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Robust learning control for robotic manipulators with an extension to a class of non-linear systems

JIAN-XIN XU,*† BADRINATH VISWANATHAN† and ZHIHUA QU‡

A robust learning control (RLC) scheme is developed for robotic manipulators by a synthesis of learning control and robust control methods. The non-linear learning control strategy is applied directly to the structured system uncertainties that can be separated and expressed as products of unknown but repeatable (over iterations) state-independent time functions and known state-dependent functions. The non-linear uncertain terms in robotic dynamics such as centrifugal, Coriolis and gravitational forces belong to this category. For unstructured uncertainties which may have non-repeatable factors but are limited by a set of known bounding functions as the only a *priori* knowledge, e.g the frictions of a robotic manipulator, robust control strategies such as variable structure control strategy can be applied to ensure global asymptotic stability. By virtue of the learning and robust properties, the new control system can easily fulfil control objectives that are difficult for either learning control or variable structure control alone to achieve satisfactorily. The proposed RLC scheme is further shown to be applicable to certain classes of non-linear uncertain systems which include robotic dynamics as a subset. Various important properties concerning learning control, such as the need for a resetting condition and derivative signals, whether using iterative control mode or repetitive control mode, are also made clear in relation to different control objectives and plant dynamics.

1. Introduction

Learning control of robotic manipulators has been well developed ever since the concept was proposed by Arimoto *et al.* (1984). Learning can be defined as a change in the system which enables it to do the same work more efficiently in the next cycle of operation. Two types of learning control methods, iterative learning control (ILC) (Arimoto 1985, Bondi *et al.* 1988, Kuc *et al.* 1992, Kurek and Zaremba 1993, Moore 1993, Amann *et al.* 1996, Lee and Bien 1996, Xu 1997) and repetitive control (Yamamoto and Hara 1988, Nakano *et al.* 1989), have been proposed and developed. The new information obtained from a recurrent control situation is considered as an experience for the controller and this experience is used to improve the quality of control whenever similar situations recur. Due to the difficulty of stability analysis, repetitive control has primarily applied to linear systems or very limited non-linear systems in comparison with ILC. Application of ILC, on the other hand, is confined by a number of factors. Contraction mapping techniques (Arimoto 1985) require the Lipschitz condition and the initial resetting condition to ensure convergence. Most learning control schemes also require measurement of the derivative signals of the states (direct transmission from input to output). Besides, most learning control strategies cannot handle system uncertainties such as exogenous disturbances which might not be periodic.

Variable structure control (VSC) is one of the robust control strategies well used for the control of highly non-linear and uncertain systems (Utkin 1978). Compared with other control strategies, less knowledge concerning the plant uncertainties is required to design a variable structure controller. In most cases it is sufficient only to know the upper bounds of the system uncertainties, regardless of whether they are constant system parameters, exogenous disturbances or non-linear functions of system state variables. However, the conservative estimate of the uncertainty bound (due to the limited plant knowledge) may result in unnecessarily large control action. It should be noted that if the operation cycle ceases after a finite period and then restarts, robust control methods only yield the same tracking profiles without any improvement.

Uncertainties in a plant dynamics can be classified as repeatable or non-repeatable from a learning control point of view. Repeatable uncertainties are those which are invariant over iterations and may be structured or unstructured. Similarly, non-repeatable uncertainties are those which are variant over iterations and may be structured or unstructured. It is important to note that learning control schemes cannot handle non-repeatable uncertainties and that robust control strategies do not show any performance improvement over iterations, even in the presence of repeatable uncertainties. In this paper we limit our attention to improving the control performance of systems which have repeatable uncertainties and non-repeatable uncertainties with known bounding functions. Robotic dynamics consists of both structured and unstructured uncertainties. The

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* Author for correspondence. e-mail: elexujx@nus.edu.sg
† National University of Singapore, Department of Electrical Engineering, 10 Kent Ridge Crescent, Singapore 119260, Singapore.
‡ University of Central Florida, Department of Electrical Engineering,m Orlando, FL 32826, USA.

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Coriolis, centrifugal and gravity terms can be expressed in parametric form, wherein the parameters are strictly repeatable over the iterations. However it is very difficult to model friction terms and there always exist some aperiodic uncertainties (disturbances) which cannot be expressed in parametric form and are not repeatable. The ultimate target of this paper is to explore the possibility of synthesizing both learning and robust control strategies to generate a new control system which can easily fulfil control objectives for plants containing both periodic and non-periodic uncertainties (which may be structured or unstructured) such as robotic manipulators.

Recently some non-linear learning control schemes have been proposed by Park *et al.* (1996) and Xu and Qu (1998). It has been shown that non-linear feedback can be incorporated into iterative learning control to achieve asymptotic convergence of tracking control for a class of non-linear systems. In this paper the non-linear learning control scheme is further developed to address both periodic and non-periodic system uncertainties. It is achieved through making learning control and robust control function in a complementary manner. It is worth noting that, owing to the essential finite time operation of learning control tasks, it is not even necessary to construct a BIBO (bounded-input–bounded-output) stable closed-loop system. In other words, a robust controller incorporated into the learning control system needs not be strictly stable as long as the finite escape problem does not occur.

Compared with robust control methods, the main advantage of synthesizing learning control and robust control is its capability of improving the system performance gradually with respect to periodic operations or repeatable control tasks with a fixed finite period. In the proposed robust learning control (RLC) scheme, the contribution from the learning control part is to *learn* and eliminate state-independent periodic uncertainties as much as possible. The contribution from the robust part is to suppress the non-periodic system uncertainties in which only the upper bounds are available for design.

It is shown that robotic dynamics, as well as some classes of non-linear dynamic systems with both periodic and non-periodic uncertainties (which can be structured as well as unstructured), can be easily dealt with by the new control scheme. The robust learning control system possesses the capability of working in either iterative or repetitive control mode for different control objectives. Through analysis of the developed control system in a systematic way, important issues regarding the objective trajectory categories, resetting condition, derivative signal requirement and their relationships have been made clear for different dynamical systems.

This paper is organized as follows. Section 2 describes the robotic manipulator trajectory tracking problem. Section 3 presents the function partition on the basis of inverse dynamics. The structured uncertainties, as the learnable part, are separated from the unstructured uncertainties which are non-periodic and are to be handled by robust control methods. The robust learning control is derived and analysed on the basis of an evaluation function method. The convergence property of the proposed RLC scheme is analysed without the initial resetting condition in § 4 and then extended to repetitive type learning control. The robust learning control scheme is then extended to more general classes of non-linear systems in § 5. Section 6 illustrates the effectiveness of the proposed control scheme on a robotic manipulator. Finally, § 7 gives the conclusion.

2. Problem formulation

The dynamics of a robotic manipulator, with n rigid bodies, can be represented by the following equations

$$\left.\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{x}_2 \\ M(\mathbf{x}_1)\dot{\mathbf{x}}_2 + \boldsymbol{h}(\bar{\mathbf{x}}) + \boldsymbol{d}(\bar{\mathbf{x}},t) &= \mathbf{u} \end{aligned}\right\} \qquad (1)$$

where $\mathbf{x}_j \in \mathcal{R}^n$, $j = 1,2$; $n \in Z_+$; $\bar{\mathbf{x}} \triangleq [\mathbf{x}_1^{\mathrm{T}}, \mathbf{x}_2^{\mathrm{T}}]^{\mathrm{T}} \in \mathcal{X} \subseteq \mathcal{R}^{2n}$ is the augmented state vector of the system, which is measurable. $\mathbf{u} \in \mathcal{R}^n$ is the control input vector of the system. $M(\mathbf{x}_1) \in \mathcal{R}^{n\times n}$ is the inertia matrix. $\boldsymbol{h}(\bar{\mathbf{x}})$ represents the Coriolis, centrifugal, gravity, coulomb friction and viscous friction terms. $\boldsymbol{d}(\bar{\mathbf{x}},t)$ represents the non-parametric friction term and other unstructured uncertainties such as modelling errors and exogenous disturbances.

We now state a few properties of the robotic manipulator.

(1) The inertia matrix is symmetric and positive definite. Each element of the matrix can be expressed as

$$m_{ij} = \boldsymbol{\phi}_{ij}^{\mathrm{T}} \boldsymbol{\eta}_{ij}(\mathbf{x}_1) \quad i,j = 1,2,\ldots,n$$

where $\boldsymbol{\phi}_{ij}^{\mathrm{T}} = [\phi_{ij}^1, \ldots, \phi_{ij}^{l_{ij}}]$ is the unknown function vector of parameters $p \in \mathcal{P}$; $\boldsymbol{\eta}_{ij}^{\mathrm{T}} = [\eta_{ij}^1, \ldots, \eta_{ij}^{l_{ij}}]$ is the vector of known non-linear functions of $\mathbf{x}_1$. $\mathcal{P}$ is the set of admissible parameters. Also, $\forall t \in [0,\infty)$

$$\left\| \frac{\partial M}{\partial x_{1,l}} \right\| \le \lambda_{l,\max} \quad l \in \{1,\ldots,n\}$$

where $\lambda_{l,\max}$ are known positive constants. Since the mass matrix is a function of only the sine and cosine terms of $\mathbf{x}_1$, its derivative with respect to the displacement is finite and hence its upper bound can be calculated.

(2) The function $\boldsymbol{h}$ represents centrifugal, Coriolis, gravity and parametrized friction terms. Each element can be factorized as

$$h_i(\bar{\mathbf{x}}) = \boldsymbol{\theta}_i^{\mathrm{T}} \boldsymbol{\xi}_i(\bar{\mathbf{x}},t) \quad i = 1,2,\ldots,n$$

where $\boldsymbol{\theta}_i^{\mathrm{T}} = [\theta_i^1,\ldots,\theta_i^{l_i}]$ is the unknown function vector of parameters $p \in \mathcal{P}$ and $\boldsymbol{\xi}_i^{\mathrm{T}} = [\xi_i^1,\ldots,\xi_i^{l_i}]$ is the known function vector of $\bar{\mathbf{x}}$ and t.

(3) The function vector $\boldsymbol{d}(\bar{\mathbf{x}},t)$ represents any unstructured uncertainties in the system, each element of which has known bounding functions such that

$$\forall t \in [0,\infty) \quad \forall \bar{\mathbf{x}} \in \mathcal{X}$$
$$d_{i,\min}(\bar{\mathbf{x}},t) \le d_i(\bar{\mathbf{x}},t) \le d_{i,\max}(\bar{\mathbf{x}},t)$$

where $d_i(\bar{\mathbf{x}},t)$ is the ith element of the function vector $\boldsymbol{d}$. $d_{i,\min}(\bar{\mathbf{x}},t)$ and $d_{i,\max}(\bar{\mathbf{x}},t)$ are known and continuous bounding functions with respect to $\bar{\mathbf{x}}$ and t.

