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Model reference robust control for MIMO systems

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Model reference robust control (MRRC) of single-input single-output (SISO) systems was introduced as a new means of designing I/O robust control (Qu *et al.* 1994). This I/O design is an extension of the recursive backstepping design in the sense that a nonlinear dynamic control (not static) is generated recursively. Backstepping entails the design of fictitious controls starting with the output state-space equation and backstepping until one arrives at the input state-space equation where the actual control can be designed. At each step the system is transformed and a fictitious control is designed to stabilize the transformed state (Naik and Kumar 1992). It is shown in this paper that MRRC of multiple input multiple output (MIMO) systems is an extension of model reference control (MRC) of MIMO systems and MRRC of SISO systems. Unwanted coupling exists in many physical MIMO systems. It is shown that MRRC decouples MIMO systems using only input and output measurements rather than state feedback. This is a very desirable property, because in many instances state information is not available. A diagonal transfer function matrix is strictly positive real (SPR) if and only if each element on the diagonal is SPR. The fact that complicates the development of robust control laws is that the recursive backstepping procedure used in non-SPR SISO systems cannot be directly applied to diagonal MIMO non-SPR systems without the introduction of the augmented matrix or a pre-compensator. MRC of systems where one has perfect plant knowledge is reviewed. Assumptions are listed for the application of model reference robust control for MIMO systems. Model selection is presented as the right Hermite normal form of the plant transfer function matrix. MRRC is derived for MIMO systems that have a right Hermite normal form which is SPR and diagonal, and then for systems whose right Hermite normal form is diagonal but not SPR. Robust control laws are generated for achieving stability using Lyapunov's second method. Future research will focus on MIMO systems which are not diagonal.

1. Introduction

Model reference control, also called model following, is a well-documented method (Chen 1984, Narendra and Annaswamy 1989, Wolovich 1974) which entails the assignment of controller poles and zeros such that the overall response of a plant plus controller asymptotically approaches that of a given reference model. One may apply normal compensator design techniques to find the solution of MRC or utilize model reference adaptive control (MRAC) techniques for the case that the plant has unknown constant parameters to automatically adjust compensator parameters to achieve model following (Landau 1979, Narendra and Annaswamy 1989, Sastry and Bodson 1989). However, MRAC techniques may have instabilities when the plant has uncertain bounded disturbances or unmodelled dynamics. In addition, MRAC requires persistent excitation (PE) for the convergence of the adaptive controller

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parameters to their desired values. Fuzzy logic techniques offer a reduction in development time due to the ability of the designer to express knowledge of a process or system in a manner suitable for fuzzy control. However, robustness properties of fuzzy systems are not well understood (Driankov *et al.* 1993). Nonlinear robust control provides both a reduction in development time and guaranteed stability in the presence of plant uncertainties and bounded disturbances. Therefore, it is important to find robust control laws that guarantee model following and stability for MIMO systems in the presence of bounded plant uncertainties and bounded disturbances.

Robust control may be classified into nonlinear robust control and linear robust control. Linear robust controllers include H^∞ controllers, H^2 controllers, etc. This paper discusses nonlinear robust control laws and is an extension of MRRC of SISO systems. The choice of nonlinear approach is based on the fact that systems under consideration have nonlinear bounded uncertainties. MRRC of SISO systems was first proposed by Qu *et al.* (1994) and the basic technique involves the following:

(*a*) Determine bounds of plant uncertainties and disturbances.

(*b*) Find a Lyapunov function candidate V.

(*c*) Take the derivative of V along the trajectories of the error system between the plant and model outputs.

(*d*) Replace terms associated with uncertainties in $\dot{V}$ by their bounds.

(*e*) Determine the control law $u_R(t)$, such that $\dot{V}$ is negative definite; if the diagonal reference model were not SPR, the control law would require derivatives of the plant output. To avoid the measurement of plant derivatives, a backstepping procedure is employed to recursively determine the robust control law.

The goal of this paper is to combine MRC of MIMO systems with robust control techniques to achieve asymptotic output tracking using only input and output information. The extension of MRRC to MIMO systems is complicated by the following.

MIMO systems have a high frequency gain matrix. As matrices do not commute this poses a problem in the recursive development of robust control. It will be shown that the robust control laws are coupled, even though in the design the error system has been decoupled.

The backstepping procedure used on SISO non-SPR systems cannot be applied directly to MIMO systems because each element of the diagonal reference model may have a different relative degree; this will necessitate the introduction of an augmented matrix.

Section 2 discusses the problem formulation by first considering the basic problem of MRC of MIMO systems. A block diagram of the MIMO system under consideration is presented with dimensions of key elements shown on the block diagram; basic assumptions are listed. MRC control design of perfectly known plants is reviewed. Section 3 considers the robust design for MIMO systems. The basic controller developed in §2 is modified to compensate for control parameter uncertainties. Bounding functions are developed for uncertainties. An augmented system used to handle the case where the plant's reference model is not SPR is presented. Robust control laws are generated and verified by Lyapunov proofs. In §3.3 simulations are presented to illustrate the concepts and effectiveness of MRRC on a 2×2 system whose reference model is SPR, a 2×2 system whose reference

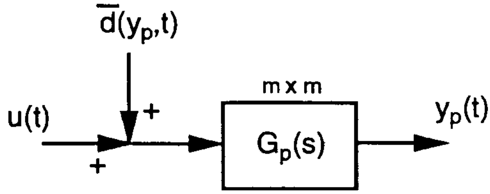

Figure 1. Basic MIMO plant with disturbance.

model is not SPR and a 2×2 system whose reference model is third order and is not SPR.

2. Problem formulation

The class of MIMO systems under consideration is given in Fig. 1 with dimension explicitly shown. The plant is assumed to have a linear and time-invariant part, and $G_p(s)$ is of full rank and strictly proper. The linear part is square and it can be represented by a right matrix fractional description as $G_p(s) = B_p(s)A_p^{-1}(s)$, where $A_p(s)$ and $B_p(s)$ are right coprime polynomial $m \times m$ matrices, and $A_p(s)$ is column proper. For the ease of subsequent discussion, let K_p be the $m \times m$ high frequency gain matrix of $G_p(s)$ defined as $K_p = \lim_{s \to \infty} H_p(s)^{-1} G_p(s)$ where $H_p(s)$ is the right Hermite normal form of $G_p(s)$. For a definition of the right Hermite normal form see Narendra and Annaswamy (1989). Nonlinearities and uncertainties (except the unknown parameters in the linear portion) in the system are lumped into $\bar{d}(y_p, t)$.

In this paper it is sufficient to consider square plants because inputs that correspond to linearly independent columns of $G_p(s)$ can be selected while setting the remaining inputs to zero. That is, to drive m outputs to arbitrary trajectories it is sufficient to consider only m control inputs.

2.1. *Assumptions and remarks*

The following assumptions are introduced for the class of MIMO systems considered in this paper.

Assumption 1: *The plant high frequency gain matrix K_p is invertible. If K_p is unknown, there is a known matrix Γ such that $K_p\Gamma + \Gamma K_p^T = Q > 0$ where Γ is a symmetric positive definite matrix for a symmetric positive definite matrix Q with $\lambda_{min}(Q) \geq 1$.*

Assumption 2: *The right Hermite normal form $H_p(s)$ of the plant transfer function matrix $G_p(s)$ is known, diagonal and stable.*

Assumption 3: *The plant observability index υ (Kailath 1980) of $G_p(s)$ is known.*

Assumption 4: *The zeros of $G_p(s)$ lie in $\mathbb{C}^-$.*

Assumption 5: *Plant parameters are elements of a compact set.*

Assumption 6: *The disturbance $\bar{d}(y_p, t)$ is bounded by a known, well-defined function $\rho(y_p, t)$ such that*

$$\|\overline{d}(y_p, t)\| \leq \rho(y_p, t)$$

where $\|\cdot\|$ denotes the euclidean norm.

With regard to the assumptions, the following observations can be made.

Remark 1: Assumption 1 is analogous to knowledge of the sign of the plant high frequency gain in the scalar case in the sense that a properly designed control can control the system in a definite direction in the output space. This requirement can be eliminated if K_p is known, because the controller can be modified by $\Gamma = 0{\cdot}5K_p^{-1}$. Then $Q = 1$. □

Remark 2: The necessary background on the Hermite normal form is discussed in the Appendix. Assumption 2 implies that the relative degrees of the elements of $G_p(s)$ are known. If the right Hermite normal form of $G_p(s)$ is not diagonal, more information concerning the elements of $G_p(s)$ must be known (Narendra and Annaswamy 1989) so that a compensator can be designed, say $G_c(s)$, which makes the right Hermite normal form of $G_p(s)G_c(s)$ diagonal. □

Remark 3: Assumption 3 is important in the sense that it determines a lower bound on the number of fictitious controls needed in a recursive design. □

Remark 4: Assumption 4 ensures there is no unstable cancellation in the design of perfect tracking control. This is equivalent to the minimum phase condition of SISO systems. The zeros of the square $G_p(s)$ are the roots of the determinant of $B_p(s)$ and do not include zeros at ∞. □

Remark 5: Assumptions 4–6 ensure that a stable controller can be designed. □

2.2. *Selection of reference model*

If the relative degree n_{ij}^* of the $(i, j)^{th}$ element of the plant transfer function matrix $G_p(s)$ is known, one can decide whether its right Hermite normal form is diagonal. If it is not, a precompensator $G_c(s)$ can be designed such that $G_p(s)G_c(s)$ has a diagonal Hermite normal form reference. As the controller contains no differentiators, any chosen reference model must have a relative degree greater than or equal to that of the plant transfer function matrix's right Hermite normal form (Singh 1985).

