Global stabilization of nonlinear systems with a class of unmatched uncertainties

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Abstract: Global and asymptotic stabilization of uncertain dynamical systems which violate the matching conditions is investigated. A class of unmatched uncertainties, called equivalently matched uncertainties, is defined. A robust controller is proposed to stabilize the uncertain system asymptotically in the large provided that the nominal system is uniformly asymptotically stable and that the unmatched uncertainties are equivalently matched. The required information about uncertain dynamics in the system is merely that the uncertainties are bounded in Euclidean norm by known functions of the system state.

Keywords: Robust control; uncertain systems; Lyapunov stability; unmatched uncertainties.

1. Introduction

Stabilization of a dynamical system with significant uncertainties has been widely studied over the last decade. A common approach is to treat the uncertainties in the system deterministically. Then, if the information about the possible size of the uncertainties is available, one can use the Lyapunov direct method to design a state feedback control. In many cases, the resulting control achieves certain stability performance for the uncertain system. In most of the research, the uncertain system is considered to be described by

\[ \dot{x} = f(x, t) + B(x, t)u, \quad x(t_0) = x_0, \]  

where \( f(\cdot): \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n \) and \( B(\cdot): \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^{n \times m} \) are known. Moreover, the so called controlled, nominal system of system (1) is defined by

\[ \dot{x} = f(x, t), \quad x(t_0) = x_0. \]  

It is well known that, if \( \xi'(x, t) = 0 \) and if the bounding function on the size of uncertainties \( \xi(x, t) \) is known and if system (3) is asymptotically stable in the large, there exist many feedback controls that yield certain types of stability for system (1). Among existing feedback control laws, the two primary ones are minmax control [7] and saturation-type control [5]. The discontinuous minmax control makes system (1) asymptotically stable in the large. The saturation-type control which is continuous was developed to guarantee the existence of a classical solution. Instead of asymptotic stability, the saturation-type control renders system (1) uniform and ultimate boundedness.

These results on global stability hold in general only if \( \xi'(x, t) = 0 \). In the case that there are unmatched uncertainties, a common approach is to apply the available controllers as if there were no unmatched uncertainties. This method will inevitably result in a threshold on the size of the unmatched uncertainties. The unmatched uncertainties is then required to be smaller than the threshold value so that a stability result holds locally with respect to the size of the uncertainties. Since unmatched uncertainties are common in control practice, it is important to design a feedback control guaranteeing global stability in the presence of significant unmatched uncertain-
ties. The first such as result was proposed in [10] which shows that stability can be achieved for any initial condition if the nominal system can be stabilized with arbitrarily large convergence rate.

The objective of this paper is to classify a class of unmatched uncertainties \( \xi'(x, t) \) such that the system (1) can be stabilized globally and asymptotically as long as a bounding function on \( \xi' \) is available. The previous result in [10] requires high convergence rate of the nominal system, the proposed new method is based on the non-uniqueness of Lyapunov functions for a stabilizable nominal system. Both these results guarantee global stability without the requirement of the threshold on the size of unmatched uncertainties. Moreover, unlike existing stability results of uniform ultimate boundedness, the proposed new continuous control renders asymptotic stability.

The paper consists of the following parts. Basic assumption and the definition of equivalently matched uncertainties are introduced in Section 2. In Section 3, a robust controller is proposed to stabilize a system with both matched and equivalently matched uncertainties. Section 4 contains an illustrative example.

2. Problem formulation

We first introduce for system (1) the following standard assumptions.

Assumption 1. The uncertain \( \xi(\cdot) \) and \( \xi(x, t) \) are bounded in Euclidean norm by known functions; namely, there are known non-negative, continuous functions \( \rho(\cdot) \) and \( \rho(x, t) \) such that

\[
\| \xi(x, t) \| \leq \rho(x, t), \quad \| \xi'(x, t) \| \leq \rho(x, t)
\]

for all \((x, t) \in \mathbb{R}^n \times \mathbb{R}\).

In addition, the functions \( \rho(\cdot) \) and \( \rho(x, t) \) are assumed without loss of any generality to be uniformly bounded with respect to time and locally uniformly bounded with respect to the state \( x \).