In this paper the inequality $A_1 \le A_2$ is defined as $\lambda_{\max}(A_1) \le \lambda_{\min}(A_2)$. $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ represent the maximum and the minimum eigenvalues, respectively. $\|A\|$ with respect to a square matrix A is the induced matrix norm, defined as

$$\|A\| = \sup\left\{\frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} \quad \text{for } \mathbf{x} \neq 0\right\}$$

Since $\|\cdot\|$ is the Euclidean norm for vectors, the corresponding induced matrix norm is

$$\|A\| = [\lambda_{\max}(A^{\mathrm{T}}A)]^{1/2}$$

for a real matrix A, and

$$\|A\| = |\lambda_{\max}(A)|$$

for a real symmetric matrix A.

2.1. *Control objective*

The control objective is to find an appropriate control input $\mathbf{u} \in \mathcal{R}^n$ for the robotic manipulator (1) such that the system state $\mathbf{x}_1(t)$ follows $\mathbf{x}_{1,\mathrm{d}}(t)$ with a prescribed accuracy ϵ as follows

$$\forall t \in [0,T_{\mathrm{f}}] \quad \|\mathbf{x}_{1,\mathrm{d}}(t) - \mathbf{x}_1(t)\| \le \epsilon \tag{2}$$

where $\mathbf{x}_{1,\mathrm{d}}$ is the desired state trajectory.

2.2. *Trajectory classification*

The desired state trajectory, which is available in control system design, can be classified into three categories, depending on whether the trajectory is repeated over a finite interval $[0,T_{\mathrm{f}}]$ or periodic over $[0,\infty)$, as well as on the alignment of the initial and terminal values.

Category I: A desired state trajectory $\mathbf{x}_{1,\mathrm{d}}(t)$, which is defined on a finite interval of time $[0,T_{\mathrm{f}}]$, is differentiable with respect to t up to the mth order, and all its higher-order derivatives are available over $t \in [0,T_{\mathrm{f}}]$.

$$\mathbf{x}_{(j+1),\mathrm{d}} = \dot{\mathbf{x}}_{j,\mathrm{d}} \tag{3}$$

Category II: In addition to the conditions of Category I, the desired trajectory of Category II satisfies the following alignment condition

$$\bar{\mathbf{x}}_{\mathrm{d}}(0) = \bar{\mathbf{x}}_{\mathrm{d}}(T_{\mathrm{f}}) \tag{4}$$

where $\bar{\mathbf{x}}_{\mathrm{d}} \triangleq [\mathbf{x}_{1,\mathrm{d}}^{\mathrm{T}},\ldots,\mathbf{x}_{m,\mathrm{d}}^{\mathrm{T}}]$.

Category III: The desired trajectory $\mathbf{x}_{1,\mathrm{d}}(t)$ of Category III is a periodic function vector with a finite period T_{f} and is smooth, or at least the mth-order derivative is continuous over $[0,\infty)$.

Remark 1: The trajectory classifications are defined for a tracking problem of an mth order non-linear system to facilitate the extension of the scheme to more general classes of non-linear systems. For the robotic dynamics, all the category classifications apply with $m = 2$.

3. Robust learning control scheme

The underlying idea of robust learning control is to learn and approximate the unknown periodic functions and suppress any unknown non-periodic functions. The learning mechanism is designed to identify all those state-independent components and leave the remaining unknowns to the robust control.

3.1. *Inverse dynamics partition*

To distinguish the required control efforts for learning and robust control respectively, the inverse model of the manipulator is needed. We define an extended tracking error which is in fact a switching surface, as

$$\boldsymbol{\sigma}(t) = \sum_{j=1}^{2} c_j[\mathbf{x}_{j,\mathrm{d}}(t) - \mathbf{x}_j(t)] \quad c_2 = 1 \tag{5}$$

where c_j $(j = 1,2)$ are coefficients of a Hurwitz polynomial.

Taking the derivative of $\boldsymbol{\sigma}$ with respect to time t and multiplying by the mass matrix yields

$$M(\mathbf{x}_1)\dot{\boldsymbol{\sigma}}(t) = M(\mathbf{x}_1)\dot{\mathbf{x}}_{2,\mathrm{d}} + M(\mathbf{x}_1)c_1\mathbf{x}_{2,\mathrm{d}} - M(\mathbf{x}_1)c_1\mathbf{x}_2 + \boldsymbol{h}(\bar{\mathbf{x}}) + \boldsymbol{d}(\bar{\mathbf{x}},t) - \mathbf{u}$$

The desired input obtained from the inverse dynamics of the system can be written as

$$\mathbf{u}(t) = M[\dot{\mathbf{x}}_{2,\mathrm{d}} + c_1\mathbf{x}_{2,\mathrm{d}} - c_1\mathbf{x}_2 - \dot{\boldsymbol{\sigma}}(t)] + \boldsymbol{h}(\bar{\mathbf{x}}) + \boldsymbol{d}(\bar{\mathbf{x}},t) \tag{6}$$

For the robotic manipulator, the inverse dynamics (6) can be rearranged as

$$\mathbf{u}(t) = A(\bar{\mathbf{x}},t)\gamma(\bar{\mathbf{x}}_\mathrm{d},t) + \mathbf{g}(\bar{\mathbf{x}},\bar{\mathbf{x}}_\mathrm{d},t) - M\dot{\sigma}(t). \tag{7}$$

$A \in \mathcal{R}^{n\times n_1}$ is a known matrix of $\bar{\mathbf{x}}$ and t. $\gamma \in \mathcal{R}^{n_1}$ is the learnable structured uncertainty which is invariant over iterations and is handled by the iterative learning control scheme. $\mathbf{g}$ is the unstructured non-periodic uncertainty function vector with a known finite upper bound and is handled by the robust control strategy. n_1 is an appropriate integer. In this arrangement, we have an option in assigning a few of the partially known terms. The terms involving $M[\dot{\mathbf{x}}_{2,\mathrm{d}} + c_1\mathbf{x}_{2,\mathrm{d}} - c_1\mathbf{x}_2]$ have known bounding function and hence can be included in $\mathbf{g}$. The same terms can also be factorized into known state-dependent functions and unknown state-independent functions and hence can be included in $A(\bar{\mathbf{x}},t)\gamma(\bar{\mathbf{x}}_\mathrm{d},t)$ also. To fully use learning control and reduce robust control efforts the second arrangement is preferred.

In detail, we have

$$\left.\begin{aligned} A(\bar{\mathbf{x}},t) &= diag(\zeta_1^\mathrm{T},\ldots,\zeta_n^\mathrm{T}) \\ \zeta_i &= [\boldsymbol{\eta}_{i,1}^\mathrm{T}\cdots\boldsymbol{\eta}_{i,n}^\mathrm{T}\vdots\boldsymbol{\eta}_{i,1}^\mathrm{T}\pi_1\cdots\boldsymbol{\eta}_{i,n}^\mathrm{T}\pi_n\vdots\xi_i^1\cdots\xi_i^{l_i}]^\mathrm{T} \\ \boldsymbol{\pi} &= [\pi_1,\ldots,\pi_n]^\mathrm{T} \triangleq -c_1\mathbf{x}_2 \end{aligned}\right\} \tag{8}$$

and correspondingly we have

$$\left.\begin{aligned} \gamma &= [\gamma_1^\mathrm{T},\ldots,\gamma_n^\mathrm{T}]^\mathrm{T} \\ \gamma_i &= [\boldsymbol{\phi}_{i,1}^\mathrm{T}\kappa_1\cdots\boldsymbol{\phi}_{i,n}^\mathrm{T}\kappa_n\vdots\boldsymbol{\phi}_{i,1}^\mathrm{T}\cdots\boldsymbol{\phi}_{i,n}^\mathrm{T}\vdots\theta_i^1\cdots\theta_i^{l_i}]^\mathrm{T} \\ \boldsymbol{\kappa} &= [\kappa_1,\ldots,\kappa_n]^\mathrm{T} \triangleq \dot{\mathbf{x}}_{2,\mathrm{d}} + c_1\mathbf{x}_{2,\mathrm{d}} \end{aligned}\right\} \tag{9}$$

The partition is arranged in such a way that all state relevant terms are assigned to the matrix A and the remaining ones to the vector γ which is to be learnt through iterations. Here again we have the flexibility of assigning the known terms $\dot{\mathbf{x}}_{2,\mathrm{d}} + c_1\mathbf{x}_{2,\mathrm{d}}(t)$ either to A or γ. For simplicity of controller construction and computation, we assign all $\bar{\mathbf{x}}_\mathrm{d}$-related terms to γ in the proposed method.

For the robotic system, the bounding function of the unstructured uncertainties is given by

$$\|\mathbf{g}\| \le l_g(\bar{\mathbf{x}},t) \triangleq \left[\sum_{j=1}^{n}(\max\{|d_{j,\min}|,|d_{j,\max}|\})^2\right]^{\frac{1}{2}} \tag{10}$$

3.2. *Iterative type RLC Algorithm*

The iterative-type robust learning control algorithm consists of two parts

$$\left.\begin{aligned} \mathbf{u}_i &= A_i\mathbf{v}_i + \mathbf{w}_i \\ A_i &\triangleq A(\bar{\mathbf{x}}_i,t) \end{aligned}\right\} \tag{11}$$

where i indicates the number of the learning trial in this and the next subsections. $\mathbf{v}_i$ is the recursive learning control part and is updated as follows

$$\mathbf{v}_i = \mathbf{v}_{i-1} + \beta_\mathrm{ff}A_{i-1}^\mathrm{T}\boldsymbol{\sigma}_{i-1} \tag{12}$$

where β_ff is a feedforward gain. $\mathbf{w}_i$ is the robust control part which can be decided through minimizing the difference of an evaluation function.

3.3. *Derivation of the robust control law*

An evaluation function approach is taken to ensure convergence of the algorithm over iterations and hence determine the appropriate robust control efforts.