The choice of reference model $G_m(s)$ is the one made from the following set (Narendra and Annaswamy 1989).

$$\mathcal{G} = \left\{G_m(s) \mid G_m(s) = H_p(s)\,V(s)\right\}$$

where $V(s) \in \Re_p^{m\times m}(s)$, $H_p(s)$ is the Hermite normal form of $G_m(s)$, and $H_p(s)$ and $V(s)$ are asymptotically stable. Specifically, one may choose $G_m(s)$ such that $G_m(s) = H_p(s)Q_m(s)$ where $Q_m(s)$ is a unimodular and asymptotically stable matrix. We shall assume that $Q_m(s) = I$, the $m \times m$ identity matrix. The Hermite normal form is unique up to an arbitrary transfer function of relative degree one.

2.3. *Control design under perfect plant knowledge*

Figure provides an overview of the control system under consideration with perfect plant knowledge and without the robust control loop. The dimensions of the matrices are shown in the figure. The structure is equivalent to a Luenberger observer followed by a state feedback gain matrix and a constant feedforward gain

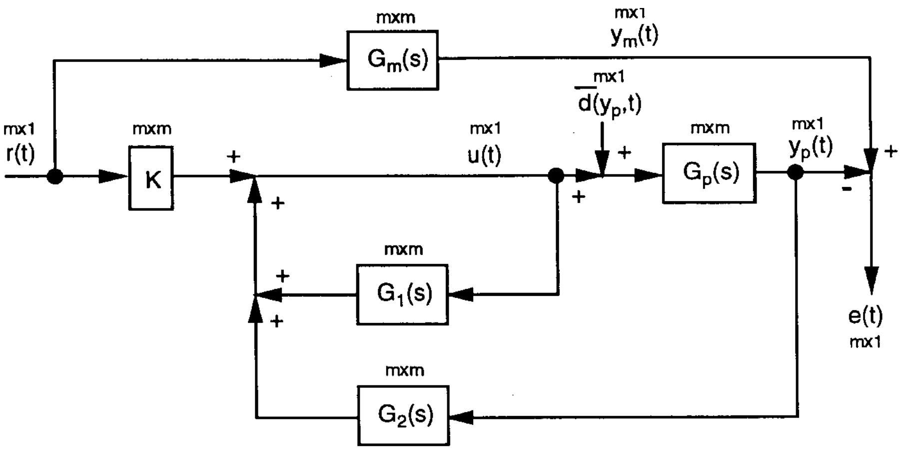

Figure 2. MIMO system description.

matrix for an MIMO plant in the state space representation (Wolovich 1974). The nonlinear disturbance is denoted by the vector $\bar{d}(y_p, t)$. The system under consideration satisfies the generalized matching conditions in the sense that the disturbance enters into the same summing node as does the control.

Referring to Fig. 2, the control law is given by

$$u(t) = G_1(s)u(t) + G_2(s)y_{pr}(t) + Kr(t) \tag{1}$$

where $G_1(s)$ and $G_2(s)$ are transfer function matrices and the other matrices are defined in Fig. 1. Equation (1) illustrates a common abuse of notation; it may be rewritten as

$$\begin{aligned} u(t) &= (G_1(s) + G_2(s)G_p(s))u(t) + Kr(t) \\ &= (I - G_1(s) - G_2(s)G_p(s))^{-1}Kr(t) \end{aligned}$$

Therefore, neglecting $\bar{d}(y_p, t)$ for the moment, we have that $y_{pr}(t)$ is given by:

$$y_{pr}(t) = G_p(s)(I - G_1(s) - G_2(s)G_p(s))^{-1}Kr(t) \tag{2}$$

By rewriting equation (1) as

$$u(t) = (I - G_1(s))^{-1}G_2(s)y_{pr}(t) + (I - G_1(s))^{-1}Kr(t) \tag{3}$$

a second derivation for $y_{pr}(t)$ reveals

$$y_{pr}(t) = \left\{ I - G_p(s)\left[I - G_1(s)\right]^{-1}G_2(s)\right\}^{-1} G_p(s)(I - G_1(s))^{-1}Kr(t) \tag{4}$$

If $r(t)$ equals zero, a derivation for the plant output due to the disturbance input $\bar{d}(y_p, t)$, yields

$$y_{pd}(t) = \left\{ I - G_p(s)\left[I - G_1(s)\right]^{-1}G_2(s)\right\}^{-1} G_p(s)\bar{d}(y_p, t) \tag{5}$$

Equation (5) may be rewritten as

$$y_{pd}(t) = \left\{ I - G_p(s)\left[I - G_1(s)\right]^{-1}G_2(s)\right\}^{-1} G_p(s)(I - G_1(s))^{-1}K\left[K^{-1}(I - G_1(s))\right]\bar{d}(y_p, t) \tag{6}$$

Equations (5) and (7) will be used in the robust control design procedure below.

The total plant output $y_p(t)$ due to both the reference and disturbance inputs is given by

$$y_p(t) = \left\{ I - G_p(s)\left[I - G_1(s) \right]^{-1} G_2(s) \right\}^{-1} G_p(s)(I - G_1(s))^{-1} K\phi(t) \tag{7}$$

where

$$\phi(t) = r(t) + \left[K^{-1}(I - G_1(s)) \right] d(y_p, t)$$

From Fig. 1 we see that the controller consists of a gain matrix $K \in \Re^{m \times m}$ in the feedforward path and two transfer functions $G_1(s)$ and $G_2(s)$ in the feedback path which may be written as $A_q^{-1}(s)B_1(s)$ and $A_q^{-1}(s)B_2(s)$ respectively. $G_i(s) = A_q^{-1}(s)B_i(s)$, $i = 1, 2$ is in left matrix fractional description (MFD) form. The matrices $B_1(s)$ and $B_2(s)$ are given by

$$B_1(s) = \sum_{i=1}^{\upsilon - 1} C_i s^{i-1}$$

$$B_2(s) = \sum_{i=0}^{\upsilon - 1} D_i s^{i}$$

See Narendra and Annaswamy (1989) for details. Using the above relationships, one may rewrite (8) as

$$y_p(t) = G_o(s)\phi(t)$$

where

$$G_o(s) = B_p(s)\left\{ \left[A_q(s) - B_1(s) \right] A_p(s) - B_2(s)B_p(s) \right\}^{-1} A_q(s)K \tag{8}$$

In (9), $A_p(s)$ and $B_p(s)$ are right coprime and therefore by the matrix Bezout identity, matrices $A_q(s) - B_1(s)$ and $B_2(s)$ of degree $\upsilon - 1$ exist such that the transfer function matrix within the brackets can be made equal to any polynomial of column degree $d_j + \upsilon - 1$ where d_j is the column degree of $A_p(s)$. If this matrix is chosen as $A_q(s)KH_p^{-1}(s)B_p(s)$, it follows that $G_o(s) = H_p(s) = G_m(s)$. The matrix Bezout identity only guarantees the existence of a solution.

To recap the Bezout identity, one wishes that

$$\left\{ A_q(s) - B_1(s) \right\} A_p(s) - B_2(s)B_p(s) = A_q(s)KH_p^{-1}(s)B_p(s) \tag{9}$$

to achieve model following. $A_q(s)$ can be chosen as $A_q(s) = a_q(s)I$, where $a_q(s)$ is a stable monic polynomial of degree $\upsilon - 1$ and I is the $m \times m$ identity matrix. After substituting the above equations into (10) and rearranging, one obtains

$$\left(\sum_{i=0}^{\upsilon - 1} D_i^* s^i \right) B_p(s) + \left(\sum_{i=1}^{\upsilon - 1} C_i^* s^{i-1} \right) A_p(s) = a_q(s)\left[A_p(s) - K^* G_m^{-1}(s) B_p(s) \right] \tag{10}$$

where $K^* = K_p^{-1}$.

Let $G_p(s) = E^{-1}(s)F(s)$ be the left MFD, not necessarily coprime, of $G_p(s)$. If $E(s)$ and $F(s)$ are left coprime with $E(s)$ row reduced and $A_p(s)$ column reduced and $\partial_{cj}\left[B_p(s) \right] < \partial_{cj}\left[A_p(s) \right]$ where $\partial_{cj}\left[B_p(s) \right]$ is the degree of the jth column of $B_p(s)$ then the solution is unique. When the above degree relation holds with equality, column reducedness of either $A_p(s)$ or $B_p(s)$ will yield a unique solution to the Bezout identity. See Singh (1985) for details.

