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Brief Paper

# Nonlinear learning control for a class of nonlinear systems☆

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**Abstract**

Based on the Lyapunov's direct method, a new learning control design is proposed. The proposed technique can be applied in two ways: it is either the standard backward recursive design or its extension. In the first case, the design yields a class of learning control with a difference learning law, under which the class of nonlinear systems is guaranteed to be asymptotically stable with respect to the number of trials in performing repeated tasks. However, implementation of the difference learning control requires derivative measurement of the state for guaranteed stability and performance, as required by most of the existing linear learning control laws. To overcome this difficulty, the proposed design extends the recursive design by employing a new state transformation and a new Lyapunov function, and it yields a class of learning control with a difference-differential learning law. Compared with the existing design methods most of which are based on linear analysis and design, the extension not only guarantees global stability and good performance but also removes such limitations as derivative measurement, Lipschitz condition, and resetting of initial conditions. In addition, the proposed design does not rely on the property of a system under consideration such as the input–output passivity. 

*Keywords:* Nonlinear learning control; Lyapunov method; Cascaded nonlinear systems

## 1. Introduction

In many control applications such as robotics and automation, one of important issues is to design control systems that achieve trajectory tracking with acceptable accuracy. Often, tracking error systems are nonlinear and contain unknown parameters or time functions. If the desired trajectory is periodic or repetitive, iterative learning control can be used to improve system performance. The intuition behind this approach is periodicity of the repeated tasks, although other functions with a given and known characteristic may be learned. From trial to trial, all periodic time functions remain to be constant at any fixed instant of local time. So, a learning control if properly designed should be able to learn constants since constants are simplest form of unknowns. Through learning unknown parameters or time functions, learning control can compensate linear as well as nonlinear dynamics so that tracking performance can be enhanced.

There have been many results reported on learning control design. A recent discussion on history and various approaches of learning control can be found in Moore (1993). In model-based learning control, there are two major approaches. First, Arimoto and his coworkers (Arimoto, Kawamura, & Miyazaki, 1984a,b; Arimoto, Kawamura, Miyazaki, & Tamaki, 1984; Kawamura, Miyazaki, & Arimoto, 1985) proposed a learning control design that updates its learning contribution from trial to trial. This approach achieves asymptotic zero tracking error by requiring derivative feedback of the state and Lipschitzian condition and by assuming the same initial conditions for all trials. Other schemes that are similar in essence to Arimoto's framework are: a high-gain, model

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reference adaptive control approach (Bondi, Casalino, & Gambardella, 1988); generalized inversion of input matrix (Hauser, 1987); linear high-gain robust control (Miller III, Glanz, & Kraft, 1987; Qu, Dorsey, Dawson, & Johnson, 1993) and robustness analysis under disturbance (Heinzinger, Fenwick, Paden, & Miyazaki, 1992); It was shown in Sugie and Ono (1991) that, if there is a direct transmission term from input to output, derivative measurement of the state is not needed. Removal of acceleration measurement were also achieved for second-order vector systems in Kuc, Lee, and Nam (1992) with Lipschitz condition and constant bound on the time derivation of the inertia matrix, and in Qu and Zhuang (1993) through non-differentiable nonlinear robust control. The second approach is the so-called adaptive learning control scheme in which an adaptation law is designed in a similar fashion as those in adaptive control. Learning controls designed using this method are updated not from trial to trial but continuously in time; for example, learning controllers proposed by Horowitz and his coworkers (Horowitz, Messner, & Moore, 1991; Messner, Horowitz, Kao, & Boals, 1991).

Despite of the progress accomplished, major limitations such as derivative feedback of the state, or resetting of initial condition, or Lipschitz condition, or their combinations remain for designing of a learning control for a class of nonlinear systems. The key to overcome these limitations and to account for nonlinear models of physical systems is to use nonlinear analysis and design tools. Recently, there was a new result that the asymptotic stability can be achieved using the input–output passivity of robotic systems without the limitations mentioned previously (Arimoto, 1996). However, it is required to generalize the design of a learning control so that it may not depend on the property of a system. For the extension, among various nonlinear methods, the Lyapunov second approach (Khalil, 1992; Rouche, Habets, & Laloy, 1977; Slotine & Li, 1991) stands out due to its universal applicability and physical implications. To successfully apply the Lyapunov method, one must find an appropriate Lyapunov function candidate using which control can be designed to ensure stability. In this paper, a new nonlinear learning design is presented, and it is an extension of the backward recursive (backstepping) design specifically improved for learning control design. This method allows us to extend the Arimoto's learning framework based on the Gronwall's inequality to the one that is based on the Lyapunov's direct method.

The proposed learning control design is applicable to the class of high-order nonlinear systems that are consisted of finite cascaded subsystems, and they include many of electrical–mechanical systems such as robots, electric motors and drivers, etc. Application results to robots (both simulation and experimentation results) can be found in Ham, Qu, and Park (1994), Ham, Qu, and Johnson (2000), Ham and Qu (1997) and Qu and Dawson (1996). However, it is the main purpose of this paper to present the nonlinear design framework of learning control in a mathematically general setting.