The evaluation function for the learning law is Euclidean norm and L_2 norm of the learning error $\gamma - \mathbf{v}_i$.

$$E_i(t) = \int_0^t \|\gamma(\tau) - \mathbf{v}_i(\tau)\|^2\,\mathrm{d}\tau \tag{13}$$

The difference of evaluation function between two successive trials is

$$\begin{aligned} \Delta E_i(t) &= E_{i+1}(t) - E_i(t) \\ &= \int_0^t \left[(\mathbf{v}_{i+1} - \mathbf{v}_i)^\mathrm{T}(\mathbf{v}_{i+1} + \mathbf{v}_i - 2\gamma)\right]\mathrm{d}\tau \end{aligned} \tag{14}$$

Substituting the learning law (12) into equation (14) yields

$$\Delta E_i(t) = \int_0^t[\beta_\mathrm{ff}^2\boldsymbol{\sigma}_i^\mathrm{T}A_iA_i^\mathrm{T}\boldsymbol{\sigma}_i - 2\beta_\mathrm{ff}\boldsymbol{\sigma}_i^\mathrm{T}A_i(\gamma - \mathbf{v}_i)]\,\mathrm{d}\tau \tag{15}$$

On the other hand, from the inverse dynamics (7) and the control law (11) we have

$$\begin{aligned} \mathbf{u}_i &= A_i\gamma + \mathbf{g}_i - M_i\dot{\boldsymbol{\sigma}}_i(t) \\ &= A_i\mathbf{v}_i + \mathbf{w}_i \end{aligned}$$

or

$$A_i(\gamma - \mathbf{v}_i) = \mathbf{w}_i - \mathbf{g}_i + M_i\dot{\boldsymbol{\sigma}}_i(t)$$

Substituting the above relation in (15) yields

$$\begin{aligned} \Delta E_i(t) &= \int_0^t[\beta_\mathrm{ff}^2\boldsymbol{\sigma}_i^\mathrm{T}A_iA_i^\mathrm{T}\boldsymbol{\sigma}_i - 2\beta_\mathrm{ff}\boldsymbol{\sigma}_i^\mathrm{T}(\mathbf{w}_i - \mathbf{g}_i + M_i\dot{\boldsymbol{\sigma}}_i(t))]\,\mathrm{d}\tau \\ &= -\beta_\mathrm{ff}\boldsymbol{\sigma}_i^\mathrm{T}M_i\boldsymbol{\sigma}_i|_0^t \\ &\quad + \int_0^t[\beta_\mathrm{ff}^2\boldsymbol{\sigma}_i^\mathrm{T}A_iA_i^\mathrm{T}\boldsymbol{\sigma}_i - 2\beta_\mathrm{ff}\boldsymbol{\sigma}_i^\mathrm{T}(\mathbf{w}_i - \mathbf{g}_i - \tfrac{1}{2}\dot{M}_i\boldsymbol{\sigma}_i)]\,\mathrm{d}\tau \\ &\le -\beta_\mathrm{ff}\boldsymbol{\sigma}_i^\mathrm{T}(t)M_i(t)\boldsymbol{\sigma}_i(t) + \beta_\mathrm{ff}\boldsymbol{\sigma}_i^\mathrm{T}(0)M_i(0)\boldsymbol{\sigma}_i(0) \\ &\quad + \int_0^{T_\mathrm{f}}[\beta_\mathrm{ff}^2\boldsymbol{\sigma}_i^\mathrm{T}A_iA_i^\mathrm{T}\boldsymbol{\sigma}_i - 2\beta_\mathrm{ff}\boldsymbol{\sigma}_i^\mathrm{T}\mathbf{w}_i + 2\beta_\mathrm{ff}\|\boldsymbol{\sigma}_i\|\cdot\|\mathbf{g}_i\| \\ &\quad + \beta_\mathrm{ff}\|\dot{M}_i\|\cdot\|\boldsymbol{\sigma}_i\|^2]\,\mathrm{d}\tau \\ &\le -\beta_\mathrm{ff}\boldsymbol{\sigma}_i^\mathrm{T}(t)M_i(t)\boldsymbol{\sigma}_i(t) + \beta_\mathrm{ff}\boldsymbol{\sigma}_i^\mathrm{T}(0)M_i(0)\boldsymbol{\sigma}_i(0) \\ &\quad + \int_0^t[\beta_\mathrm{ff}^2\boldsymbol{\sigma}_i^\mathrm{T}A_iA_i^\mathrm{T}\boldsymbol{\sigma}_i - 2\beta_\mathrm{ff}\boldsymbol{\sigma}_i^\mathrm{T}\mathbf{w}_i + 2\beta_\mathrm{ff}l_\mathrm{g}\|\boldsymbol{\sigma}_i\| \\ &\quad + \beta_\mathrm{ff}\|\dot{M}_i\|\cdot\|\boldsymbol{\sigma}_i\|^2]\,\mathrm{d}\tau \end{aligned} \tag{16}$$

The upper bound of $\|\dot{M}\|$ can be computed on the basis of the manipulator properties as follows

$$\|\dot{M}\| \le \sum_{l=1}^{n} \left\| \frac{\mathrm{d}M}{\mathrm{d}x_{1,l}} \right\| |x_{2,l}| \le \sum_{l=1}^{n} \lambda_{l,\max} |x_{2,l}| \triangleq \rho_{BB}$$

The robust control part $\mathbf{w}_i$ can be designed so as to ensure the decay of the evaluation function to zero and hence ensure the convergence of the learning algorithm. To make ΔE_i as negative as possible, the robust control part is designed as

$$\mathbf{w}_i = \tfrac{1}{2}\beta_{\mathrm{ff}} A_i A_i^{\mathrm{T}} \boldsymbol{\sigma}_i + \left(\tfrac{1}{2}\rho_{BB} + \beta_{\mathrm{fb}}\right)\boldsymbol{\sigma}_i + l_g \mathbf{sgn}(\boldsymbol{\sigma}_i) \tag{17}$$

Here β_{fb} is a constant feedback gain. Substituting $\mathbf{w}_i$ from equation (17) into (16) we have

$$\Delta E_i(t) \le -\beta_{\mathrm{ff}} \boldsymbol{\sigma}_i^{\mathrm{T}}(t) M_i(t) \boldsymbol{\sigma}_i(t) + \beta_{\mathrm{ff}} \boldsymbol{\sigma}_i^{\mathrm{T}}(0) M_i(0) \boldsymbol{\sigma}_i(0) - 2\int_0^t \beta_{\mathrm{ff}} \beta_{\mathrm{fb}} \|\boldsymbol{\sigma}_i\|^2 \,\mathrm{d}\tau \tag{18}$$

The difference in evaluation function for the final time instant $t = T_{\mathrm{f}}$ is given by

$$\Delta E_i \triangleq \Delta E_i(T_{\mathrm{f}}) \le -\beta_{\mathrm{ff}} \boldsymbol{\sigma}_i^{\mathrm{T}}(T_{\mathrm{f}}) M_i(T_{\mathrm{f}}) \boldsymbol{\sigma}_i(T_{\mathrm{f}}) + \beta_{\mathrm{ff}} \boldsymbol{\sigma}_i^{\mathrm{T}}(0) M_i(0) \boldsymbol{\sigma}_i(0) - 2\int_0^{T_{\mathrm{f}}} \beta_{\mathrm{ff}} \beta_{\mathrm{fb}} \|\boldsymbol{\sigma}_i\|^2 \,\mathrm{d}\tau \tag{19}$$

On the basis of inequality (19), we can make the following conclusion.

Theorem 1 (iterative RLC with resetting)**:** *Assume that the initial zeroing condition $\bar{\mathbf{x}}_i(0) = \bar{\mathbf{x}}_{\mathrm{d}}(0)$ is available for all trials. Then the learning control law* (12) *and robust control law* (17) *guarantee that the robotic manipulator tracks the desired trajectory of Category I asymptotically while all state variables are globally and uniformly bounded.*

Proof: The proof is provided in Appendix A. □

Remark 2: Both learning control gain β_{ff} and feedback gain β_{fb} can be adjusted sufficiently high to achieve the fast convergence of the extended tracking error $\boldsymbol{\sigma}_i$. The upper limitation on the value is imposed by sampled data implementation.

Remark 3: In the proposed RLC scheme, the controller implementation needs neither the measurement of acceleration nor its estimation. The dynamics of most motion control systems is similar to that of the robotic manipulator and can be expressed as in (1). Hence, acceleration measurement is not required for the proposed robust learning control scheme.

Remark 4: Chattering in variable structure control arises because of the infinite gain requirement across the switching surface $\boldsymbol{\sigma} = 0$. This is due to the presence of the uncertainties $\mathbf{d}$ which may not be zero when $\boldsymbol{\sigma} = 0$. This chattering problem in variable structure control can be minimized by increasing the sampling rate. This problem can be eliminated by inserting a smoothing factor (which may be in the form of a saturation function) as explained in Slotine and Li (1991).

Remark 5: The non-repeatable uncertainties are handled by the robust control scheme and the repeatable uncertainties are learnt by the learning control scheme. If the system is strictly non-repeatable, then robust control is the only approach and there is no performance improvement over iterations. If the non-repeatable uncertainties are small, the control performance is good since the learnable components dominate. However, as the contribution of the non-repeatable uncertainties increases, the repeatable component decreases and consequently the learning effect decreases.

4. RLC without resetting and repetitive control mode

4.1. *Iterative RLC without resetting*

It has been shown in Lee and Bien (1991) that iterative learning control schemes are very sensitive to the initial zeroing condition. Incomplete resetting or non-zero initial error, no matter how small the initial error is, may result in divergence of the learning control system. Therefore, from a practical point of view, it would be more important and interesting to investigate the condition under which the resetting requirement can be removed for iterative-type learning control. This is concluded in the following theorem.

Theorem 2 (iterative RLC without resetting)**:** *Assume that the alignment of the system state variables $\bar{\mathbf{x}}_i(T_{\mathrm{f}}) = \bar{\mathbf{x}}_{i+1}(0)$ is ensured for any two consecutive trials. Then the learning control law* (12) *and robust control law* (17) *guarantee that the robotic manipulator* (1) *tracks the desired trajectory of Category II asymptotically while all state variables are globally and uniformly bounded.*

Proof: The proof is provided in Appendix B. □

Remark 6: The assumption imposed on the initial and terminal system states $\bar{\mathbf{x}}_i(T_{\mathrm{f}}) = \bar{\mathbf{x}}_{i+1}(0)$ for any two consecutive trials is very reasonable for most motion control systems as the final position of the previous trial naturally becomes the initial position of the new trial. As a consequence, we can remove the resetting mechanism, which is indispensable for conventional ILC schemes. ILC with resetting can be used when an accurate resetting mechanism is available. If initial resetting ($\mathbf{e}(0) = 0$) cannot be ensured, then it is suggested that the problem be handled at the task planning level by making use of ILC with alignment condition.

4.2. *Repetitive type RLC scheme*

In the case where the control process cannot be ended or the desired trajectory requires continuous operation, repetitive learning control is the only appropriate method which is able to control and *learn* to improve the system performance, as far as the periodic tracking problem is concerned. In such case a necessary condition for learning control is that, all the unknown functions to be learned, namely those unknown state-independent functions γ in (7), must be either constant or periodic with period T_{f}. Besides, it is necessary that the state-dependent functions in vector $\mathbf{g}$ of (7) are uniformly upper bounded with respect to t in (10) if they are explicit time functions. It is easy to verify that robotic dynamics meets the above conditions.

Now we are in the position to show that the developed RLC scheme can work in repetitive control mode.