Let

$$\omega_i(t) = \frac{s^{i-1}}{a_q(s)} u(t), \; \omega_j(t) = \frac{s^{j-\upsilon}}{a_q(s)} y_p(t), \; i = 1, \ldots, \upsilon - 1, \; j = \upsilon \ldots, 2\upsilon - 1$$

and let the $2m\upsilon$ vector ω and the $m \times 2m\upsilon$ matrix Θ be defined as

$$\omega(t) = \left[r, \omega_1^T, \ldots, \omega_{\upsilon-1}^T, \omega_\upsilon^T, \ldots, \omega_{2\upsilon-1}^T\right]^T$$
$$\Theta = \left[K, C_1, \ldots, C_{\upsilon-1}, D_0, \ldots, D_{\upsilon-1}\right]$$

where K, C_i and D_j are $m \times m$ matrices for $i = 1, \ldots, \upsilon - 1$ and $j = 0 = 1, \ldots, \upsilon - 1$. The control input to the plant can be written compactly as

$$u(t) = \Theta \omega(t) \tag{11}$$

Under perfect knowledge $\Theta = \Theta^*$. A 2×2 example in the Appendix will serve to illustrate these points.

3. Robust control design

If one does not have perfect knowledge of the plant transfer function matrix, (11) may be rewritten with a robust control term as

$$u(t) = \Theta^* \omega(t) - \tilde{\Theta} \omega(t) + u_R \tag{12}$$

where Θ is an arbitrary estimate of Θ^*, $\tilde{\Theta} = \Theta^* - \Theta$ represents the effect of lacking exact knowledge of plant parameters, and u_R is the robust control to be designed. By expanding the first term, (12) may be rewritten as

$$u(t) = K^*\left[r(t) + K^{*-1}(u_R - \tilde{\Theta}\omega(t))\right] + \sum_{i=1}^{\upsilon-1} C_i^* \omega_i(t) + \sum_{j=0}^{\upsilon-1} D_j^* \omega_{j+\upsilon}(t) \tag{13}$$

The term $r(t) + K^{*-1}(u_R - \tilde{\Theta}\omega(t))$ can be considered a reference input to a plant where one has perfect knowledge. The plant output under both reference and disturbance inputs may be written as

$$y_p(t) = G_m(s)\left\{r(t) + K^{*-1}\left[u_R - \tilde{\Theta}\omega(t) + (I - G_1(s))\bar{d}(y_p, t)\right]\right\} \tag{14}$$

Figure 3 shows the proposed modification to the plant using a robust control, in which $g(\cdot)$ is a bounding function to be developed for the unknown terms in (14).

Superposition may be used because although the disturbance input may be an unknown nonlinear function, the plant transfer function matrix is linear. The disturbance input term can be justified by noting the relationship of (7). The robust control u_R must compensate for plant parameter uncertainties represented by $\tilde{\Theta}$ and bounded disturbances represented by $(I - G_1(s))\bar{d}(y_p, t)$. Let $e(t) = y_m(t) - y_p(t)$. From (14) one obtains the error system as

$$\begin{aligned} e(t) &= G_m(s)K^{*-1}\left[\tilde{\Theta}\omega(t) - u_R - (I - G_1(s))\bar{d}(y_p, t)\right] \\ &= G_m(s)K_p\left[\tilde{\Theta}\omega(t) - u_R - (I - G_1(s))\bar{d}(y_p, t)\right] \end{aligned} \tag{15}$$

Qu *et al.* (1994) defined a bounding function, BND. In this paper BND is denoted as $|||\cdot|||$. The definition of BND is repeated here for clarity.

Definition: Let $\psi(y_p, t)$ be a known continuous function. Then $|||\psi(y_p, t)|||$ is a continuous nonnegative function that bounds the magnitude (or euclidean norm)

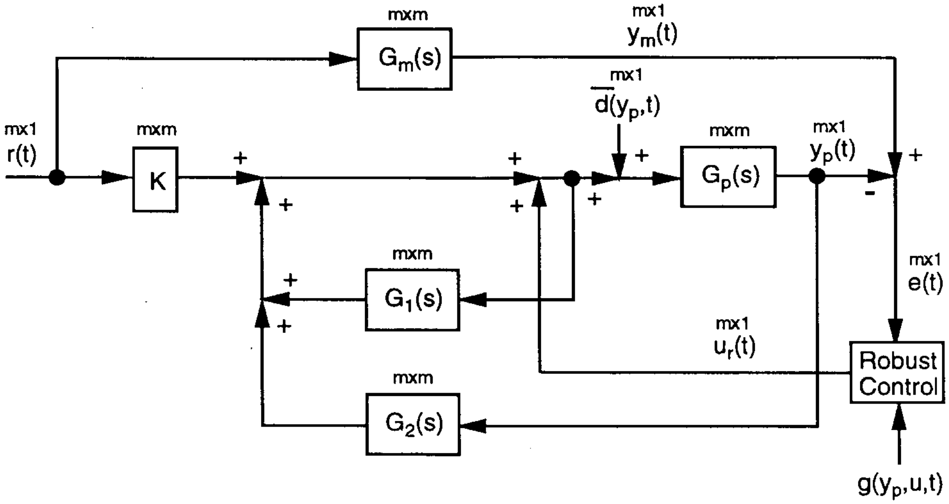

Figure 3. MIMO revised system description.

of $\psi(y_p, t)$. That is

$$\|\psi(y_p,t)\| \leq |||\psi(y_p,t)|||, \ \forall (y_p,t) \qquad \square$$

The robust control law must dominate the term

$$K_p\left[\tilde{\Theta}\,\omega(t) - (I - G_1(s))\bar{d}(y_p,t)\right]$$

which implies that a bounding function must be found for this term. Let

$$\zeta = \left[\tilde{\Theta}\,\omega(t) - (I - G_1(s))\bar{d}(y_p,t)\right]$$

Remark 6: A bounding function for $K_p\zeta$ can be obtained as follows:

$$\begin{aligned}
K_p\zeta &= K_p\left[\tilde{\Theta}\,\omega(t) - (I - G_1(s))\bar{d}(y_p,t)\right] \\
&\leq |||K_p|||\ |||\tilde{\Theta}\,\omega(t) - (I - G_1(s))\bar{d}(y_p,t)||| \\
&= |||K_p|||\left|\left|\left|\tilde{K}r(t) + \sum_{i=1}^{\upsilon-1}\tilde{C}_i\omega_i(t) + \sum_{j=0}^{\upsilon-1}\tilde{D}_j\omega_{j+\upsilon}(t) - \bar{d}(y_p,t) + G_1(s))\bar{d}(y_p,t)\right|\right|\right| \\
&\leq |||K_p|||\Bigg[|||\tilde{K}|||\ \|r(t)\| + \sum_{i=1}^{\upsilon-1}|||\tilde{C}_i|||\ \|\omega_i(t)\| + \sum_{j=0}^{\upsilon-1}|||\tilde{D}_j|||\ \|\omega_{j+\upsilon}(t)\| + \rho(y_p,t) \\
&\quad + \sum_{i=1}^{\upsilon-1}\int_{t_0}^{t}|||\tilde{C}_i|||(\|C_{2i}\exp\left[A_{2i}(t-\tau)B_{2i}\right]\|)\rho(y_p,t)\Bigg] \triangleq \frac{g(y_p,u,t)}{2}
\end{aligned}$$

where $\{A_{2i}, B_{2i}, C_{2i}\}$ is a minimal realization of $s^{i-1}/a_q(s)$ for $i = 1,\ldots,\upsilon - 1$. Until now the initial conditions have not been considered. Due to the similarities between the SISO case and this one, one may assume that the initial conditions are zero. This simplifies the model following problems in the following sections. $\square$

3.1. *MRRC for SPR systems*

The robust control proposed in the case where $G_m(s)$ is SPR is

$$u_R = \frac{\Gamma\mu(e,y,u,t)\|\mu(e,y,u,t)\|^{\tau}}{2\left(\|\mu(e,y,u,t)\|^{\tau+1} + \epsilon^{\tau+1}\exp(-\beta(\tau+1)t)\right)}g(y_p,u,t) \qquad (16)$$

where $\mu(e,y,u,t) = g(y_p,u,t)e(t)$,

$$g(y_p,u,t) = 2\|K_p\|\left(\|\tilde{\Theta}\|\|\omega(t)\| + \|(I - G_1(s))\bar{d}(y_p,t)\|\right)$$

and Γ is that given in assumption 1. The terms τ, ϵ and β are design parameters. The design parameter β affects the convergence of the tracking error, ϵ affects the initial control magnitude and, if $\beta = 0$, determines the tracking accuracy and τ ensures that the first-order partial derivatives of u_R are well defined. This form of nonlinear robust control is called a saturation control (Qu *et al.* 1994) because the magnitude of the control u_R is bounded. This can be seen by an examination of (19). Generating the control law follows Lyapunov's second method, and the derivation details are as follows. The reference model may be chosen as

$$G_m(s) = \frac{1}{s+a}I$$

where $a > 0$ and I is the $m \times m$ identity matrix. One may write the dynamics of the error system as

$$\dot{e}(t) = -ae(t) + K_p\left[\tilde{\Theta}\omega(t) - u_R - (I - G_1(s))\bar{d}(y_p,t)\right] \qquad (17)$$

$$= -ae(t) + K_p\zeta - K_p u_R \qquad (18)$$

where $\zeta = \left[\tilde{\Theta}\omega(t) - (I - G_1(s))\bar{d}(y_p,t)\right]$ as before. Let $V(t) = \|e(t)\|^2$ be a Lyapunov function candidate. This function is positive definite and radially unbounded. Taking the time derivative of $V(t)$ along the trajectories of the system yields

