g_1(x)u_1+g_2u_2$, or,

$$\dot{x}_1=u_1, \tag{30}$$

$$\dot{x}_2=g_{21}(x)u_1+g_{22}u_2, \tag{31}$$

where $x=[x_1, x_2^{\mathrm{T}}]^{\mathrm{T}} \in \Re^n$, $x_1 \in \Re$, $x_2=[x_{21}, \ldots, x_{2,n-1}]^{\mathrm{T}} \in \Re^{n-1}$, $g_{21}=[x_{22}, x_{23} \cdots x_{2,n-1} 0]^{\mathrm{T}}$, $g_{22}=[0\ 0 \cdots 0\ 1]^{\mathrm{T}}$, $g_1=[1, g_{21}^{\mathrm{T}}]^{\mathrm{T}}$, and $g_2=[0, g_{22}^{\mathrm{T}}]^{\mathrm{T}}$. By checking that

$$[g_1, g_2]=[0, -1-1 \cdots, -10]^{\mathrm{T}} \notin \operatorname{span}\{g_1, g_2\},$$

we know that a chained system given by (30) and (31) is not feedback linearizable. In what follows, its stabilization problem is solved using the proposed cascaded feedback linearization method.

Lemma 1. *A chained system given by* (30) *and* (31) *is cascaded feedback linearizable. In particular, if the exogenous dynamics are chosen to be*

$$\dot{w}=-w \in \Re, \quad w(0) \neq 0, \tag{32}$$

the chained system is transformed into

$$\begin{aligned} &\dot{z}_1=z_1+v_1, \\ &\dot{z}_2=A_2' z_2+B_2' v_2+(A_2'-\operatorname{diag}\{n-2, n-3, \ldots, 1, 0\}) z_2 v_1, \end{aligned} \tag{33}$$

where

$$\begin{aligned} &z_1=\frac{x_1-w}{w}, \quad v_1=\frac{u_1+w}{w}, \\ &z_2=\left[\begin{array}{llll} \frac{x_{21}}{w^{n-2}} & \cdots & \frac{x_{2,n-2}}{w} & x_{2,n-1} \end{array}\right]^{\mathrm{T}}, \quad v_2=u_2, \end{aligned} \tag{34}$$

$$A_2'=\left[\begin{array}{ccccc} n-2 & 1 & 0 & \cdots & 0 \\ 0 & n-3 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 0 \end{array}\right] \in \Re^{(n-1) \times(n-1)},$$

$$B_2'=\left[\begin{array}{llll} 0 & 0 & \cdots & 1 \end{array}\right]^{\mathrm{T}} \in \Re^{(n-1) \times 1},$$

and the pair $\{A_2', B_2'\}$ *is controllable.*

Proof. It follows from Theorem 2 that, under the state and input transformations with

$$z_1=\frac{x_1-w}{w}, \quad v_1=\frac{u_1+w}{w},$$

$f_2'=-g_{21} w$ and the corresponding distributions of vector fields are

$$[G_{10}]=1,$$

and

$$\begin{aligned} &\left[\begin{array}{llll} G_{20} & G_{21} & \cdots & G_{2,n-2} \end{array}\right] \\ &=\left[\begin{array}{ccccccc} 0 & 0 & 0 & 0 & \cdots & 0 & (-1)^{n-1} w^{n-2} \\ 0 & 0 & 0 & 0 & \cdots & (-1)^{n-2} w^{n-3} & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & w^3 & \cdots & * & * \\ 0 & 0 & -w^2 & * & \cdots & * & * \\ 0 & w & * & * & \cdots & * & * \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right], \end{aligned}$$

which are of rank 1 and $n-1$, respectively. It is trivial that $\{G_{10}\}$ is involutive. In addition, it follows from $ad_{G_{2i}} G_{2j}=0$ for all $i, j=0, \ldots, n-3$ that $\{G_{20}\ G_{21}\ \cdots\ G_{2,n-3}\}$ is involutive. Thus, by Theorem 2, the chained system is cascaded feedback linearizable.

It is straightforward to verify that, under the transformation, the chained system is mapped into (33) which can further be transformed into the canonical form (2) and (3). □

The following theorem on stabilization of the chained system follows directly from Lemma 1 and Theorem 3.

Theorem 4. *Consider a chained system given by* (30) *and* (31). *Then, the system state converges to the origin exponentially under the following control*:

$$u_1=-r_1^{-1} p_1(x_1-w)+w, \tag{35}$$

$$u_2=-r_2^{-1} B_2'^{\mathrm{T}} P_2 z_2, \tag{36}$$

where w *and* z_2 *are defined by* (32) *and* (34) *with* $w(0)=\|x(0)\|^\rho$ *for some constant* $0<\rho<1$; $r_1>0$, $r_2>0$, $q_1>0$ *and* $q_2>0$ *are design constants*; $p_1=r_1+\sqrt{r_1^2+q_1 r_1}$, P_2 *is the solution to algebraic Riccati equation*

$$P_2 A_2'+A_2'^{\mathrm{T}} P_2-P_2 B_2' r_2^{-1} B_2'^{\mathrm{T}} P_2+C_2'^{\mathrm{T}} q_2 C_2'=0,$$

and $C_2'=[1, \ldots, 0,0] \in \Re^{1 \times(n-1)}$.