Theorem 3 (repetitive RLC): *Design learning control law (20) and robust control law (21), which are now defined over* $[0, \infty)$, *as follows*

$$\mathbf{v}(t) = \begin{cases} 0 & \text{if } t < T_{\mathrm{f}} \\ \mathbf{v}(t - T_{\mathrm{f}}) + \beta_{\mathrm{ff}} A^{\mathrm{T}}(\cdot, t - T_{\mathrm{f}})\boldsymbol{\sigma}(t - T_{\mathrm{f}}) & \text{otherwise} \end{cases} \tag{20}$$

$$\mathbf{w}(t) = \frac{\beta_{\mathrm{ff}}}{2} A(\cdot, t) A(\cdot, t)^{\mathrm{T}} \boldsymbol{\sigma}(t) + l_g \mathbf{sgn}(\boldsymbol{\sigma}(t)) + \left(\tfrac{1}{2}\rho_{BB}(t) + \beta_{\mathrm{fb}}\right)\boldsymbol{\sigma}(t) \tag{21}$$

where $A(\cdot, t) \triangleq A(\bar{\mathbf{x}}(t), t)$. *Then the repetitive-type RLC guarantees that the robotic manipulator* (1) *tracks the desired trajectory of Category III asymptotically while all state variables are globally and uniformly bounded.*

Proof: The proof is provided in Appendix C. □

5. Extension to a class of non-linear high-order systems

Consider a class of higher-order MIMO non-linear dynamical uncertain systems described by

$$\left.\begin{aligned} \dot{\mathbf{x}}_j &= \mathbf{x}_{j+1} \\ \dot{\mathbf{x}}_m &= \boldsymbol{f}(\bar{\mathbf{x}}, \mathbf{p}, t) + \boldsymbol{h}(\bar{\mathbf{x}}, \mathbf{p}, t) + \boldsymbol{d}(\bar{\mathbf{x}}, \mathbf{p}, t, \omega) + B(\mathbf{z}, \mathbf{p}, t)\mathbf{u} \end{aligned}\right\} \tag{22}$$

where $\mathbf{x}_j \in \mathcal{R}^{n\times 1}$, $j = 1, \ldots, m$; $m, n \in Z_+$; $\bar{\mathbf{x}} \triangleq [\mathbf{x}_1^{\mathrm{T}}, \mathbf{x}_2^{\mathrm{T}}, \ldots, \mathbf{x}_m^{\mathrm{T}}]^{\mathrm{T}} \in \mathcal{X} \subseteq \mathcal{R}^{nm\times 1}$ is the measurable state vector of the system. $\mathbf{p} \in \mathcal{P}$ is an unknown system parameter vector. $\mathcal{P}$ is the set of admissible system parameters. $\boldsymbol{d}(\mathbf{x}, \mathbf{p}, t, \omega)$ is also a function of ω, where ω represents any aperiodic factors such as aperiodic exogenous disturbance or system noise. $\mathbf{u} \in \mathcal{R}^{n\times 1}$ is the control input vector of the system. $B(\mathbf{z}, \mathbf{p}, t) \in \mathcal{R}^{n\times n}$ is the input distribution matrix. $\mathbf{z} \triangleq [\mathbf{x}_1^{\mathrm{T}}, \mathbf{x}_2^{\mathrm{T}}, \ldots, \mathbf{x}_{m-1}^{\mathrm{T}}]^{\mathrm{T}} \in \mathcal{Z}$, where $\mathcal{Z}$ is a subset of the state space $\mathcal{X}$ with dimension $q = n \times (m - 1)$.

In this paper, we make the following assumptions about the system.

Assumption 1: *The unknown non-linear function vector* $\boldsymbol{f}$ *satisfies the following Lipschitz-like condition*

$$\forall t \in [0, \infty) \quad \forall \bar{\mathbf{x}}_{\mathrm{d}}, \bar{\mathbf{x}} \in \mathcal{X} \quad \forall p \in \mathcal{P}$$

$$\|\boldsymbol{f}(\bar{\mathbf{x}}_{\mathrm{d}}, \mathbf{p}, t) - \boldsymbol{f}(\bar{\mathbf{x}}, \mathbf{p}, t)\| \le l_f(\bar{\mathbf{x}}_{\mathrm{d}}, \bar{\mathbf{x}}, t)\|\bar{\mathbf{x}}_{\mathrm{d}} - \bar{\mathbf{x}}\|$$

where $l_f(\bar{\mathbf{x}}_{\mathrm{d}}, \bar{\mathbf{x}}, t)$ *is a known and continuous scalar bounding function with respect to all its arguments.*

Note that l_f becomes a Lipschitz constant if it is invariant with respect to the arguments.

Assumption 2: *Each element of the unknown function vector* $\boldsymbol{h}$ *can be expressed as*

$$\boldsymbol{\theta}_i^{\mathrm{T}}(\mathbf{p}, t)\boldsymbol{\xi}_i(\bar{\mathbf{x}}, t), \quad i = 1, 2, \ldots, n$$

where $\boldsymbol{\theta}_i^{\mathrm{T}} = [\theta_i^1, \ldots, \theta_i^{l_i}]$ *is the unknown function vector of* $\mathbf{p}$ *and t; and* $\boldsymbol{\xi}_i^{\mathrm{T}} = [\xi_i^1, \ldots, \xi_i^{l_i}]$ *is the known function vector of* $\bar{\mathbf{x}}$ *and t.*

Assumption 3: *The non-linear function vector* $\boldsymbol{d}$ *is bounded such that*

$$\forall t \in [0, \infty) \ \forall \bar{\mathbf{x}} \in \mathcal{X} \ \forall p \in \mathcal{P}$$

$$d_{i,\min}(\bar{\mathbf{x}}, t) \le d_i(\bar{\mathbf{x}}, \mathbf{p}, t) \le d_{i,\max}(\bar{\mathbf{x}}, t)$$

where $d_i(\bar{\mathbf{x}}, t)$ *is the ith element of the function vector* $\boldsymbol{d}$. $d_{i,min}(\bar{\mathbf{x}}, t)$ *and* $d_{i,max}(\bar{\mathbf{x}}, t)$ *are known and continuous bounding functions with respect to* $\bar{\mathbf{x}}$ *and t.*

Assumption 4: *The input distribution matrix B is positive definite for all* $t \in [0, \infty)$, $\mathbf{z} \in \mathcal{Z}$, $\mathbf{p} \in \mathcal{P}$ *and satisfies the following inequalities* $\forall t \in [0, \infty)$

$$0 < \lambda_{\min} I \le B$$

$$\left\|\frac{\partial B}{\partial z_l}\right\| \le \lambda_{l,\max} \quad l = 1, \ldots, q$$

where $\lambda_{\min}$ *and* $\lambda_{l,\max}$ *are known positive constants. Each element of the matrix* B^{-1} *can be expressed as*

$$\boldsymbol{\phi}_{ij}^{\mathrm{T}}(\mathbf{p}, t)\boldsymbol{\eta}_{ij}(\mathbf{z}, t) \quad i, j = 1, 2, \ldots, n$$

where $\boldsymbol{\phi}_{ij}^{\mathrm{T}} = [\phi_{ij}^1, \ldots, \phi_{ij}^{l_{ij}}]$ *is the unknown function vector of* $\mathbf{p}$ *and t;* $\boldsymbol{\eta}_{ij}^{\mathrm{T}} = [\eta_{ij}^1, \ldots, \eta_{ij}^{l_{ij}}]$ *is the vector of known non-linear functions of* $\mathbf{z}$ *and t.*

Remark 7: The purpose of expressing the non-linear uncertainties $\boldsymbol{f}$, $\boldsymbol{h}$ and $\boldsymbol{d}$ separately in equation (22) is to show clearly how robust control and learning control work in a complementary manner and henceforth increase the application range. Most existing learning control or robust control methods (including VSC) fail to work for the plant (22). Conventional ILC methods could only apply to the vector $\boldsymbol{f}$ which satisfies the Lipschitz condition. VSC is effective only for the vector $\boldsymbol{d}$ associated with known bounding functions. Note that $\boldsymbol{d}$ may include non-periodic arguments. Besides, neither VSC nor ILC can process the vector $\boldsymbol{h}$, which is highly non-linear, and the size of the uncertainties is not available.

5.1. *Modelling of inverse dynamics*

For the mth-order dynamic system, define an extended tracking error as

$$\sigma(t) = \sum_{j=1}^{m} c_j[\mathbf{x}_{j,\mathrm{d}}(t) - \mathbf{x}_j(t)] \quad c_m = 1, \tag{23}$$

in which c_j $(j = 1, \ldots, m)$ are coefficients of a Hurwitz polynomial.

Taking the derivative of σ with respect to time t yields

$$\dot{\sigma}(t) = \sum_{j=1}^{m} c_j \mathbf{x}_{(j+1),d} - \sum_{j=1}^{m-1} c_j \mathbf{x}_{j+1} - \boldsymbol{f}(\bar{\mathbf{x}}, \mathbf{p}, t)$$
$$- \boldsymbol{h}(\bar{\mathbf{x}}, \mathbf{p}, t) - \boldsymbol{d}(\bar{\mathbf{x}}, \mathbf{p}, t, \omega) - B(\mathbf{z}, \mathbf{p}, t)\mathbf{u}$$

In order to partition the system uncertainties into factors of known state-dependent functions, unknown state-independent functions and unknown state-dependent functions with known bounds, rearrange the above error dynamics as follows

$$\mathbf{u} = B^{-1}(\mathbf{z}, \mathbf{p}, t)\left[\sum_{j=1}^{m} c_j \mathbf{x}_{(j+1),\mathrm{d}} - \sum_{j=1}^{m-1} c_j \mathbf{x}_{j+1}\right.$$
$$\left. - \boldsymbol{f}(\bar{\mathbf{x}}_{\mathrm{d}}, \mathbf{p}, t) - \boldsymbol{h}(\bar{\mathbf{x}}, \mathbf{p}, t)\right]$$
$$+ B^{-1}(\mathbf{z}, \mathbf{p}, t)[\boldsymbol{f}(\bar{\mathbf{x}}_{\mathrm{d}}, \mathbf{p}, t) - \boldsymbol{f}(\bar{\mathbf{x}}, \mathbf{p}, t)$$
$$- \boldsymbol{d}(\bar{\mathbf{x}}, \mathbf{p}, t, \omega)] - B^{-1}(\mathbf{z}, \mathbf{p}, t)\dot{\sigma}(t)$$
$$= A(\bar{\mathbf{x}}, t)\gamma(\bar{\mathbf{x}}_{\mathrm{d}}, t) + \mathbf{g}(\bar{\mathbf{x}}, \bar{\mathbf{x}}_{\mathrm{d}}, t) - B^{-1}\dot{\sigma}(t) \tag{24}$$