$$\dot{V} = -2a\|e(t)\|^2 + 2e(t)^T K_p\zeta - 2e(t)^T K_p u_R$$

$$\leq -2a\|e(t)\|^2 + \|e(t)\|g - 2e(t)^T K_p u_R$$

$$= -a\|e(t)\|^2 + \|e(t)\|g - \frac{e(t)^T K_p\Gamma\mu(e,y,u,t)\|\mu(e,y,u,t)\|^{\tau}}{\|\mu(e,y,u,t)\|^{\tau+1} + \epsilon^{\tau+1}\exp(-\beta(\tau+1)t)}g(y_p,u,t)$$

$$= -a\|e(t)\|^2 + \|\mu(e,y,u,t)\| - \frac{\mu(e,y,u,t)^T K_p\Gamma\mu(e,y_p,u,t)\|\mu(e,y_p,u,t)\|^{\tau}}{\|\mu(e,y_p,u,t)\|^{\tau+1} + \epsilon^{\tau+1}\exp(-\beta(\tau+1)t)}$$

$$= -a\|e(t)\|^2 + \frac{\|\mu(e,y_p,u,t)\|^{\tau}\left[\mu(e,y_p,u,t)^T\left(I - K_p\Gamma\right)\mu(e,y_p,u,t)\right]}{\|\mu(e,y_p,u,t)\|^{\tau+1} + \epsilon^{\tau+1}\exp(-\beta(\tau+1)t)}$$

$$+ \frac{\|\mu(e,y_p,u,t)\|\epsilon^{\tau+1}\exp(-\beta(\tau+1)t)}{\|\mu(e,y_p,u,t)\|^{\tau+1} + \epsilon^{\tau+1}\exp(-\beta(\tau+1)t)}$$

$$
\begin{aligned}
&= -a\|e(t)\|^2 + \frac{\|\mu(e,y_p,u,t)\|^\tau \left[\mu(e,y_p,u,t)^T\left(I - \frac{Q}{2}\right)\mu(e,y_p,u,t)\right]}{\|\mu(e,y_p,u,t)\|^{\tau+1} + \epsilon^{\tau+1}\exp(-\beta(\tau+1)t)} \\
&\quad + \frac{\|\mu(e,y_p,u,t)\|\epsilon^{\tau+1}\exp(-\beta(\tau+1)t)}{\|\mu(e,y_p,u,t)\|^{\tau+1} + \epsilon^{\tau+1}\exp(-\beta(\tau+1)t)} \\
&\le -a\|e(t)\|^2 + \frac{\|\mu(e,y_p,u,t)\|\epsilon^{\tau}\exp(-\beta\tau t)}{\|\mu(e,y_p,u,t)\|^{\tau+1} + \epsilon^{\tau+1}\exp(-\beta(\tau+1)t)}\epsilon\exp(-\beta t) \\
&\le -a\|e(t)\|^2 + \epsilon\exp(-\beta t) \\
&= -aV + \epsilon\exp(-\beta t).
\end{aligned}
$$

Let

$$
s(t) = \dot{V} + \lambda V - \epsilon\exp(-\beta t)
$$

where $\lambda \triangleq a$. Note that $s(t) \le 0$. Solving this differential equation yields.

$$
\begin{aligned}
V(t) &= \exp\left[-\lambda(t-t_0)\right]V(t_0) + \int_{t_0}^{t}\exp\left[-\lambda(t-\tau)\right]\left[s(\tau) + \epsilon\exp(-\beta\tau)\right]d\tau \\
&\le \exp\left[-\lambda(t-t_0)\right]V(t_0) \\
&\quad + \begin{cases} \dfrac{\epsilon\exp(-\beta t_0)}{\lambda-\beta}\left(\exp\left[-\beta(t-t_0)\right] - \exp\left[-\lambda(t-t_0)\right]\right) & \text{if } \lambda \ne \beta \\ \epsilon(t-t_0)\exp(-\beta t) & \text{if } \lambda = \beta \end{cases} \\
&\to \begin{cases} 0 & \text{if } \beta > 0 \\ \dfrac{\epsilon}{\lambda} & \text{if } \beta = 0 \end{cases}
\end{aligned}
$$

Therefore one has that $V(t)$ converges to zero exponentially. As $V(t)$ converges to zero the output tracking error $e(t)$ converges exponentially to either zero or a residue set. Because the tracking error is bounded one can conclude that the plant output $y_p(t)$ is uniformly bounded.

3.2. *MRRC for systems with higher relative degree*

If $G_m(s)$ is not SPR then it is of the form

$$
G_m(s) = \begin{bmatrix} \dfrac{1}{\pi^{n_1}(s)} & & 0 \\ 0 & \ddots & \\ \vdots & & \ddots \\ 0 & & \dfrac{1}{\pi^{n_m}(s)} \end{bmatrix}
$$

where $\pi(s) = s + a$ is a stable polynomial of degree one. This form of $G_m(s)$ is not suitable for the recursive backstepping procedure because each diagonal element is potentially of different relative degree. The recursive backstepping procedure entails breaking up a transfer function matrix into n sections each one being SPR where n represents the relative degree of that transfer function matrix. To utilize the backstepping procedure one may modify $G_m(s)$ as follows. Let $n = \max n_i, i = 1, \cdots, m$

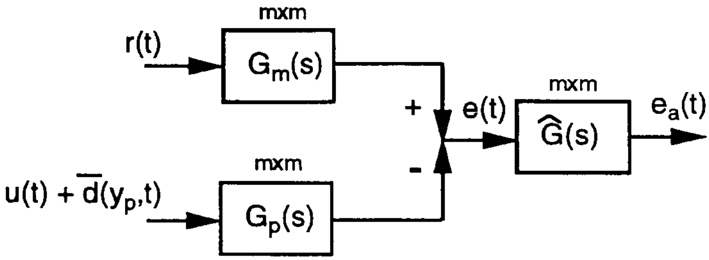

Figure 4. MIMO system modification.

and let

$$G_a(s) = \hat{G}(s) G_m(s)$$

where

$$\hat{G}(s) = \begin{bmatrix} \dfrac{1}{\pi^{n-n_1}(s)} & & 0 \\ 0 & \ddots & \\ \vdots & & \ddots \\ 0 & & \dfrac{1}{\pi^{n-n_m}(s)} \end{bmatrix}$$

Figure 4 shows the result of this modification.

After this modification, one has that

$$G_a(s) = \begin{bmatrix} \dfrac{1}{\pi^{n}(s)} & & 0 \\ 0 & \ddots & \\ \vdots & & \ddots \\ 0 & & \dfrac{1}{\pi^{n}(s)} \end{bmatrix} = \frac{1}{(s+a)^n} I$$

Thus $G_a(s)$ is diagonal with equal elements. Although $G_a(s)$ is not SPR, it can be divided into n factors, each of which is SPR. Let the variables $\bar{\omega}(t)$, $\bar{u}_R(t)$, and let $\alpha(s)$ be defined as

$$\bar{\omega}(t) = \frac{1}{\alpha(s)} \omega(t)$$

$$\bar{u}_R(t) = \frac{1}{\alpha(s)} u_R$$

$$\alpha(s) = (s+a)^{n-1}$$

The dynamics of the augmented output tracking error can be written as

$$\begin{aligned} e_a(t) &= G_a(s) K^{*-1} \left[\tilde{\Theta} \omega(t) - u_R - (I - G_1(s)) \bar{d}(y_p, t) \right] \\ &= \frac{1}{(s+a)^n} K_p \left[\tilde{\Theta} \omega(t) - u_R - (I - G_1(s)) \bar{d}(y_p, t) \right] \end{aligned} \tag{19}$$

If the above analysis is applied to systems with relative degree greater than one, the actual control would require derivatives of y_p up to $l = n - 1$, where n is the relative degree of $G_a(s)$. To avoid measuring derivatives of the output, we apply the recursive backstepping procedure. The first step is to rewrite (19) as

$$e_a(t) = \frac{1}{s+a} K_p \left[\tilde{\Theta} \bar{\omega}(t) - \bar{u}_R - \frac{1}{\alpha(s)} (I - G_1(s)) \bar{d}(y_p, t) \right] \tag{20}$$

Let

$$\begin{aligned} z_1 &= \bar{u}_R \\ \dot{z}_1 &= -az_1 + z_2 \\ \dot{z}_i &= -az_i + z_{i+1}, \quad \forall i = 2, \cdots, l-1 \\ \dot{z}_l &= -az_l + u_R \end{aligned}$$

Secondly, using the state variable z_1 one rewrites (25) as

$$\dot{e}_a = -ae_a + K_p \left[\tilde{\Theta} \bar{\omega}(t) - z_1 - \frac{1}{\alpha(s)} (I - G_1(s)) \bar{d}(y_p, t) \right] \tag{21}$$

The next step is to substitute the fictitious control v_1 into (20), resulting in

$$\dot{e}_a = -ae_a + K_p \bar{\zeta} - K_p v_1 - K_p(z_1 - v_1) \tag{22}$$

where

$$\bar{\zeta} = \tilde{\Theta} \bar{\omega}(t) - \frac{1}{\alpha(s)} (I - G_1(s)) \bar{d}(y_p, t)$$