It is worthy mentioning that, with little change, the proposed method is applicable to the following generalized nonholonomic systems [27]:

$$\dot{x}_1=u_1, \quad \dot{x}_2=\left[\begin{array}{c} 1 \\ \xi_2(x_1) \\ \vdots \\ \xi_n(x_1) \end{array}\right] u_2,$$

where $\xi_i(x_1)$ are analytic functions vanishing at the origin with $\partial^{i-1} \xi_i / \partial x_1^{i-1} \neq 0$ for $i=2, \ldots, n$. The proposed cascaded feedback linearization idea is also applicable to high-order chained system. For instance, consider the second-order

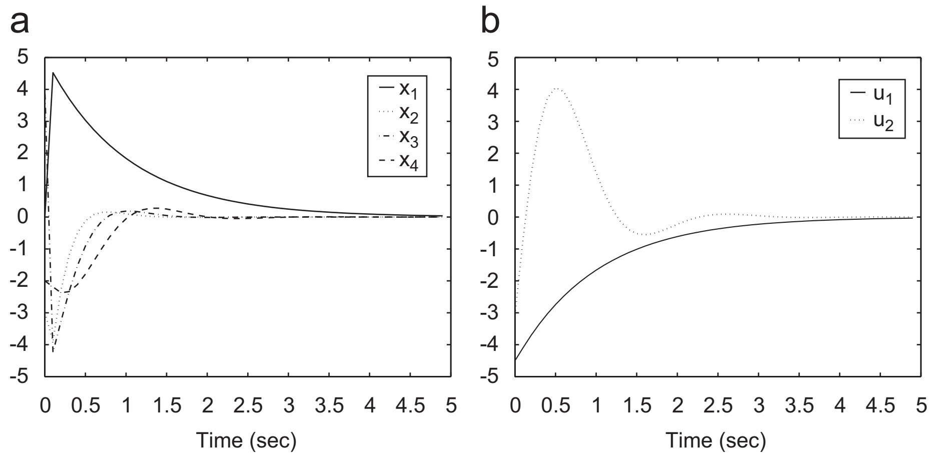

Fig. 1. Simulation results: (a) State trajectories; (b) Control inputs.

chained form system given by

$$\dddot{\xi}_1 = u_1, \quad \dddot{\xi}_2 = u_2, \quad \dddot{\xi}_i = \xi_{i-1} u_1, \quad i = 3, 4, \ldots, n,$$

where $\xi = [\xi_1, \xi_2, \ldots, \xi_n]^{\mathrm{T}} \in \Re^n$ and $\dot{\xi} = [\dot{\xi}_1, \dot{\xi}_2, \ldots, \dot{\xi}_n]^{\mathrm{T}} \in \Re^n$ denote the configuration variables and their derivative, respectively, and $[u_1, u_2]^{\mathrm{T}} \in \Re^2$ is the vector control inputs. Defining the coordinates transformation:

$$x_1 = \xi_1, \quad x_2 = \dot{\xi}_1, \quad x_3 = \xi_n,$$

$$x_4 = \dot{\xi}_n, \ldots, x_{2n-1} = \xi_2, x_{2n} = \dot{\xi}_2,$$

we have

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u_1, \tag{37}$$

$$\dot{x}_3 = x_4, \quad \dot{x}_4 = x_5 u_1, \ldots, \dot{x}_{2i-3} = x_{2i-2},$$

$$\dot{x}_{2i-2} = x_{2i-1} u_1, \quad \dot{x}_{2n} = u_2. \tag{38}$$

By following the same arguments for system (30) and (31), it can be shown that system (37) and (38) is cascaded feedback linearizable.

7. Simulation

In this section, a simulation result is given for smooth feedback stabilization of the (4, 2) chained system according to the control designed in Theorem 4. In the simulation, the initial states are given by $x(0) = [0, -3, 4, -2]^{\mathrm{T}}$. Fig. 1(a) shows the convergence of the states and the boundedness of the control inputs is depicted in Fig. 1(b).

8. Conclusion

In this paper, the problem of cascaded feedback linearization has been formulated and addressed for a class of nonlinear systems which are not standard feedback linearizable. A set of conditions have been found to check the cascaded feedback linearizability for a class of two-inputs affine nonlinear systems. As an illustrative application, the proposed cascaded feedback linearization technique renders a new solution to the stabilization problem (and smooth feedback control design) of nonholonomic chained systems.

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