Hence the inverse dynamics for some classes of non-linear systems can be rearranged in the same form as equation (7). In detail, we have

$$A(\bar{\mathbf{x}}, t) = diag(\zeta_1^{\mathrm{T}}, \ldots, \zeta_n^{\mathrm{T}})$$
$$\zeta_i = [\boldsymbol{\eta}_{i,1}^{\mathrm{T}} \cdots \boldsymbol{\eta}_{i,n}^{\mathrm{T}} \vdots \boldsymbol{\eta}_{i,1}^{\mathrm{T}}\pi_1 \cdots \boldsymbol{\eta}_{i,n}^{\mathrm{T}}\pi_n \vdots \boldsymbol{\eta}_{i,1}^{\mathrm{T}}\xi_1^1$$
$$\cdots \boldsymbol{\eta}_{i,1}^{\mathrm{T}}\xi_1^{l_1} \vdots \boldsymbol{\eta}_{i,2}^{\mathrm{T}}\xi_2^1 \cdots\cdots \vdots \boldsymbol{\eta}_{i,n}^{\mathrm{T}}\xi_n^1 \cdots \boldsymbol{\eta}_{i,n}^{\mathrm{T}}\xi_n^{l_n}]^{\mathrm{T}}$$
$$\pi = [\pi_1 \cdots \pi_n]^{\mathrm{T}} \triangleq -\sum_{j=1}^{m-1} c_j \mathbf{x}_{j+1}$$

and correspondingly we have

$$\gamma = [\gamma_1^{\mathrm{T}} \cdots \gamma_n^{\mathrm{T}}]^{\mathrm{T}} \tag{25}$$
$$\gamma_i(\mathbf{x}_{\mathrm{d}}, \mathbf{p}, t) = [\boldsymbol{\phi}_{i,1}^{\mathrm{T}}\kappa_1 \cdots \boldsymbol{\phi}_{i,n}^{\mathrm{T}}\kappa_n \vdots \boldsymbol{\phi}_{i,1}^{\mathrm{T}} \cdots \boldsymbol{\phi}_{i,n}^{\mathrm{T}} \vdots \boldsymbol{\phi}_{i,1}^{\mathrm{T}}\theta_1^1$$
$$\cdots \boldsymbol{\phi}_{i,1}^{\mathrm{T}}\theta_1^{l_1} \vdots \boldsymbol{\phi}_{i,2}^{\mathrm{T}}\theta_2^1 \cdots\cdots \vdots \boldsymbol{\phi}_{i,n}^{\mathrm{T}}\theta_n^1 \cdots \boldsymbol{\phi}_{i,n}^{\mathrm{T}}\theta_n^{l_n}]^{\mathrm{T}}$$
$$\boldsymbol{\kappa} = [\kappa_1 \cdots \kappa_n]^{\mathrm{T}} \triangleq \sum_{j=1}^{m}[c_j \mathbf{x}_{(j+1),\mathrm{d}}] - \mathbf{f}(\mathbf{x}_{\mathrm{d}}, \mathbf{p}, t)$$

The partition is arranged in such a way that all state-relevant terms are assigned to the matrix A, and the remaining ones to the vector γ which is to be learned through iterations.

To design variable structure control it is necessary to find the bounding functions of the vector $\mathbf{g}$ which are functions of $\bar{\mathbf{x}}_{\mathrm{d}}$, $\bar{\mathbf{x}}$, $\mathbf{p}$, t and ω. Taking the Euclidean norm to both sides of $\mathbf{g}$,

$$\|\mathbf{g}\| \le \|B^{-1}(\mathbf{z}, \mathbf{p}, t)\| \cdot [\|\mathbf{f}(\bar{\mathbf{x}}_{\mathrm{d}}, \mathbf{p}, t) - \mathbf{f}(\bar{\mathbf{x}}, \mathbf{p}, t)\|$$
$$+ \|\mathbf{d}(\bar{\mathbf{x}}, \mathbf{p}, t, \omega)\|]$$

From Assumption 1

$$\|\mathbf{f}(\bar{\mathbf{x}}_{\mathrm{d}}, \mathbf{p}, t) - \mathbf{f}(\bar{\mathbf{x}}, \mathbf{p}, t)\| \le l_f(\bar{\mathbf{x}}_{\mathrm{d}}, \bar{\mathbf{x}}, t)\|\bar{\mathbf{x}}_{\mathrm{d}} - \bar{\mathbf{x}}\|$$

From Assumption 3

$$\|\mathbf{d}(\bar{\mathbf{x}}, \mathbf{p}, t, \omega)\| \le l_d(\bar{\mathbf{x}}, t)$$
$$\triangleq \left[\sum_{i=1}^{n}\left(\max\{|d_{i,\min}|, |d_{i,\max}|\}\right)^2\right]^{1/2}$$

From Assumption 4, we have

$$0 \le B^{-1} \le \lambda_{\min}^{-1} I$$

and hence

$$\|B^{-1}(\mathbf{z}, \mathbf{p}, t)\| \le \max_i |\lambda_i(B^{-1})| \le |\lambda_{\min}^{-1}| \triangleq l_B(\mathbf{z}, t)$$

Finally, the bounding function of the system uncertainty $\mathbf{g}$ is

$$\|\mathbf{g}\| \le l_g(\bar{\mathbf{x}}_{\mathrm{d}}, \bar{\mathbf{x}}, t)$$
$$\triangleq l_B(\mathbf{z}, t)[l_f(\mathbf{x}_{\mathrm{d}}, \mathbf{x}, t)\|\mathbf{x}_{\mathrm{d}} - \mathbf{x}\| + l_d(\mathbf{x}, t)] \tag{26}$$

Remark 8: The system (22) represents classes of non-linear systems which include robotics as a subset. The lipschitz continuous function $\boldsymbol{f}$ can be handled alone by conventional ILC schemes. However, in the presence of $\boldsymbol{h}$ and $\boldsymbol{d}$, handling $\boldsymbol{f}$ requires a completely new approach. The non-linear uncertain system can have unknown explicit time-varying functions in $\boldsymbol{\phi}$ and θ, unlike the robotic manipulator whose parameter vector is independent of time. The known state-dependent functions in ξ_i can include highly non-linear terms and are not limited to quadratic terms as in (1).

5.2. *Robust learning control law*

The learning control law (12) and the robust control law (17) are as defined in §4. The parameters in the control laws and for all subsequent discussions in this section are defined as A_i by equation (25), l_g by equation (26), σ_i by equation (23), ρ_{BB} is the upper bound of $\|B^{-1}\dot{B}B^{-1}\|$, to be derived, and β_{fb} is a constant feedback gain. Again in this subsection we use the subscript 'i' to denote the learning iteration.

Defining an evaluation function as in equation (13) and proceeding in the same way, we can derive that

$$\begin{aligned}\Delta E_i \le & -\beta_{\mathrm{ff}}\boldsymbol{\sigma}_i^{\mathrm{T}}(T_{\mathrm{f}})B_i^{-1}(\mathbf{z}(T_{\mathrm{f}}),T_{\mathrm{f}})\boldsymbol{\sigma}_i(T_{\mathrm{f}}) \\ & + \beta_{\mathrm{ff}}\boldsymbol{\sigma}_i^{\mathrm{T}}(0)B_i^{-1}(\mathbf{z}(0),0)\boldsymbol{\sigma}_i(0) \\ & - 2\int_0^{T_{\mathrm{f}}} \beta_{\mathrm{ff}}\beta_{\mathrm{fb}}\|\boldsymbol{\sigma}_i\|^2\,\mathrm{d}\tau \qquad (27)\end{aligned}$$

The upper bound of $\|B^{-1}\dot{B}B^{-1}\|$ is derived as follows

$$B_i^{-1}\dot{B}_iB_i^{-1} = B_i^{-1}\sum_{l=1}^{q}\frac{\partial B_i}{\partial z_l}\dot{z}_lB_i^{-1}$$

From the matrix norm property we have

$$\begin{aligned}\|B_i^{-1}\dot{B}_iB_i^{-1}\| &\le \|B_i^{-1}\|^2\cdot\|\dot{B}_i\| \\ &\le \|B_i^{-1}\|^2\sum_{l=1}^{q}\left\|\frac{\partial B_i}{\partial z_l}\right\|\cdot|\dot{z}_l| \\ &\le |\lambda_{\min}^{-1}|^2\sum_{l=1}^{q}|\lambda_{l,\max}|\cdot|\dot{z}_l| \triangleq \rho_{BB} \qquad (28)\end{aligned}$$

Remark 9: In the calculation of the bounding function ρ_{BB} in (28), the derivative signal of the system state $\dot{z}_l$ is needed. It is then easy to observe that, if $\mathbf{x}_n$ is not included in $\mathbf{z}$, all components of $\dot{\mathbf{z}}$ are in fact measurable system states. Most iterative learning control schemes require the measurement (or estimation) of derivatives of the system states. The derivative signals of the system states are not needed if the proposed RLC scheme is used.

On the basis of equation (27), we can make the following conclusions.

Theorem 4 (iterative RLC with resetting): *Assume that the initial zeroing condition* $\bar{\mathbf{x}}_i(0) = \bar{\mathbf{x}}_{\mathrm{d}}(0)$ *is available for all trials, the learning control law* (12) *and robust control law* (17) *guarantee that the non-linear system* (22) *tracks the desired trajectory of Category I asymptotically while all state variables are globally and uniformly bounded.*

Proof: The proof is similar to that of Theorem 1. □

For RLC without resetting, the following assumption is needed.

Assumption 5: *For any element of the matrix B which is an explicit function of t, it is also a periodic function with period* T_{f}, *that is*

$$B(\mathbf{z},\mathbf{p},0) = B(\mathbf{z},\mathbf{p},T_{\mathrm{f}}) \qquad (29)$$

Theorem 5 (iterative RLC without resetting)**:** *Assume that the alignment of the system state variables* $\bar{\mathbf{x}}_i(T_{\mathrm{f}}) = \bar{\mathbf{x}}_{i+1}(0)$ *is ensured for any two consecutive trials and that Assumption 5 is satisfied; then learning control law* (12) *and robust control law* (17) *guarantee that the non-linear system* (22) *tracks the desired trajectory of Category II asymptotically while all state variables are globally and uniformly bounded.*

Proof: The proof is provided in Appendix D. □

Remark 10: If B matrix in (22) is autonomous, namely no explicit time function in B, as in most motion control systems including robotic dynamics, then the periodicity Assumption 5 is not necessary.

To extend the repetitive-type RLC for the non-linear system (22), it is necessary that γ should consist of constants or periodic components only. Also, $\|\mathbf{g}\|$ should be uniformly upper bounded by a positive gain l_g.