Assumption 2. The known functions \( f(\cdot) \) and \( B(\cdot) \), as well as the unknown functions \( \xi(\cdot) \) and \( \xi'(\cdot) \), are continuous, uniformly bounded with respect to time, and locally uniformly bounded with respect to the state \( x \).

Assumption 3. The uncontrolled, nominal system \( \dot{x} = f(x, t) \) is uniformly asymptotically stable in the large. More specifically, there is a family of Lyapunov functions, \( V(x, t) \), for systems (3). That is, there are a \( C^1 \) function \( V(\cdot) \in \mathbb{R}(x, t) : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^+ \), continuous, strictly increasing, scalar functions \( \gamma_i(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \), \( i = 1, 2 \), and a continuous, positive definite, scalar function \( \gamma_3(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \), which satisfy

\[
\gamma_i(0) = 0, \quad i = 1, 2, 3,
\]

\[
\lim_{s \to \infty} \gamma_i(s) = \infty, \quad i = 1, 2,
\]

such that for all \((x, t) \in \mathbb{R}^n \times \mathbb{R}\),

\[
\gamma_1(\| x \|) \leq V(x, t) \leq \gamma_2(\| x \|),
\]

\[
\frac{\partial V(x, t)}{\partial t} + \nabla_x^T T V(x, t) f(x, t) \leq -\gamma_3(\| x \|).
\]

The state feedback control developed in the next section will depend on this set of Lyapunov functions.

It is worth noting that Assumption 2 is made to guarantee the existence of a classical solution for system (1) under any control that is continuous and locally uniformly bounded. It is also worth noting that Assumption 3 is equivalent to the assumption that the nominal system (2) is uniformly asymptotically stabilizable.

If \( \xi'(x, t) = 0 \), it is well known that the controller proposed in [2,5] guarantees uniform ultimate boundedness. If \( \xi'(x, t) \neq 0 \), the existing technique employed in [2,3,4] is basically to apply the same control as if there were no unmatched uncertainties. Namely, if there exist unmatched uncertainties, a control law is first designed as if there were no unmatched uncertainties; and then the norm of the unmatched uncertainties is required to be smaller than a certain threshold value which is determined by the control law so that a similar stability result holds.

In order to achieve global stability in the presence of unmatched uncertainties, we introduce the following definition. As shown later, the uncertainties given in the following definition are
not required to satisfy any given threshold on their size.

**Definition 1.** The unmatched uncertain function \( \xi'(\cdot) \) is called *equivalently matched* if there is a Lyapunov function of the nominal system, \( V(\cdot) \in \mathcal{V}(x, t) \) and if there exists \( 0 \leq \phi_H < \infty \) such that for any given constant \( \epsilon > 0 \),

\[
\| \nabla_x^T V(x, t) \xi'(x, t) \| \leq \phi_H \epsilon
\]

\( \forall (x, t) \in \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : \| \nabla_x V(x, t) B(x, t) \| = \epsilon \} , \)

where \( H \subset \mathbb{R}^n \times \mathbb{R} \) is an arbitrarily given compact set.

It is easy to see from Definition 1 that a necessary condition for uncertainties to be equivalently matched is that

\[
\nabla_x^T V(x, t) \xi'(x, t) = 0
\]

if \( (x, t) \in \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : \| \nabla_x V(x, t) B(x, t) \| = 0 \} . \)

It is also apparent that a sufficient condition for uncertainties to be equivalently matched is that

\[
\| \nabla_x^T V(x, t) \| \| \xi'(x, t) \| \leq \phi_H \epsilon
\]

\( \forall (x, t) \in \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : \| \nabla_x V(x, t) B(x, t) \| = \epsilon \} , \)

This paper is to investigate robust control of uncertain systems with unmatched but equivalently matched uncertainties. Therefore, we introduce the following assumption.