The next step is to design the fictitious control v_1. Let $v_1 = v_{n1} + v_{r1}$ where v_{n1} is a linear control and v_{r1} is a nonlinear robust control. Therefore one has that

$$K_p v_1 = v_{n1} + K_p v_{r1} + \left(K_p - I\right) v_{n1}$$

We choose the linear control v_{n1} to be

$$v_{n1} = (\gamma - a) e_a$$

where $\gamma \geq a > 0$. The selection of γ allows the designer to speed the convergence rate of the tracking error. The robust control v_{r1} is next to be designed; it must dominate the uncertainties in (28). Similarly to the SPR case, we design v_{r1} to be

$$v_{r1} = \frac{\Gamma \mu_1(e_a, y_p, u, t) \|\mu_1(e_a, y_p, u, t)\|^{\tau_1}}{2\left(\|\mu_1(e_a, y_p, u, t)\|^{\tau_1 + 1} + \epsilon_1^{1+\tau_1} \right)} g_1(y_p, u, t) \tag{23}$$

where $\mu_1(e_a, y_p, u, t) = e_a(t) g_1(y_p, u, t)$ and

$$g_1(y_p, u, t) = 2 \left\| |K_p| \right\| \left(\left\| |\tilde{\Theta}| \right\| \|\bar{\omega}(t)\| + \left\| \left| \frac{1}{\alpha(s)} (I - G_1(s)) \bar{d}(y_p, t) \right| \right\| + \|v_{n1}\| \right) + 2\|v_{n1}\| \tag{24}$$

Note that (28) contains the term $-K_p(z_1 - v_1)$. For (28) to be stable the term $z_1 - v_1$ must be stable, i.e. must tend to zero. Let $w_1 = z_1 - v_1$. Examining the dyamics of w_1 yields

$$\begin{aligned} \dot{w}_1 &= \dot{z}_1 - \dot{v}_1 \\ &= -az_1 + z_2 - \dot{v}_1 + v_2 - v_2 \\ &= -az_1 - \dot{v}_1 + v_2 + w_2 \end{aligned}$$

where $w_2 = z_2 - v_2$ and v_2 is a fictitious control to be designed. To stablize w_1, we choose v_2 to be

$$v_2 = az_1 - \gamma w_1 + v_{r2} \tag{25}$$

where

$$v_{r2} = \frac{\mu_2(e_a, y_p, u, t)\|\mu_2(e_a, y_p, u, t)\|^{\tau_2}}{2\left(\|\mu_2(e_a, y_p, u, t)\|^{\tau_2+1} + \epsilon_2^{1+\tau_2}\right)} g_2(y_p, u, t) \tag{26}$$

and where

$$g_2(y, u, t) = 2\|\|\dot{v}_1\|\| + \|\|K_p\|\|^2 + \|e_a(t)\|^2$$

$$\mu_2(e_a, y_p, u, t) = (v_1 - z_1)g_2(y_p, u, t)$$

$$= - w_1 g_2(y_p, u, t)$$

Continuing the backstepping procedure, the next l mappings are defined as

$$v_i = - w_{i-2} + az_{i-1} - \gamma w_{i-1} + v_{ri} \tag{27}$$

where

$$v_{ri} = \frac{\mu_i(e_a, y_p, u, t)\|\mu_i(e_a, y_p, u, t)\|^{\tau_i}}{2\left(\|\mu_i(e_a, y_p, u, t)\|^{\tau_i+1} + \epsilon_i^{1+\tau_i}\right)} g_i(y_p, u, t) \tag{28}$$

$$v_{l+1} = - w_{l-1} + az_l - \gamma w_l + v_{r(l+1)}$$

$$= u_R$$

$$v_{r(l+1)} = \frac{\mu_{l+1}(e_a, y_p, u, t)\|\mu_{l+1}(e_a, y_p, u, t)\|^{\tau_{l+1}}}{2\left(\|\mu_{l+1}(e_a, y_p, u, t)\|^{\tau_{l+1}+1} + \epsilon_{l+1}^{1+\tau_{l+1}}\right)} g_{l+1}(y_p, u, t) \tag{29}$$

for $i = 3, \cdots, l$ with

$$g_i(y, u, t) = 2\|\|\dot{v}_{i-1}\|\|$$

$$\mu_i(e_a, y_p, u, t) = - w_{i-1} g_i(y_p, u, t)$$

$$g_{l+1}(y, u, t) = 2\|\|\dot{v}_l\|\|$$

$$\mu_{l+1}(e_a, y_p, u, t) = - w_l g_{l+1}(y_p, u, t)$$

where for the latter two equations $i = 3, \cdots, l$. As in the SPR case, the ϵ_i are design parameters that control magnitude and perhaps tracking accuracy, and the τ_i are constants chosen such that the first-order partial derivatives of v_i are well defined.

In the backstepping or parameter projection procedure one wishes that z_i should track v_i. The procedure proceeds by back-stepping through $1/\alpha(s)$.

The Lyapunov proof follows with $V = V_1 + \sum_{i=1}^{l} w_i^T w_i$, where

$$V_1(t) = e_a^T e_a = \|e_a\|^2$$

Taking the time derivative of V along the trajectories of the system yields

$$\dot{V} = 2e_a^T \dot{e}_a + 2\sum_{i=1}^{l} w_i^T \dot{w}_i$$

$$\begin{aligned}\dot{V} = {} & 2e_a^T(-\gamma e_a + K_p\bar{\zeta} + K_p v_{n1} - v_{n1} - K_p v_{r1} - K_p w_1) \\ & + 2w_1^T(-\gamma w_1 - \dot{v}_1 + v_{r2} + w_2) \\ & + 2\sum_{i=2}^{l-1} w_i^T(-\gamma w_i - \dot{v}_i + v_{r(i+1)} - w_{i-1} + w_{i+1}) \\ & + 2w_l^T(-\gamma w_l - \dot{v}_l + v_{r(l+1)} - w_{l-1}) \\ = {} & -2\gamma\|e_a\|^2 - 2\gamma\sum_{i=1}^{l}\|w_i\|^2 \\ & + 2e_a^T(K_p\bar{\zeta} + K_p v_{n1} - v_{n1} - K_p v_{r1}) \\ & + 2w_1^T(-k_p^T e_a - \dot{v}_1 + v_{r2}) \\ & + 2\sum_{i=2}^{l-1} w_i^T(\gamma w_i - \dot{v}_i + v_{r(i+1)}) \\ & + 2w_l^T(-\gamma w_l - \dot{v}_l + v_{r(l+1)})\end{aligned}$$

We know from the previous proof that

$$2e_a^T(K_p\bar{\zeta} + K_p v_{n1} - v_{n1} - K_p v_{r1}) \le \epsilon_1$$

Similarly, one may show that

$$2w_1^T(-k_p^T e_a - \dot{v}_1 + v_{r2}) \le \epsilon_2$$

$$2w_i^T(\gamma w_i - \dot{v}_i + v_{r(i+1)}) \le \epsilon_{i+1}, i \in 2, \cdots, l-1$$

$$2w_l^T(-\gamma w_l - \dot{v}_l + v_{r(l+1)}) \le \epsilon_{l+1}$$

Therefore one obtains the result

$$\dot{V} \le -2\gamma\|e_a\|^2 - 2\gamma\sum_{i=1}^{l}\|w_i\|^2 + \sum_{j=1}^{l+1}\epsilon_j = -\lambda V + \lambda\epsilon$$

where

$$\lambda = 2\gamma, \epsilon = \frac{1}{\lambda}\sum_{j=1}^{l+1}\epsilon_j$$

Similarly to before, define

$$s(t) = \dot{V} + \lambda V - \lambda\epsilon$$

Note that $s(t) \le 0$. Solving the differential equation yields

$$\begin{aligned} V(t) &= \exp\left[-\lambda(t-t_0)\right]V(t_0) + \int_{t_0}^{t} \exp\left[-\lambda(t-\tau)\right]\left[s(\tau)+\lambda\epsilon\right]d\tau \\ &\le \exp\left[-\lambda(t-t_0)\right]V(t_0) + \epsilon\int_{t_0}^{t} \exp\left[-\lambda(t-\tau)\right]\lambda d\tau \\ &= \exp\left[-\lambda(t-t_0)\right]V(t_0) + \epsilon(1-\exp\left[-\lambda(t-t_0)\right]) \\ &\to \epsilon, \text{ as } t \to \infty. \end{aligned}$$

Therefore V is uniformly ultimately bounded by ϵ. All the variables in V are globally and uniformly ultimately bounded including the augmented tracking error $e_a(t)$. As the states z_i are globally uniformly ultimately bounded, $e(t)$ is globally uniformly ultimately bounded.

Remark 7: To simplify the understanding of the Lyapunov proof, the parameter γ was chosen to be the same in all the fictitious control equations. That restriction is not necessary, however. The designer is free to choose different convergence parameters in the design of the fictitious controls, say γ_1, γ_2, etc. In addition, the γ and the ϵ can be made time varying to reduce the magnitude of the control law to initial conditions while maintaining overall tracking accuracy. □

Remark 8: Bounding functions must be found for $\dot{v}_1, \ldots, \dot{v}_l$.