Assumption 6: *Any explicit time function in* $\boldsymbol{f}$, $\boldsymbol{h}$ *and B of* (22) *should be periodic and bounded.*

Theorem 6 (repetitive RLC)**:** *For learning control law* (20) *and robust control law* (21), *under Assumption 6, the non-linear system* (22) *tracks the desired trajectory of Category III asymptotically while all state variables are globally and uniformly bounded.*

Proof: By segmenting $[0,\infty)$ into a series of even intervals $[iT_{\mathrm{f}},(i+1)T_{\mathrm{f}}]$, $i = 0,1,\cdots$, and proceeding as in Theorem 3, the global and asymptotic convergence of the tracking error can be proved. □

6. Illustrative example

In this section, the following two-link robotic manipulator is considered.

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22}\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1 \\ \ddot{\theta}_2\end{bmatrix} + \begin{bmatrix} -2h\dot{\theta}_1\dot{\theta}_2 - h\dot{\theta}_2^2 + g_1 + f_c + f_v\dot{\theta}_1 \\ h\dot{\theta}_1^2 + g_2 + f_c + f_v\dot{\theta}_2\end{bmatrix} + \begin{bmatrix} d_1 \\ d_2\end{bmatrix} = \begin{bmatrix} u_1 \\ u_2\end{bmatrix}$$

with $\boldsymbol{\theta} = [\theta_1, \theta_2]^{\top}$ being the two joint angles, $\mathbf{u} = [u_1, u_2]^{\mathrm{T}}$ being the joint inputs and

$$
\begin{aligned}
m_{11} &= m_1 l_{c1}^2 + I_1 + m_2[l_1^2 + l_{c2}^2 + 2l_1 l_{c2}\cos(\theta_2) + I_2] \\
&\triangleq b_1 + b_2\cos(\theta_2) \\
m_{22} &= m_2 l_{c2}^2 + I_2 \triangleq b_3 \\
m_{12} &= m_{21} = m_2 l_1 l_{c2}\cos(\theta_2) + m_2 l_{c2}^2 + I_2 \\
&\triangleq b_3 + b_4\cos(\theta_2) \\
h &= m_2 l_1 l_{c2} sin(\theta_2) \triangleq b_4 sin(\theta_2) \\
g_1 &= m_1 l_{c1} g\cos(\theta_1) + m_2 g[l_{c2}\cos(\theta_1+\theta_2) + l_1\cos(\theta_1)] \\
&\triangleq b_5\cos(\theta_1) + b_6\cos(\theta_1+\theta_2) \\
g_2 &= m_2 l_{c2} g\cos(\theta_1+\theta_2) \triangleq b_6\cos(\theta_1+\theta_2)
\end{aligned}
$$

In the model, the second term represents all centrifugal, Coriolis, gravity , coulomb and viscous friction terms ($\boldsymbol{h}$). The friction for each link can be represented by a Gaussian model (Brian 1991) as $f_c + (f_s - f_c)e^{-|\dot{\theta}/\theta_s|^{\delta_s}} + f_v\dot{\theta}$. The unstructured uncertainties hence consist of the non-parametric friction terms

$$
\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} (f_s - f_c)\mathrm{e}^{-|\frac{\dot{\theta}_1}{\theta_s}|^{\delta_s}} \\ (f_s - f_c)\mathrm{e}^{-|\frac{\dot{\theta}_2}{\theta_s}|^{\delta_s}} \end{bmatrix}
$$

The pairs $\{m_1, m_2\}$, $\{I_1, I_2\}$, $\{l_1, l_2\}$ and $\{l_{c1}, l_{c2}\}$ are the masses, moments of inertia , lengths and center of gravity co-ordinates of the two robotic arms respectively. The coefficients are appropriate unknown constants. The desired trajectory for tracking is given by

$$
\begin{aligned}
\theta_{\mathrm{d},1} &= \theta_0\sin^3(\pi\tau) - \frac{\pi}{2} \\
\theta_{\mathrm{d},2} &= \theta_0\sin^3(\pi\tau)
\end{aligned}
$$

where $\tau = t/T_{\mathrm{f}}$. The parameters are chosen as follows: $m_1 = 4\,\mathrm{kg}$, $m_2 = 3\,\mathrm{kg}$, $l_1 = 0.5\,\mathrm{m}$, $l_2 = 0.5\,\mathrm{m}$, $l_{c1} = 0.3\,\mathrm{m}$, $l_{c2} = 0.25\,\mathrm{m}$, $I_1 = 0.4\,\mathrm{kg.m^2}$, $I_2 = 0.25\,\mathrm{kg.m^2}$, $f_c = 3.5$, $f_s = 3.65$, $f_v = 1.06$, $\theta_s = 0.1$, $\delta_s = 0.05$, $\theta_0 = 10^{\circ}$, $\epsilon = 0.05'$ and $T_{\mathrm{f}} = 1\,\mathrm{s}$. The desired trajectory belongs to Category 1. By defining the same trajectory over the entire time period, it can be seen that the trajectory satisfies Category 3 also.

In the following, we illustrate the detailed procedure of controller design. The switching surface is chosen as in equation (5) with $c_1 = 2$. On the basis of the inverse dynamics partition and the factoring of the structured uncertainties, we have

$$
\zeta_1^{\mathrm{T}}\gamma_1 = \begin{bmatrix} 1 \\ \cos(\theta_2) \\ \pi_1 \\ \pi_1\cos(\theta_2) \\ \pi_2 \\ \pi_2\cos(\theta_2) - \sin(\theta_2)(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \\ \cos(\theta_1) \\ \cos(\theta_1+\theta_2) \\ \dot{\theta}_1 \end{bmatrix}^{\mathrm{T}} \cdot \begin{bmatrix} b_1\kappa_1 + b_3\kappa_2 + f_c \\ b_2\kappa_1 + b_3\kappa_2 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ f_v \end{bmatrix} \tag{30}
$$

$$
\zeta_2^{\mathrm{T}}\gamma_2 = \begin{bmatrix} 1 \\ \cos(\theta_2) \\ \pi_1 + \pi_2 \\ \pi_1\cos(\theta_2) + \sin(\theta_2)\dot{\theta}_1^2 \\ \cos(\theta_1+\theta_2) \\ \dot{\theta}_2 \end{bmatrix}^{\mathrm{T}} \cdot \begin{bmatrix} b_3(\kappa_1+\kappa_2) + f_c \\ b_4\kappa_1 \\ b_3 \\ b_4 \\ b_6 \\ f_v \end{bmatrix} \tag{31}
$$

where

$$
\kappa = \begin{bmatrix} \ddot{\theta}_{\mathrm{d},1} + c\dot{\theta}_{\mathrm{d},1} \\ \ddot{\theta}_{\mathrm{d},2} + c\dot{\theta}_{\mathrm{d},2} \end{bmatrix} \quad \pi = \begin{bmatrix} -c_1\dot{\theta}_1 \\ -c_1\dot{\theta}_2 \end{bmatrix} \tag{32}
$$

The bound of each element of the unstructured uncertainties (friction terms) can be calculated as

$$
\|g_i\| = f_s - f_c \tag{33}
$$

The upper bound of the unstructured uncertainties is hence calculated as $l_g = (g_1^2 + g_2^2)^{0.5}$. The mass matrix is a constant with respect to θ_1 and hence $\lambda_{1,\max} = 0$. Taking the derivative of the mass matrix with respect to θ_2, we have $\lambda_{2,\max} = (b_2^2 + 4b_4^2)^{0.5} = 0.88$. An eigenvalue of $\lambda_{2,\max} = 1.2$ was chosen for simulation due to the uncertainties in the parameter knowledge. It can be observed that acceleration measurement is not required for the RLC scheme.

A potential problem of VSC is its chattering phenomenon. To reduce chattering, the switching function $sgn(\sigma)$ can be replaced by the following continuous saturation function

$$sat(\sigma) = \left\{ \begin{array}{cc} sgn(\sigma) & \text{if } |\sigma| > \delta \\ \dfrac{\sigma}{\delta} & \text{otherwise} \end{array} \right\} \tag{34}$$

To meet the control specification $\|\theta_{1,\mathrm{d}} - \theta_1\| \le \epsilon$, it is necessary to find the bound of the quantity δ. Since the desired tracking accuracy $\epsilon = 0.05$, the bound is $\delta = c_1 \epsilon = 0.1$. For the simulation we chose $\beta_{\mathrm{ff}} = 10$, $\beta_{\mathrm{fb}} = 10$ and a sampling period of 1 ms.

6.1. *RLC with resetting*

The simulation is performed for the iterative robust learning control mode with resetting condition. Figures 1 and 2 illustrate the convergence of the maximum tracking error for the first link (θ_1) and second link (θ_2) respectively over iterations. The control scheme ensures the convergence of the extended tracking error and consequently the convergence of the system states. However, parameter convergence in terms of the individual components of $\mathbf{v}_i$ cannot be ensured without additional assumptions such as persistent excitation condition. The simulation results further verify the fact that tracking error convergence can be achieved even though parameter convergence cannot be ensured.

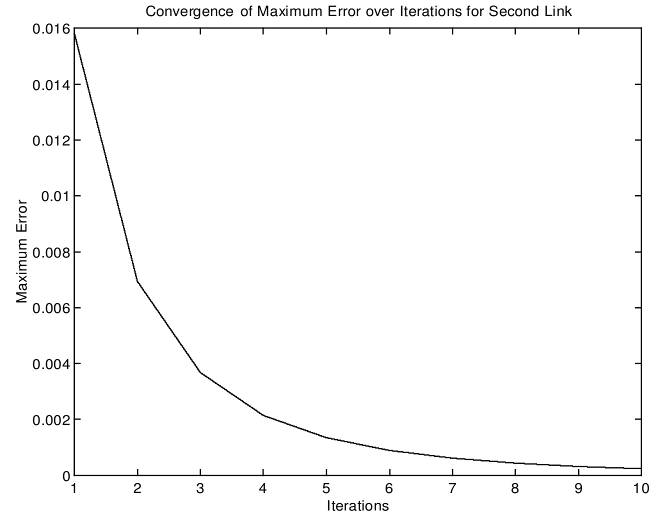

Figure 2. Maximum value of θ_2 tracking error over iterations (RLC in iterative mode with resetting).

6.2. *RLC in repetitive mode*

The RLC is then applied in repetitive control mode for the periodic trajectory of Category 3. The convergence of the maximum tracking error for the first link (θ_1) and the second link (θ_2) are given by figures 3 and 4 respectively. The iteration axis in these figures stands for each period of the repetitive action. The maximum tracking error in each periodic operation interval is plotted against the iteration axis.

It is confirmed that the effectiveness of the new robust learning control scheme is explicit for the purpose of tracking control in the presence of such high system non-linearities and uncertainties.