**Assumption 4.** The uncertain function \( \xi'(\cdot) \) is equivalently matched. That is, the bounding function \( \rho'(\cdot) \) defined by

\[
\| \nabla_x^T V(x, t) \xi'(x, t) \| \leq \rho'(x, t)
\]

for all \( (x, t) \in \mathbb{R}^n \times \mathbb{R} \), where \( V(x, t) \) is defined in Assumption 3, has the property that the function

\[
\frac{\rho'(x, t)}{\| \nabla_x^T V(x, t) B(x, t) \|}
\]

is continuous, uniformly bounded with respect to \( t \), and locally uniformly bounded with respect to \( x \).

The robust control guaranteeing global stability in the presence of equivalently matched uncertainties is the subject of the next section.

3. Global stability

Motivated by minmax control [7], and saturation-type control [5], we propose the following feedback controller. For any \( \epsilon, \beta > 0 \), the controller is defined by

\[
u(x, t) = u_1(x, t) + u_2(x, t), \tag{4}
\]

\[
u_1(x, t) = -\rho(x, t) \frac{\mu_1(x, t)}{\| \mu_1(x, t) \| + \epsilon e^{-\beta t}}, \tag{5}
\]

\[
u_2(x, t) = -\frac{\rho'(x, t)}{\| \mu_2(x, t) \| + \epsilon e^{-\beta t}}, \tag{6}
\]

where

\[
\mu_1(x, t) \triangleq B^T(x, t) \nabla_x V(x, t) \rho(x, t),
\]

\[
\mu_2(x, t) \triangleq B^T(x, t) \nabla_x V(x, t)
\]

\[
\rho'(x, t)
\]

\[
\| \nabla_x^T V(x, t) B(x, t) \|
\]

**Remark 1.** It is apparent that \( u \) is continuous and that

\[
\| u \| \leq \| u_1 \| + \| u_2 \|, \quad \| u_1 \| \leq \rho(x, t),
\]

\[
\| u_2 \| \leq \frac{\rho'(x, t)}{\| \nabla_x^T V(x, t) B(x, t) \|},
\]

which shows that the control \( u(x, t) \) is locally uniformly bounded.

**Remark 2.** The difference between the standard robust control in [5] and the control (4) is that a time function \( \epsilon e^{-\beta t} \) instead of a constant \( \epsilon \) is used in (4). The idea of including time function \( e^{-\beta t} \) into robust control was first proposed in [6] to deal with the case, using the adaptive version of robust control, that bounding functions for
uncertainties contain unknown constants. The recent results in [11,12] extend the choice of $e^{-\beta t}$ to a class of time functions, and guarantee asymptotic or exponential stability.

The following theorem shows asymptotic stability of system (1) under control (4).

**Theorem.** Consider system (1) satisfying Assumption 1 to 4. Under the control (4), the system always has a unique classical solution and the solution is asymptotically stable in the large. Furthermore, the control given in (4) is uniformly bounded for any given initial conditions.

**Proof.** It follows from Theorem 3.1 on p. 18 of [8] that the existence and unicity of classical solutions of the system (1) under (4) is guaranteed.

Using the same Lyapunov function as given Assumption 4, we have

$$
\dot{V}(x, t) := \frac{\partial V(x, t)}{\partial t} + \nabla_x^T V(x, t) \dot{x} \\
= \frac{\partial V(x, t)}{\partial t} \\
+ \nabla_x^T V(x, t) \{f(x, t) + \xi'(x, t) \\
+ B(x, t) \{\xi(x, t) + u\} \} \\
\leq -\gamma_3(\|x\|) + \nabla_x^T V(x, t) \xi'(x, t) \\
+ \nabla_x^T V(x, t) B(x, t) \{\xi(x, t) + u\} \\
\leq -\gamma_3(\|x\|) + \|\nabla_x^T V(x, t) B(x, t)\| \\
\cdot \|\xi(x, t)\| \\
+ \nabla_x^T V(x, t) \xi'(x, t) \\
+ \|\nabla_x^T V(x, t) B(x, t)\| \\
\leq -\gamma_3(\|x\|) + \|\mu_1\| \\
+ \|\nabla_x^T V(x, t) B(x, t)\| u_1 \\
+ \lambda'(x, t) + \nabla_x^T V(x, t) B(x, t) u_2.
$$

(7)