One knows that

$$\dot{v}_1 = \frac{\partial v_1}{\partial e_a}\dot{e}_a + \frac{\partial v_1}{\partial g_1}\dot{g}_1$$

$$\vdots$$

$$\dot{v}_i = \frac{\partial v_i}{\partial e_{i-1}}\dot{e}_{i-1} + \frac{\partial v_i}{\partial g_i}\dot{g}_i$$

for $i = 2, \ldots, l$ where $e_i = v_{i-1} - z_{i-1}$. For $\dot{v}_1$ we have that

$$\frac{\partial v_1}{\partial e_a} = \begin{bmatrix} \frac{\partial v_{11}}{\partial e_{a1}} & \cdots & \frac{\partial v_{11}}{\partial e_{an}} \\ \vdots & \ddots & \vdots \\ \frac{\partial v_{1n}}{\partial e_{a1}} & \cdots & \frac{\partial v_{1n}}{\partial e_{an}} \end{bmatrix}$$

$$\frac{\partial v_1}{\partial g_1} = \begin{bmatrix} \frac{\partial v_{11}}{\partial g_1} \\ \vdots \\ \frac{\partial v_{1n}}{\partial g_n} \end{bmatrix}$$

As an example of finding a bounding function for v_1 let $\tau_1 = 1$. After taking the derivative of v_1, one obtains

$$\frac{\partial v_1}{\partial e_a} = \frac{\left(\epsilon_1^2 + g_1^2\|e_a\|^2\right) g_1^3\|e_a\|\Gamma + \left(\frac{\epsilon_1^2}{\|e_a\|} - g_1^2\|e_a\|\right) g_1^3\Gamma e_a e_a^T}{2\left(\|e_a\|^2 g_1^2 + \epsilon_1^2\right)^2}$$

$$\frac{\partial v_1}{\partial g_1} = \frac{\left(3\epsilon_1^2 + \|e_a\|^2 g_1^2\right)\|e_a\| g_1^2 \Gamma e_a}{2\left(\|e_a\|^2 g_1^2 + \epsilon_1^2\right)^2}$$

A bounding function for $\dot{e}_a(t)$ may be found as

$$\begin{aligned}\|\dot{e}_a\| &\le a\|e_a\| + |||K_p|||\|z_1\| + |||K_p|||\|\bar{\zeta}\| \\ &\triangleq |||\dot{e}_a|||\end{aligned}$$

where

$$\bar{\zeta} = \left[\tilde{\Theta}\bar{\omega}(t) - \frac{1}{\alpha(s)}(I - G_1(s))\bar{d}(y_p, t)\right]$$

as before. □

Remark 9: Instead of using an augmented error matrix, one may achieve nonlinear robust control of non-SPR systems by post multiplying $G_p(s)$ by a known transfer function matrix, say $G_c(s)$, such that $G_p(s)G_c(s)$ has a Hermite normal form which is diagonal and has equal elements. The reference model $G_m(s)$ is this Hermite normal form. □

Remark 10: Using the method of Remark 9, one may choose the reference model as

$$G_m(s) = \begin{bmatrix} \frac{1}{(s+a_1)(s+a_2)\cdots(s+a_l)} & & 0 \\ 0 & \ddots & \\ \vdots & & \ddots \\ 0 & & \frac{1}{(s+a_1)(s+a_2)\cdots(s+a_l)} \end{bmatrix}$$

where the relative degree of $G_m(s)$ is l. The above reference model allows the designer flexibility to choose distinct pole locations $a_1, a_2, \ldots, a_l$. In this case, the fictitious control signals must be modified appropriately. □

3.3. *Simulation examples*

Example 1—Simulation of SPR MIMO system using Matlab/Simulink©: The reference model chosen for this example is given by

$$G_m(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

The plant to be simulated is given by

$$\begin{aligned} G_p(s) &= \begin{bmatrix} \frac{0{\cdot}5(s+1)}{(s-1)(s+1{\cdot}5)} & \frac{0{\cdot}5}{(s-2)(s+0{\cdot}5)} \\ \frac{1}{(s-1)(s+1{\cdot}5)} & \frac{s+1}{(s-2)(s+0{\cdot}5)} \end{bmatrix} \\ &= \begin{bmatrix} 0{\cdot}5(s+1) & 0{\cdot}5 \\ 1 & s+1 \end{bmatrix}\begin{bmatrix} (s-1)(s+1{\cdot}5) & 0 \\ 0 & (s-2)(s+0{\cdot}5) \end{bmatrix}^{-1} \\ &= B_p(s)A_p^{-1}(s) \end{aligned}$$

The plant has considerable coupling. The minimum relative degree of each row is one. After multiplying each row by s and letting $s \to \infty$, one obtains

$$\begin{bmatrix} 0{\cdot}5 & 0 \\ 0 & 1 \end{bmatrix}$$

Because this matrix is non-singular, the Hermite normal form of the plant is diagonal, with elements equal to the minimum relative degree of each row of $G_p(s)$.

The observability index υ of the plant is two. The polynomial $a_q(s)$ of degree $\upsilon - 1$ was chosen as $a_q(s) = s + 3$. The disturbance and reference inputs for this simulation were

$$\bar{d}(y_p, t) = \begin{bmatrix} 0{\cdot}5\sin(t) + 0{\cdot}2\cos(y_{p1}(t)) + y_{p2}^2(t)\cos(t) \\ 0{\cdot}5\cos(t) + 0{\cdot}4\sin(y_{p1}(t)) + y_{p1}^2(t)\sin(t) + y_{p2}^2(t) \end{bmatrix}$$

$$r(t) = \begin{bmatrix} \cos(t) \\ \sin(3t) \end{bmatrix}$$

respectively. The bound ρ on the disturbance is given by

$$\rho = \left[\left(1 + y_{p2}^2(t)\right)^2 + \left(1 + \|y_p(t)\|^2\right)^2\right]^{1/2}$$

and ϵ and β were both chosen as $0{\cdot}2$. The bounding function $g(y_p, u, t)$ is given by

$$g(y_p, u, t) = 2\left(\|r\|^2 + \|\omega_1\|^2 + \|\omega_2\|^2 + \|\omega_3\|^2 + \rho^2 + \bar{\rho}^2 + 1{\cdot}0\right)$$

The simulation step size was selected as $0{\cdot}001$ and the error tolerance was selected as $1{\cdot}0e-6$. The tracking error and control law plots are shown in Figs 5 and 6, respectively. As one can see, the tracking error converges to zero very rapidly.

The control law and auxiliary signal generator was coded in C code and embedded into the Matlab simulation.

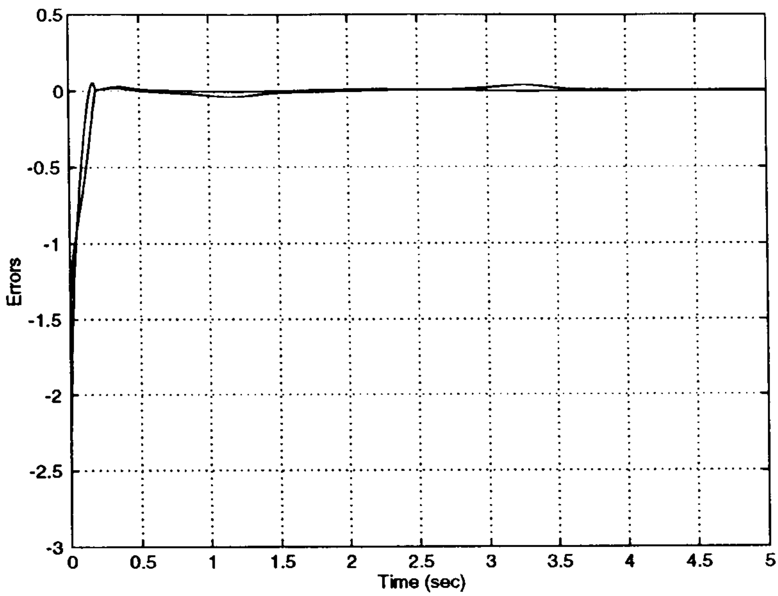

Figure 5. Tracking error plot.

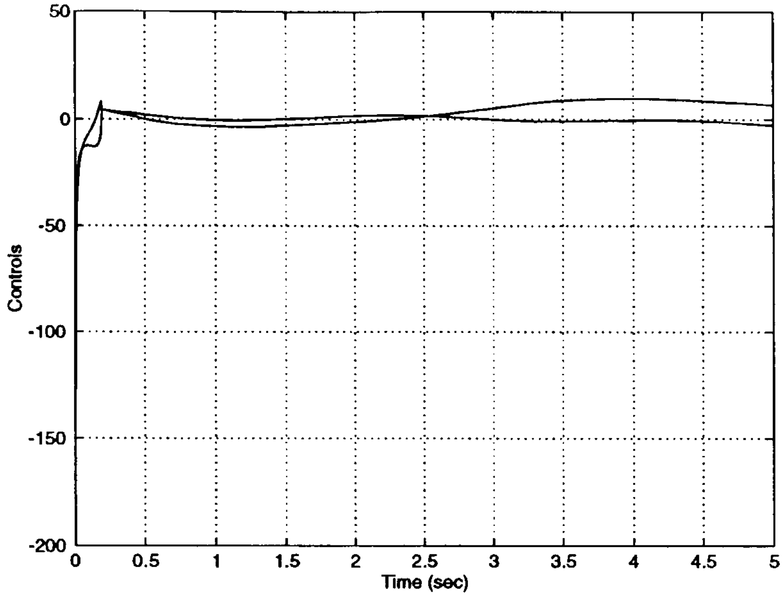

Figure 6. Control law plot.