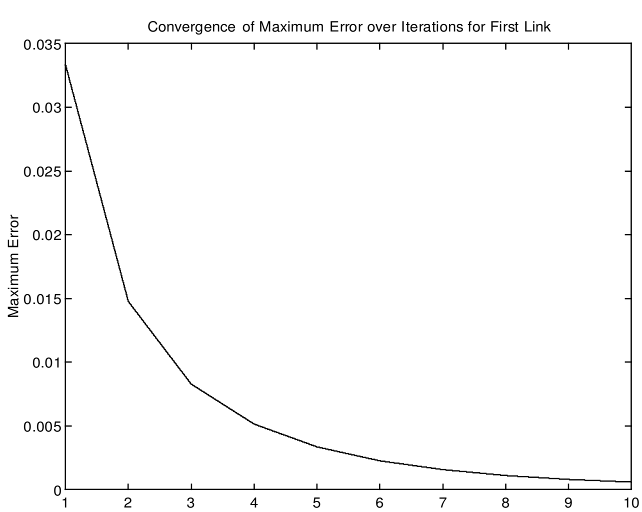

Figure 1. Maximum value of θ_1 tracking error over iterations (RLC in iterative mode with resetting).

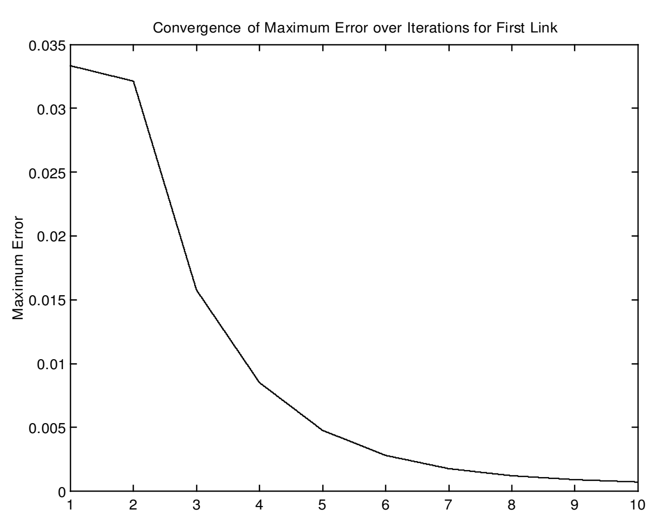

Figure 3. Maximum value of θ_1 tracking error over each interval of operation (RLC in repetitive mode).

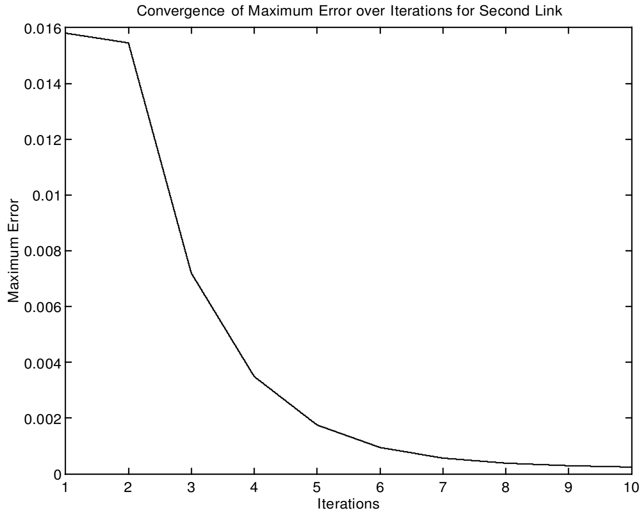

Figure 4. Maximum value of θ_2 tracking error over each interval of operation (RLC in repetitive mode).

7. Conclusion

In this paper a new control scheme, the robust learning control scheme, is developed by incorporating variable structure control approaches into the learning control system. The proposed RLC system possesses both learning and robustness properties, and thereby is able to handle robotic systems as well as certain classes of non-linear and uncertain dynamic systems. The robust learning control system illustrates the capability of working in either iterative or repetitive control mode with respect to the different control objectives. Theoretical analyses and substantial discussions have been presented to disclose the inherent relationships between the plant non-linearities and uncertainties, objective trajectory categories, resetting condition, use of derivative signals and learning control modes.

Appendix A: Proof of Theorem 1

Under the resetting condition

$$\sigma_i^{\mathrm{T}}(0)M_i(0)\sigma_i(0) = 0 \quad i = 1, 2, \ldots$$

is ensured. It follows from the relationship (19) that

$$\Delta E_i \le -\beta_{\mathrm{ff}}\sigma_i^{\mathrm{T}}(T_{\mathrm{f}})M_i(T_{\mathrm{f}})\sigma_i(T_{\mathrm{f}}) - 2\int_0^{T_{\mathrm{f}}} \beta_{\mathrm{ff}}\beta_{\mathrm{fb}}\|\sigma_i\|^2\,\mathrm{d}\tau \tag{A1}$$

which is negative definite when $\sigma_i(t) \ne 0, t \in [0, T_{\mathrm{f}}]$. Now, taking the summation of ΔE_i up to k yields

$$\sum_{i=0}^{k} \Delta E_i = E_{k+1} - E_0$$
$$\le -\sum_{i=0}^{k} \beta_{\mathrm{ff}}\sigma_i^{\mathrm{T}}(T_{\mathrm{f}})M_i(T_{\mathrm{f}})\sigma_i(T_{\mathrm{f}})$$
$$-\sum_{i=0}^{k} 2\beta_{\mathrm{ff}}\beta_{\mathrm{fb}}\int_0^{T_{\mathrm{f}}} \|\sigma_i\|^2\,\mathrm{d}\tau$$

Consequently we have

$$\sum_{i=0}^{k}\int_0^{T_{\mathrm{f}}} \|\sigma_i\|^2\,\mathrm{d}\tau \le E_0/(2\beta_{\mathrm{ff}}\beta_{\mathrm{fb}}) < \infty$$

which leads to

$$\lim_{i\to\infty}\int_0^{T_{\mathrm{f}}} \|\sigma_i\|^2\,\mathrm{d}\tau = 0$$

Since the extended tracking error dynamics (5) is selected to be stable, $\sigma_i = 0$ and the initial resetting condition ($\bar{\mathbf{x}}_{\mathrm{d}}(0) = \bar{\mathbf{x}}_i(0)$) ensures the global convergence of $\bar{\mathbf{x}}_i$ to $\bar{\mathbf{x}}_{\mathrm{d}}$ asymptotically. The finiteness of $E_i(T_{\mathrm{f}})$ is ensured by the negative definiteness of ΔE_i. Since $E_i(t) \le E_i(T_{\mathrm{f}})$, the finiteness of $E_i(t)$ is also ensured. Suppose the tracking error has a finite escape time $t_{\mathrm{e}} \in [0, T_{\mathrm{f}}]$, from equation (18)

$$\Delta E_i(t_{\mathrm{e}}) \le -\beta_{\mathrm{ff}}\sigma_i^{\mathrm{T}}(t_{\mathrm{e}})B_i^{-1}(t_{\mathrm{e}})\sigma_i(t_{\mathrm{e}}) - \int_0^{t_{\mathrm{e}}} 2\beta_{\mathrm{ff}}\beta_{\mathrm{fb}}\|\sigma_i\|^2\,\mathrm{d}\tau$$
$$\le -\infty \tag{A2}$$

The evaluation function $E_i(t)$ at $t = t_{\mathrm{e}}$ will become negative definite, which contradicts the finite positive definite character of the evaluation function. Hence finite escape time is not possible in subsequent iterations. The boundedness of the system signals is hence ensured in the sense of L_∞ norm. □

Appendix B: Proof of Theorem 2

Under the alignment conditions $\bar{\mathbf{x}}_i(T_{\mathrm{f}}) = \bar{\mathbf{x}}_{i+1}(0)$ and $\bar{\mathbf{x}}_{\mathrm{d}}(T_{\mathrm{f}}) = \bar{\mathbf{x}}_{\mathrm{d}}(0)$ for the trajectory of Category 2, it follows that

$$\sigma_i(T_{\mathrm{f}}) = \sigma_{i+1}(0) \tag{B1}$$

is satisfied for all trials. From inequality (19) we know that

$$\Delta E_i \le -\beta_{\mathrm{ff}}\sigma_i^{\mathrm{T}}(T_{\mathrm{f}})M_i(\mathbf{x}_1(T_{\mathrm{f}}))\sigma_i(T_{\mathrm{f}})$$
$$+\beta_{\mathrm{ff}}\sigma_i^{\mathrm{T}}(0)M_i(\mathbf{x}_1(0))\sigma_i(0) - 2\int_0^{T_{\mathrm{f}}} \beta_{\mathrm{ff}}\beta_{\mathrm{fb}}\|\sigma_i\|^2\,\mathrm{d}\tau$$

Again, taking summation of ΔE_i up to k and using condition (B1) yields

$$
\begin{aligned}
\sum_{i=0}^{k} \Delta E_i &= E_{k+1} - E_0 \\
&\le -\sum_{i=0}^{k} \beta_{\mathrm{ff}} \boldsymbol{\sigma}_i^{\mathrm{T}}(T_{\mathrm{f}}) M_i(\mathbf{x}_1(T_{\mathrm{f}})) \boldsymbol{\sigma}_i(T_{\mathrm{f}}) \\
&\quad + \sum_{i=0}^{k} \beta_{\mathrm{ff}} \boldsymbol{\sigma}_i^{\mathrm{T}}(0) M_i(\mathbf{x}_1(0)) \boldsymbol{\sigma}_i(0) \\
&\quad - \sum_{i=0}^{k} 2\beta_{\mathrm{ff}}\beta_{\mathrm{fb}} \int_0^{T_{\mathrm{f}}} \|\boldsymbol{\sigma}_i\|^2 \, \mathrm{d}\tau \\
&= -\sum_{i=1}^{k+1} \beta_{\mathrm{ff}} \boldsymbol{\sigma}_i^{\mathrm{T}}(0) M_i(\mathbf{x}_1(0)) \boldsymbol{\sigma}_i(0) \\
&\quad + \sum_{i=0}^{k} \beta_{\mathrm{ff}} \boldsymbol{\sigma}_i^{\mathrm{T}}(0) M_i(\mathbf{x}_1(0)) \boldsymbol{\sigma}_i(0) \\
&\quad - \sum_{i=0}^{k} 2\beta_{\mathrm{ff}}\beta_{\mathrm{fb}} \int_0^{T_{\mathrm{f}}} \|\boldsymbol{\sigma}_i\|^2 \, \mathrm{d}\tau \\
&= -\beta_{\mathrm{ff}} \boldsymbol{\sigma}_{k+1}^{\mathrm{T}}(0) M_{k+1}(\mathbf{x}_1(0)) \boldsymbol{\sigma}_{k+1}(0) \\
&\quad + \beta_{\mathrm{ff}} \boldsymbol{\sigma}_0^{\mathrm{T}}(0) M_0(\mathbf{x}_1(0)) \boldsymbol{\sigma}_0(0) \\
&\quad - \sum_{i=0}^{k} 2\beta_{\mathrm{ff}}\beta_{\mathrm{fb}} \int_0^{T_{\mathrm{f}}} \|\boldsymbol{\sigma}_i\|^2 \, \mathrm{d}\tau \qquad \text{(B2)}
\end{aligned}
$$