It follows from (5) and from the inequality

$$
0 \leq \frac{ab}{a + b} \leq a, \quad \forall a, b \geq 0,
$$

that

$$
\|\mu_1\| + \nabla_x^T V(x, t) B(x, t) u_1 \\
= \|\mu_1\| - \frac{\|\mu_1(x, t)\|^2}{\|\mu_1(x, t)\| + \varepsilon e^{-\beta t}} \\
= \frac{\|\mu_1(x, t)\| \varepsilon e^{-\beta t}}{\|\mu_1(x, t)\| + \varepsilon e^{-\beta t}} \\
\leq \varepsilon e^{-\beta t}.
$$

(8)

Similarly, it follows from (6) that

$$
\rho'(x, t) + \nabla_x^T V(x, t) B(x, t) u_2 \\
= \|\mu_2(x, t)\| - \frac{\|\mu_2(x, t)\|^2}{\|\mu_2(x, t)\| + \varepsilon e^{-\beta t}} \\
= \frac{\|\mu_2(x, t)\| \varepsilon e^{-\beta t}}{\|\mu_2(x, t)\| + \varepsilon e^{-\beta t}} \\
\leq \varepsilon e^{-\beta t}.
$$

(9)

Applying the control (4) in (7), we know from (8) and (9) that for all $(x, t) \in \mathbb{R}^n \times \mathbb{R}$,

$$
\dot{V}(x, t) \leq -\gamma_3(\|x\|) + 2\varepsilon e^{-\beta t}.
$$

(10)

It follows from Assumption 3 and (10) that, for any $t \geq t_0$,

$$
0 \leq \gamma_1(\|x(t)\|) \\
\leq V(x, t) \\
= V(x_0, t_0) + \int_{t_0}^{t} \dot{V}(x, \tau) \, d\tau \\
\leq \gamma_2(\|x_0\|) - \int_{t_0}^{t} \gamma_3(\|x(\tau)\|) \, d\tau \\
+ 2\varepsilon \int_{t_0}^{t} e^{-\beta \tau} \, d\tau \\
= \gamma_2(\|x_0\|) - \int_{t_0}^{t} \gamma_3(\|x(\tau)\|) \, d\tau \\
+ \frac{2\varepsilon}{\beta} (1 - e^{-\beta t}).
$$

(11)

We can obtain two intermediate results from (11). First, taking the limit as $t$ approaches infinity on both sides of the inequality (11), we have

$$
0 \leq \gamma_2(\|x_0\|) - \lim_{t \to \infty} \int_{t_0}^{t} \gamma_3(\|x(\tau)\|) \, d\tau + \frac{2\varepsilon}{\beta}.
$$
Second, it follows from (11) that
\[ 0 \leq \gamma_1(\|x(t)\|) \leq \gamma_2(\|x_0\|) + \frac{2\varepsilon}{\beta}, \]
which implies that \( x(t) \) is uniformly bounded. Since \( x(\cdot) \) has been proved to be continuous, it follows from Assumption 2, Remark 1, and (1) that \( x(\cdot) \) is then uniformly continuous. Therefore, \( \gamma_1(\|x\|) \) is uniformly continuous. Applying the Barbalat lemma [13] to (12) yields that
\[ \lim_{t \to \infty} \gamma(\|x\|) = 0, \]
and consequently \( \lim_{t \to \infty} \|x\| = 0 \) since \( \gamma_1(\cdot) \) is a positive definite function.

It follows from \( \|x\| \) being uniformly bounded that the control \( u(x, t) \) is uniformly bounded. \( \Box \)

**Remark 3.** The asymptotic stability result in the above theorem does not mean that the origin is an equilibrium point of system (1). In fact, an uncertain system has in general no equilibrium point. By convection, stability of uncertain system is then studied with respect to the equilibrium point of the nominal system (3). Asymptotic stability can be stated because the origin is the unique equilibrium point of the nominal system based on Assumption 3. As an example, the asymptotic stability result of uncertain systems stated in [7] is of the same nature.