Example 2—Simulation of non-SPR MIMO system using Matlab/Simulink©: The reference model chosen for this example is

$$G_m(s) = \begin{bmatrix} \dfrac{1}{(s+1)} & 0 \\ 0 & \dfrac{1}{(s+1)^2} \end{bmatrix}$$

and the plant to be simulated is given by

$$G_p(s) = \begin{bmatrix} \dfrac{1}{s+2} & \dfrac{1}{(s-3)(s+2)} \\ \dfrac{1}{(s+2)^2} & \dfrac{1}{(s-3)(s+2)} \end{bmatrix} \tag{30}$$

The right coprime factorization of $G_p(s)$ is given by

$$\begin{aligned} G_p(s) &= \begin{bmatrix} s+2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (s+2)^2 & 0 \\ 0 & (s-3)(s+2) \end{bmatrix}^{-1} \\ &= B_p(s) A_p^{-1}(s) \end{aligned}$$

To obtain the observability index one determines the left coprime factorization of $G_p(s)$. It is given by

$$G_p(s) = \begin{bmatrix} (s-3)(s+2) & 0 \\ 0 & (s+2)^2(s-3) \end{bmatrix}^{-1} \begin{bmatrix} s-3 & 1 \\ s-3 & s+2 \end{bmatrix}$$

Using the right coprime factorization of $G_p(s)$ one obtains the controllable canonical form as

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 6 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u$$

$$y_p = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} x$$

The minimum relative degree of the first row of $G_p(s)$ is one, and the minimum relative degree of the second row of $G_p(s)$ is two. After multiplying each row by s raised to the minimum relative degree of that row and letting $s \to \infty$ one obtains

$$K_p = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

where K_p is the plant high frequency gain matrix. Because this matrix is non-singular, the Hermite normal form of the plant is diagonal with elements equal to the minimum relative degree of each row of $G_p(s)$. Note that the Hermite normal form is not SPR. The observability index υ of the plant is three as determined by observing the highest degree in the denominator of the left coprime factorization of $G_p(s)$. The polynomial $a_q(s)$ of degree $\upsilon - 1$ was chosen as $a_q(s) = s^2 + 10s + 25$. The augmented matrix $\hat{G}_m(s)$ is

$$\hat{G}_m(s) = \begin{bmatrix} \dfrac{1}{(s+1)} & 0 \\ 0 & 1 \end{bmatrix}$$

The disturbance was the same as the previous example and $\epsilon_1 = \epsilon_2 = 5.0$. The bounding functions are given as

$$g_1 = \left(\|r\|^2 + \|\omega_1\|^2 + \|\omega_2\|^2 + \|\omega_3\|^2 + \|\omega_4\|^2 + \|\omega_5\|^2 + 2\bar{\rho} + \bar{\rho}_f + 12.0 \right)$$

$$g_2 = 0{\cdot}75\left(g_1 + g_1^2\right)$$

where $\bar{\rho}_f = [1/(s+1)]\bar{\rho}$. The gain matrix Γ was chosen as

$$\Gamma = \begin{bmatrix} 1{\cdot}5 & -0{\cdot}75 \\ -0{\cdot}75 & 2{\cdot}25 \end{bmatrix}$$

The simulation step size was selected as 0·001 and the error tolerance was selected as $1.0e-6$. Simulation results are shown in Figs 7 and 8.

Example 3—Simulation of non-SPR MIMO system using Matlab/Simulink©: The reference model chosen for this example is

$$G_m(s) = \begin{bmatrix} \dfrac{1}{(s+1)^3} & 0 \\ 0 & \dfrac{1}{(s+1)^2} \end{bmatrix}$$

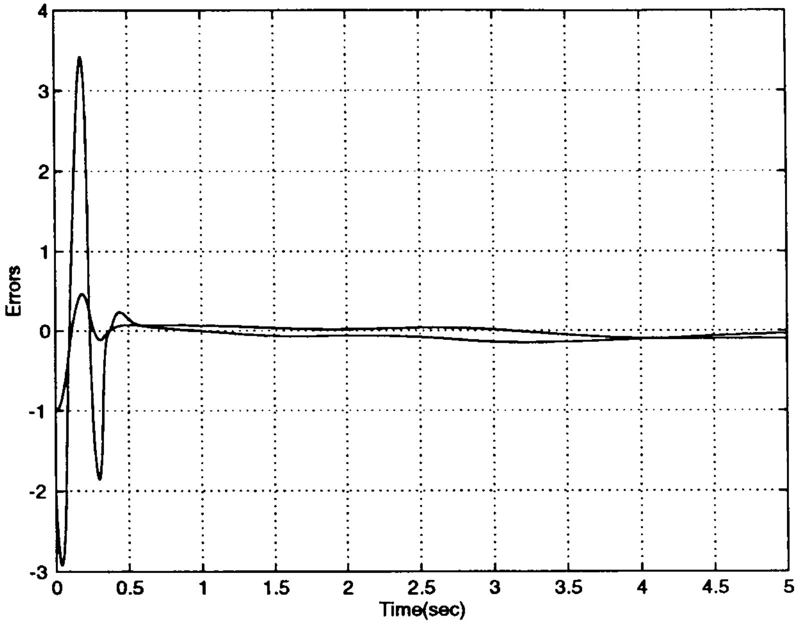

Figure 7. Tracking error plot.

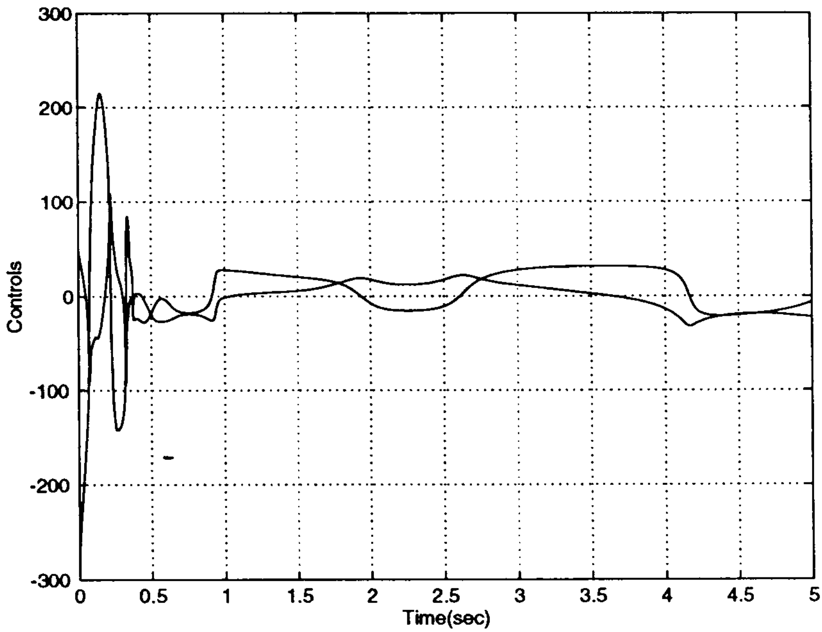

Figure 8. Control law plot.

and the plant to be simulated is given by

$$G_p(s) = \begin{bmatrix} \dfrac{1}{(s-3)(s+2)^2} & \dfrac{1}{(s+2)^3} \\ \dfrac{1}{(s-3)(s+2)^2} & \dfrac{1}{(s+2)^2} \end{bmatrix} \tag{31}$$

This example displays results for a system whose reference after multiplication by the augmented matrix is of third order. The augmented matrix is given by

$$G_a(s) = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1}{(s+1)} \end{bmatrix} \tag{32}$$

The right coprime factorization of $G_p(s)$ is given by

$$\begin{aligned} G_p(s) &= \begin{bmatrix} 1 & 1 \\ 1 & s+2 \end{bmatrix} \begin{bmatrix} (s-3)(s+2)^2 & 0 \\ 0 & (s+2)^3 \end{bmatrix}^{-1} \\ &= B_p(s) A_p^{-1}(s) \end{aligned}$$

and the left coprime factorization of $G_p(s)$ is given by

$$G_p(s) = \begin{bmatrix} (s-3)(s+2)^3 & 0 \\ 0 & (s-3)(s+2)^2 \end{bmatrix}^{-1} \begin{bmatrix} s+2 & s-3 \\ 1 & s-3 \end{bmatrix}$$

Using the right coprime factorization of $G_p(s)$ one obtains the controllable canonical form as

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 12 & 8 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -8 & -12 & -6 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u$$

$$y_p = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 & 1 & 0 \end{bmatrix} x$$

The observability index $\upsilon = 4$. The bounding functions are given as

$$g_1 = (\|r\|^2 + \|\omega_1\|^2 + \|\omega_2\|^2 + \|\omega_3\|^2 + \|\omega_4\|^2 + \|\omega_5\|^2 + \|\omega_6\|^2 + \|\omega_7\|^2 + 2\bar{\rho} + \bar{\rho}_f + \|e_a(t)\|)$$

$$g_2 = 0{\cdot}7 g_1 + 0{\cdot}7 g_1^2 + 0{\cdot}7\|e_a(t)\|^2$$

$$g_3 = 10{\cdot}0 g_2$$

where $\bar{\rho}_f = \left[1/(s+1)^2\right]\bar{\rho}$. To reduce control law magnitude while maintaining tracking error, the ϵ and λ parameters were made time varying. The parameters ϵ_1, ϵ_2 and ϵ_3 were selected as $100{\cdot}0 + \left[100{\cdot}0/(1+10{\cdot}0t)\right]$ $100{\cdot}0 + \left[100{\cdot}0/(1+10{\cdot}0t)\right]$