From the above formula we can derive

$$
\sum_{i=0}^{k} 2\beta_{\mathrm{ff}}\beta_{\mathrm{fb}} \int_0^{T_{\mathrm{f}}} \|\boldsymbol{\sigma}_i\|^2 \, \mathrm{d}\tau \le E_0 + \beta_{\mathrm{ff}} \boldsymbol{\sigma}_0^{\mathrm{T}}(0) M_0(\mathbf{x}_1(0)) \boldsymbol{\sigma}_0(0) \qquad \text{(B3)}
$$

in which the right-hand side remains constant as k increases. Therefore, similarly to Theorem 1, $\boldsymbol{\sigma}_i = 0$ can be ensured as the number of iterations approaches infinity. Hence we can obtain $\|\boldsymbol{\sigma}_i(T_f)\| < \|\boldsymbol{\sigma}_i(0)\|$. The initial alignment condition leads to $\|\boldsymbol{\sigma}_{i+1}(0)\| < \|\boldsymbol{\sigma}_i(0)\|$. This contraction mapping ensures the convergence of the system states to their desired values. Equation (B2) also ensures the finiteness of $E_i(T_{\mathrm{f}})$. The finiteness of the system signals can hence be ensured, as in the proof of Theorem 1. □

Appendix C: Proof of Theorem 3

Choose the evaluation function

$$
E = \int_0^{\infty} \|\gamma(\tau) - \mathbf{v}(\tau)\|^2 \, \mathrm{d}\tau \qquad \text{(C1)}
$$

By segmenting $[0, \infty)$ into a series of even intervals $[iT_{\mathrm{f}}, (i+1)T_{\mathrm{f}}]$, $i = 0, 1, \cdots$, and noticing the periodicity of the function vector $\gamma(t)$, evaluation function (C1) can be rewritten as

$$
\begin{aligned}
E = \sum_{i=0}^{\infty} E_i &= \sum_{i=0}^{\infty} \int_{iT_{\mathrm{f}}}^{(i+1)T_{\mathrm{f}}} \|\gamma(\tau) - \mathbf{v}(\tau)\|^2 \, \mathrm{d}\tau \\
&= \sum_{i=0}^{\infty} \int_0^{T_{\mathrm{f}}} \|\gamma(\tau) - \mathbf{v}_i(\tau)\|^2 \, \mathrm{d}\tau \qquad \text{(C2)}
\end{aligned}
$$

where

$$
\mathbf{v}_i(t) \triangleq \mathbf{v}(iT_{\mathrm{f}} + t) \quad t \in [0, T_{\mathrm{f}}]
$$

is the learning control law. Using the above definition, learning control $\mathbf{v}(t)$ in (20) can be expressed in the following recursive form

$$
\mathbf{v}_i(t) = \mathbf{v}_{i-1}(t) + \beta_{\mathrm{ff}} A_{i-1}^{\mathrm{T}}(\cdot, t) \boldsymbol{\sigma}_{i-1}(t) \qquad \text{(C3)}
$$

Now both evaluation functions E_i and learning control law $\mathbf{v}_i(t)$ can be treated as defined over a finite interval $[0, T_{\mathrm{f}}]$. By comparison, we find that both evaluation functions in (13) and (C2), and the learning control laws $\mathbf{v}_i$ in (12) and (C3) have exactly the same forms. By virtue of the uniform upper bound of $\|\mathbf{g}\|$, it is possible to design a gain $l_g(\bar{\mathbf{x}}, t)$ with uniform upper bound to ensure global stability. Therefore, the proof of Theorem 2 can be applied directly to show the global and asymptotic convergence of system states to their desired value. □

Appendix D: Proof of Theorem 5

Under the condition $\bar{\mathbf{x}}_i(T_{\mathrm{f}}) = \bar{\mathbf{x}}_{i+1}(0)$ and the property of the desired trajectory which also ensures $\bar{\mathbf{x}}_{\mathrm{d}}(T_{\mathrm{f}}) = \bar{\mathbf{x}}_{\mathrm{d}}(0)$, it follows that

$$
\boldsymbol{\sigma}_i(T_{\mathrm{f}}) = \boldsymbol{\sigma}_{i+1}(0) \qquad \text{(D1)}
$$

is satisfied for all trials. On the basis of inequality (27), taking summation of ΔE_i up to k and using conditions (29) and (D1) yields

$$
\begin{aligned}
\sum_{i=0}^{k} \Delta E_i &= E_{k+1} - E_0 \\
&\le -\sum_{i=0}^{k} \beta_{\mathrm{ff}} \boldsymbol{\sigma}_i^{\mathrm{T}}(T_{\mathrm{f}}) B^{-1}(\mathbf{z}_i(T_{\mathrm{f}}), T_{\mathrm{f}}) \boldsymbol{\sigma}_i(T_{\mathrm{f}}) \\
&\quad + \sum_{i=0}^{k} \beta_{\mathrm{ff}} \boldsymbol{\sigma}_i^{\mathrm{T}}(0) B^{-1}(\mathbf{z}_i(0), 0) \boldsymbol{\sigma}_i(0) \\
&\quad - \sum_{i=0}^{k} 2\beta_{\mathrm{ff}}\beta_{\mathrm{fb}} \int_0^{T_{\mathrm{f}}} \|\boldsymbol{\sigma}_i\|^2 \, \mathrm{d}\tau \\
&= -\sum_{i=1}^{k+1} \beta_{\mathrm{ff}} \boldsymbol{\sigma}_i^{\mathrm{T}}(0) B^{-1}(\mathbf{z}_i(0), 0) \boldsymbol{\sigma}_i(0) \\
&\quad + \sum_{i=0}^{k} \beta_{\mathrm{ff}} \boldsymbol{\sigma}_i^{\mathrm{T}}(0) B^{-1}(\mathbf{z}_i(0), 0) \boldsymbol{\sigma}_i(0)
\end{aligned}
$$

$$- \sum_{i=0}^{k} 2\beta_{\mathrm{ff}}\beta_{\mathrm{fb}} \int_0^{T_{\mathrm{f}}} \|\boldsymbol{\sigma}_i\|^2 \, \mathrm{d}\tau$$

$$= -\beta_{\mathrm{ff}} \boldsymbol{\sigma}_{k+1}^{\mathrm{T}}(0) B^{-1}(\mathbf{z}_{k+1}(0), 0)\boldsymbol{\sigma}_{k+1}(0)$$
$$+ \beta_{\mathrm{ff}} \boldsymbol{\sigma}_0^{\mathrm{T}}(0) B^{-1}(\mathbf{z}_0(0), 0)\boldsymbol{\sigma}_0(0)$$
$$- \sum_{i=0}^{k} 2\beta_{\mathrm{ff}}\beta_{\mathrm{fb}} \int_0^{T_{\mathrm{f}}} \|\boldsymbol{\sigma}_i\|^2 \, \mathrm{d}\tau \quad \text{(D2)}$$

From the above formula we can derive

$$\sum_{i=0}^{k} 2\beta_{\mathrm{ff}}\beta_{\mathrm{fb}} \int_0^{T_{\mathrm{f}}} \|\boldsymbol{\sigma}_i\|^2 \, \mathrm{d}\tau \leq E_0 + \beta_{\mathrm{ff}} \boldsymbol{\sigma}_0^{\mathrm{T}}(0) B^{-1}(\mathbf{z}_0(0), 0)\boldsymbol{\sigma}_0(0) \quad \text{(D3)}$$

in which the right-hand side remains constant as k increases. The asymptotic convergence of the tracking error and the boundedness of the system states are hence ensured as in Theorem 2. □

References

AMANN, N., OWENS, D. H., and ROGERS, E., 1996, Iterative learning control using optimal feedback and feedforward actions. *International Journal of Control*, **65**, 277–293.

ARIMOTO, S., 1985, Mathematical theory of learning with applications to robot control. In K. S. Narendra (Ed.), *Adaptive and Learning Systems: Theory and Applications* (New Haven, CT: Yale University).

ARIMOTO, S., KAWAMURA, S., and MIYAZAKI, F., 1984, Bettering operation of robots by learning. *Journal of Robotic Systems*, **1**, 123–140.

BONDI, P., CASALINO, G., and GAMBARDELLA, L., 1988, On the iterative learning control theory for robotic manipulators. *IEEE Journal of Robotics and Automation*, **4**, 14–22.

BRIAN, A.-H., 1991, Friction as a function of velocity. In *Control of Machines with Friction* (London: Kluwer), pp. 63–94.

KUC, Y. Y., LEE, S. J., and NAM, K. H., 1992, An iterative learning control theory for a class of nonlinear dynamic systems. *Automatica*, **28**, 1215–1221.

KUREK, J. E., and ZAREMBA, 1993, Iterative learning control synthesis based on 2-d system theory. *IEEE Transactions on Automatic Control*, **38**, 121–125.

LEE, K., and BIEN, Z., 1991, Initial condition problem of learning control. *IEE Proceedings, Part D*, **138**, 525–528.

LEE, K., and BIEN, Z., 1996, Study of robustness of iterative learning control with non-zero initial error. *International Journal of Control*, **64**, 349–359.

MOORE, K., 1993, *Iterative Learning Control for Deterministic Systems* (London: Springer-Verlag).

NAKANO, M., INOUE, T., TAMAMOTO, Y., and HARA, S., 1989, Repetitive control. *SICE* (in Japanese).

PARK, B. H., KUC, T.-Y., and LEE, J. S., 1996, Adaptive learning control of uncertain robotic systems. *International Journal of Control*, **65**, 725–744.

SLOTINE, J.-J. E., and LI. W., 1991, *Applied Nonlinear Control* (New York: Prentice-Hall International).

UTKIN, V. I., 1978, Variable structure systems with sliding modes. *IEEE Transactions on Automatic Control*, **22**, 212–222.

XU, J.-X., 1997, Analysis of iterative learning control for a class of nonlinear discrete-time systems. *Automatica*, **33**, 1905–1907.

XU, J.-X., and QU, Z., 1998, Robust iterative learning control for a class of nonlinear systems. *Automatica*, **34**, 983–988.

YAMAMOTO, Y., and HARA, S., 1988, Relationships between internal and external stability for infinite-dimensional systems with applications to a servo problem. *IEEE Transactions on Automatic Control*, **33**, 1044–1052.