**Remark 4.** The above discussion can be easily extended to the case where there are input-related, unmatched uncertainties. Say, if the equation of uncertain system is given by
\[ \dot{x} = f(x, t) + \xi'(x, t) + \Delta B'(x, t)u + B(x, t)[\xi(x, t) + \Delta B(x, t)u + u], \]
where \( \Delta B(x, t) \) is 'matched' [5], \( \Delta B'(x, t) \) is 'unmatched', and both of them are input-related uncertainties. Then, a robust control guaranteeing global stability can be obtained in a similar fashion if there exists \( V(\cdot) \in \mathcal{V}(x, t) \) such that
\[ \inf_{(x, t) \in \mathbb{R}^n \times \mathbb{R}} \left\{ \lambda_{\min}\left( I + \frac{1}{2}(\Delta B + \Delta B^T) \right) \right\} \right) \geq \eta, \]
for some \( \eta > 0 \).

If the inequality (14) is satisfied, the uncertainties \( \Delta B' \) is called to the **equivalently matched**. Furthermore, if Assumption 4 and condition (14) can be satisfied for the same Lyapunov function \( V(\cdot) \), the robust control guaranteeing global and asymptotical stability is given by
\[ u(x, t) = u_1(x, t) + u_2(x, t), \]
\[ u_1(x, t) = -\rho(x, t) \frac{\mu_1(x, t)}{\eta(\|\mu_1(x, t)\| + \varepsilon e^{-\beta t})}, \]
\[ u_2(x, t) = -\frac{\rho'(x, t)}{\eta(\|\mu_2(x, t)\| + \varepsilon e^{-\beta t})}, \]
where \( \mu_1 \) and \( \mu_2 \) are the same as those defined in (4).

### 4. Illustrative example

To illustrate the concept of equivalently matched uncertainties, let us consider a simple system whose dynamics is described by
\[ \dot{x}_1 = x_2 + x_3 \xi_1(x_1, x_2, x_3, t) \]
\[ + x_4 \Delta B_1(x_1, x_2, x_3, t)u, \quad (16a) \]
\[ \dot{x}_2 = -2x_2 - x_1 - x_3 \xi_2(x_1, x_2, x_3, t), \quad (16b) \]
\[ \dot{x}_3 = \xi_1(x_1, x_2, x_3, t) + (1 + x_1^2 + x_2^2)u, \quad (16c) \]
where \( u \) is the control, \( \xi_1(\cdot) \) is the matched uncertainties, and \( \xi_1(\cdot), \xi_2(\cdot), \) and \( \Delta B_1(\cdot) \) are the unmatched uncertainties. The existence of unmatched uncertainties is obvious by comparing equations (16) and (13). These unmatched uncer-
Uncertainties are bounded by nonnegative bounding functions as
\[
|\xi(x_1, x_2, x_3, t)| \leq \rho(x, t),
|\Delta B'_1(x_1, x_2, x_3, t)| \leq b,
|\xi'_1(x_1, x_2, x_3, t)| \leq \rho'_1(x, t),
|\xi'_2(x_1, x_2, x_3, t)| \leq \rho'_2(x, t),
\]
where \(b > 0\) is a given constant,
\[
x = [x_1, x_2, x_3]^T
\]
is the state vector of the system.

It is obvious that the nominal system given below is asymptotically stabilizable.
\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -2x_2 - x_1, \\
\dot{x}_3 &= (1 + x_1^2 + x_2^2)u.
\end{align*}
\]

Let the control be of the form
\[
u(x, t) = \frac{1}{1 + x_1^2 + x_2^2} \left[ u_0(x, t) + u_1(x, t) + u_2(x, t) \right],
\]
where \(u_0\) is the nominal control to stabilize the nominal system, \(u_1\) is one part of robust control to compensate for \(\xi(\cdot)\), and \(u_2\) is the other part of robust control to compensate for \(\xi'_1(\cdot)\) and \(\xi'_2(\cdot)\).