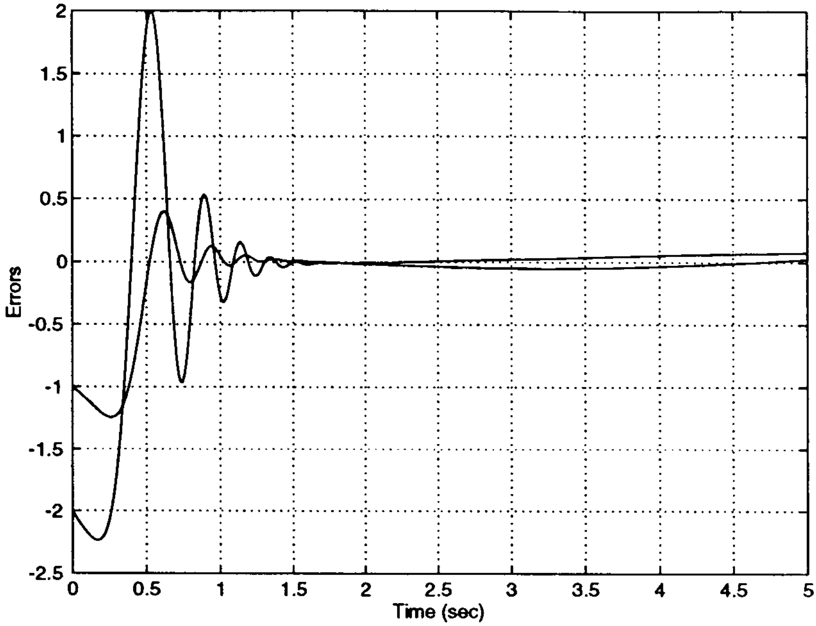

Figure 9. Tracking error plot.

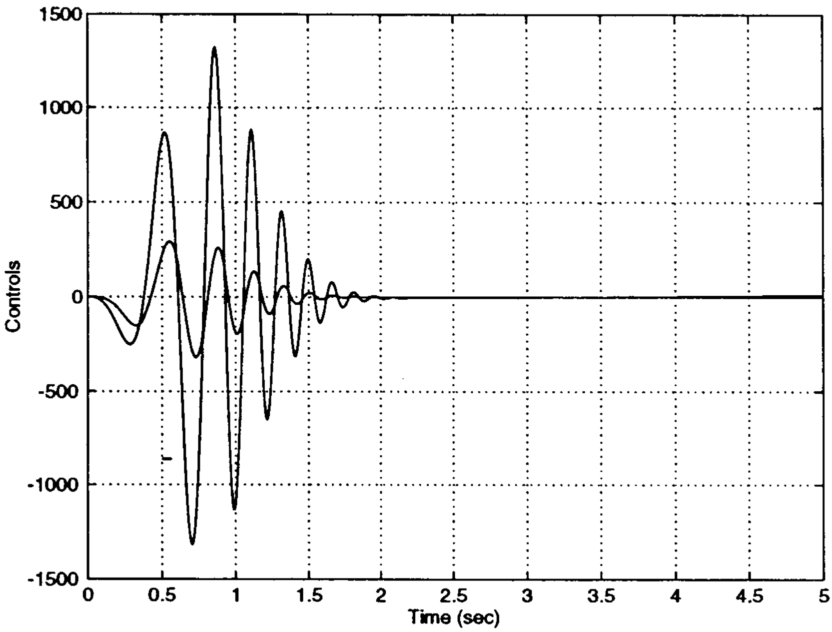

Figure 10. Tracking error plot.

and $100{\cdot}0 + [100{\cdot}0/(1 + 10{\cdot}0t)]$ respectively and the parameters λ_1, λ_2 and λ_3 were selected as $100{\cdot}0(1 - \exp(-0{\cdot}2t))$, $150{\cdot}0(1 - \exp(-0{\cdot}2t))$ and $200{\cdot}0(1 - \exp(-0{\cdot}2t))$ respectively. The simulation step size was selected as 0·0005 and the error tolerance was selected as $1.0e-6$.

4. Conclusions

Model reference robust control of MIMO plants has been examined. The method was shown to be an extension of model reference robust control for SISO systems. Control laws for SPR and non-SPR systems were derived. A development was given which introduced the augmented matrix so that a backstepping procedure for the situation where the plant's reference model is not SPR could be used. AsympTotic stability was proven for SPR systems, and uniform ultimate boundedness was proven for non-SPR systems. Simulations were performed on SPR and non-SPR systems, which illustrated the principles of MRRC for MIMO systems.

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Appendix

Hermite normal forms

A non-singular $m \times m$ matrix $G_p(s)$ can be transformed into a lower triangular form $H_p(s)$ known as the right Hermite normal form, by performing elementary column operations. This operation is equivalent to multiplying $G_p(s)$ on the right by an appropriate unimodular matrix. $H_p(s)$ has the form

$$H_p(s) = \begin{bmatrix} \dfrac{1}{\pi^{n_1}(s)} & & 0 \\ h_{21}(s) & \ddots & \\ \vdots & & \ddots \\ h_{m1}(s) & & \dfrac{1}{\pi^{n_m}(s)} \end{bmatrix}$$

The procedure for determining $H_p(s)$ is as follows. Let $t_{ij}(s)$ denote the ijth element of $G_p(s)$.

(*a*) By the interchange of columns, move the element with lowest relative degree in the first row to the (1,1) position.

(*b*) Subtract a multiple of the first column from the second, third, ... etc. column to ensure $r(t_{ij}) < r(t_{ii})$ for $j = 2, \ldots, m$ where $r(t_{ij})$ is the relative degree of the ijth element of $H_p(s)$.

(*c*) If one or more of the $t_i j$ for $j = 2, \ldots, m$ is nonzero, go to (*a*). Else proceed.

(*d*) Temporarily delete the first row and column.

(*e*) Repeat the procedure of steps (*a*)–(*d*) $(m - 1)$ times, each time on the remaining matrix. This leaves the temporarily deleted rows and columns unchanged.

(f) Subtract a multiple of column 2 from column 1, multiples of column 3 for columns 2 and 1, and so on, ensuring that $r(t_{ii}) > r(t_{ij})$, for $j < 1$. The multiples in steps (b) and (d) are the quotients chosen according to the division algorithm.

For further details see Hung and Anderson (1979) and Singh (1985).

Bezout identity example

Let

$$G_p(s) = \begin{bmatrix} \dfrac{0\cdot5(s+1)}{(s-1)(s+5)} & \dfrac{0\cdot5}{(s-2)(s+0\cdot5)} \\ \dfrac{1}{(s-1)(s+5)} & \dfrac{s+1}{(s-2)(s+0\cdot5)} \end{bmatrix}$$

The right coprime factorization of $G_p(s)$ is given by

$$G_p(s) = \begin{bmatrix} 0\cdot5(s+1) & 0\cdot5 \\ 1 & (s+1) \end{bmatrix} \begin{bmatrix} (s-1)(s+5) & 0 \\ 0 & (s-2)(s+0\cdot5) \end{bmatrix}^{-1}$$

The A, B, C and D matrices of the plant are determined from the right coprime factorization as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 5 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1\cdot5 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0\cdot5 & 0\cdot5 & 0\cdot5 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The observability index $\upsilon = 2$. This is determined by the knowledge of the A and C matrices determined above. Let $A_q^{-1}(s)$ be given by

$$A_q^{-1}(s) = \begin{bmatrix} \dfrac{1}{a_q(s)} & 0 \\ 0 & \dfrac{1}{a_q(s)} \end{bmatrix}$$

where $a_q(s) = s + 3$. By using (11), one obtains

$$[D_1 s + D_0] B_p(s) + C_1 A_p(s) = \begin{bmatrix} 2s^2 - 18 & -s^2 - 4s - 3 \\ -s^2 - 4s - 3 & 3\cdot5s^2 - 12\cdot5s - 6 \end{bmatrix}$$

After matching coefficients, one obtains

$$[D_0 \quad C_1 \quad D_1]\begin{bmatrix} 0{\cdot}5 & 0{\cdot}5 & 0{\cdot}5 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ -5 & 0 & 4 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1{\cdot}5 & 0 & 1 \\ 0 & 0 & 0{\cdot}5 & 0{\cdot}5 & 0{\cdot}5 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -18 & -3 & 0 & -4 & 2 & -1 \\ -3 & -6 & -4 & -12.5 & -1 & 3.5 \end{bmatrix}$$

Therefore

$$C_1 = \begin{bmatrix} 3{\cdot}8571 & 4{\cdot}2857 \\ 0{\cdot}7619 & 6{\cdot}8095 \end{bmatrix}$$

$$D_0 = \begin{bmatrix} -16{\cdot}5714 & 9{\cdot}5714 \\ -3{\cdot}9524 & 2{\cdot}7857 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} -3{\cdot}7143 & -5{\cdot}2857 \\ -3{\cdot}5238 & -3{\cdot}3095 \end{bmatrix}$$

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