It follows from the Lyapunov direct method that, under the nominal control
\[
u_0 = -k_1x_1 - k_2x_2 - k_3x_3
\]
with \(k_1, k_2, k_3 \geq 0\), the nominal system can be stabilized. There are in fact infinite number of Lyapunov functions of the form \(V = \frac{1}{2}x^TPx\) for the nominal system. The matrix \(P\) can be determined by the following two steps. First, choose a positive definite matrix \(Q\). Then, by properly picking the control gains \(k_i\), solve for a positive definite matrix solution \(P\) from the Lyapunov equation
\[
\frac{1}{2}(PA + A^TP) = -Q,
\]
where
\[
A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ -k_1 & -k_2 & -k_3 \end{bmatrix}.
\]

Of these infinite choices of Lyapunov functions, there is one important class which can be obtained as follows. First, we choose the gains in \(u_0\) such that \(k_1 = k_2 = 0\) and \(k_3 > 0\). Then, we pick the Lyapunov function matrix \(P\) to be
\[
P = \text{diag}\{P_1, 1\},
\]
where \(P_1\) is a solution of the equation
\[
\frac{1}{2}(P_1A_1 + A_1^TP_1) = Q_1
\]
for some positive definite \(Q_1\), and
\[
A_1 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}.
\]

It is evident that \(P_1\) is positive definite for any positive choice of \(Q_1\). Finally, select a \(Q_1\) such that \(\lambda_{\text{max}}(P_1) \leq 1/2b\) which can be achieved easily by multiplying a small constant on both sides of the above Lyapunov equation.

Any choice of Lyapunov function in the above class makes \(\xi'_1(\cdot)\), \(\xi'_2(\cdot)\), and \(\Delta B'(\cdot)\) be equivalently matched. To see this, let us note that
\[
\begin{align*}
\nabla_x^T V(x, t)B(x, t) &= (1 + x_1^2 + x_2^2)x_3, \\
\|\nabla_x^T V(x, t)\xi'(x, t)\| \\
\|\nabla_x^T V(x, t)B(x, t)\|
\end{align*}
\]
\[
\leq \lambda_{\text{max}}(P_1)\left[ \rho'_2(x, t) + |x_3| \rho'_1(x, t) \right] \\
\leq \frac{1}{2b}\left[ \rho'_2(x, t) + |x_3| \rho'_1(x, t) \right] \\
= \rho'_3(x, t),
\]
\[
\begin{align*}
\|\nabla_x^T V(x, t)\Delta B'(x, t)\| \\
\|\nabla_x^T V(x, t)B(x, t)\|
\end{align*}
\]
\[
\leq \lambda_{\text{max}}(P_1)b \frac{|z_1|}{1 + x_1^2 + x_2^2} \\
\leq \frac{1}{2},
\]
where
\[z = [x_1 \ \ x_2]^T, \quad \xi'_0 = [x_3^2 \xi'_1 \ \ x_3^2 \xi'_3]^T,\]
and the symbol := denotes the operation of defining a new function. It is then easy to see that Assumption 4 and condition (14) are satisfied with \(\eta = \frac{1}{2}\). As a consequence, we have the following robust control which guarantees global and asymptotic stability for all bounded uncertainties of the form shown in the system equation (16). The overall controller, nominal control plus robust control, is
\[
u(x, t) = \frac{1}{1 + x_1^2 + x_2^2} \left[ -k_3 x_3 + u_1(x, t) + u_2(x, t) \right],
\]
\[u_1(x, t) = -\rho(x, t) \frac{2x_3 \rho(x, t)}{\|x_3 \rho(x, t)\| + \epsilon e^{-\beta t}},\]
\[u_2(x, t) = -\rho_3(x, t) \frac{2x_3 \rho_3'(x, t)}{\|x_3 \rho_3'(x, t)\| + \epsilon e^{-\beta t}},\]
where \(k_3 > 0\).

**Conclusion**

Global and asymptotic stabilization of uncertain systems in the absence of matching conditions is considered. A class of unmatched uncertainties, called equivalently matched uncertainties, is defined. It has been shown that, if the uncontrolled, nominal system is uniformly asymptotically stabilizable, the proposed robust controller, which is a continuous and uniformly bounded feedback control, guarantees asymptotic stability of the uncertain system for all matched and equivalently matched uncertainties as long as bounded functions on the size of the uncertainties are available. These results are achieved by using the fact that existence of Lyapunov function is not unique and by making a judicious choice of Lyapunov function such that the unmatched uncertainties become equivalently matched.

**References**