The proposed learning control scheme contains two parts: a feedforward/feedback part and a learning part. The latter is described either by a difference equation or by a differential-difference equation. The learning control is designed to be robust in the sense that it ensures global stability in the presence of unknown dynamics in the system so long as the unknown is bounded by a known nonlinear function of the state.

This paper is organized as follows. In Section 2, nonlinear learning control design method is introduced. The new approach is illustrated by an example in Section 3. Conclusions are made in Section 4.

## 2. Nonlinear learning control

In this section, a learning control design is introduced to achieve asymptotic stabilization for a class of cascaded nonlinear systems in the form that, for $i = 1, \dots, m-1$,

$$\dot{z}_{i,j} = A_{i,j}(z_{1,j}, \dots, z_{i,j}, t) + B_{i,j}(z_{1,j}, \dots, z_{i-1,j}, t) z_{i+1,j} \quad (1)$$

and

$$\dot{z}_{m,j} = A_{m,j}(z_{1,j}, \dots, z_{m,j}, t) + B_{m,j}(z_{1,j}, \dots, z_{m-1,j}, t) u_j, \quad (2)$$

where $z_{i,j} \in \Re^n$ is the state of the $i$th subsystem, $u_j \in \Re^n$ is the control variable, subscripts $i$ and $j$ are the indices of subsystems and learning trials, respectively.

Systems in the above class may have both unknown time-varying parameters and nonlinear uncertainties. For these systems, robust control developed in Qu (1993) can always be applied if uncertainties are bounded by well-defined functions and if matrix $B_{i,j}(\cdot)$ is positive definite. In this paper, a learning control is shown to be applicable to the above systems provided that, in a proper transformed state space, nonlinear uncertainties that do not vanish as the transformed state converges to zero are periodic. A learning control is to compensate for uncertain periodic time functions. Compared with robust control method in Qu (1993), the proposed learning design method yields simpler control and, by taking advantage of periodicity, ensures better stability result (asymptotic stability in contrast to stability of uniform ultimate boundedness). The proposed framework of learning control design is based on the following assumptions.

**Assumption 1.** *Vector time function $z^{\mathrm{d}}(t): [0, \delta T] \to \Re^n$ denotes the desired trajectory for system output $z_{1,j}$ to track repeatedly in all trials. It is smooth in the sense that*

$$\sup_{t \in [0, \delta T]} \left\{ \|z^{\mathrm{d}}\|, \|\dot{z}^{\mathrm{d}}\|, \dots, \left\| \frac{d^m z^{\mathrm{d}}}{dt^m} \right\| \right\}$$

*is bounded. Furthermore, it is assumed that, unless resetting of initial condition of the system is done after each trial,* $z^{d}(0) = z^{d}(\delta T)$ *in order to have a continued, repetitive, desired motion.*

**Assumption 2.** *Dynamics of a system described above can asymptotically be learned if, for any fixed values of* $z_{i,j} \in \Re^n$, *vector/matrix time functions* $A_{i,j}(z_{1,j}, \dots, z_{i,j}, t)$ *and* $B_{i,j}(z_{1,j}, \dots, z_{i-1,j}, t)$, $i = 1, \dots, m$, *are periodic in time with respect to* $\delta T$.[1]

**Assumption 3.** *Functions* $A_{i,j}(z_{1,j}, \dots, z_{i,j}, t)$ *and* $B_{i,j}(z_{1,j}, \dots, z_{i-1,j}, t)$ *are of known expression with respect to the state variables. The unknowns in* $A_{i,j}(z_{1,j}, \dots, z_{i,j}, t)$ *and* $B_{i,j}(z_{1,j}, \dots, z_{i-1,j}, t)$ *are time functions that either periodic (as defined in assumption 2) or bounded in norm by known constants. Furthermore, matrix* $B_{i,j}(\cdot)$ *is positive definite with known constant upper and lower bounds, and its partial derivatives are bounded in norm by known functions of its arguments.*[2]

Based on the three assumptions, learning control can be designed analytically using Lyapunov's direct method. The main feature of our design is that learning control will be devised in closed form, to be of same expression for all trials $j$, and recursively with respect to index $i$ of subsystems. That is, the Lyapunov method is applied to each subsystem in order to find a proper fictitious control as if it could be controlled independently. For each subsystem, control design is done in four minor steps: a translational transformation is first applied, followed by re-grouping of dynamics, then properties of these parts of dynamics are developed, and finally the control is given in closed form for all $t$ and for all $j$.

For the $i$th subsystem, the state transformation is from $z_{i,j}$ to $x_{i,j}$ and is defined by

$$x_{1,j} = z_{1,j} - z^{d}, \quad x_{i,j} = z_{i,j} - u_{i-1,j}$$
$$\text{for } i = 2, \dots, m \text{ and for all } j, \tag{3}$$

where $u_{i-1,j}$ is the so-called fictitious control designed for the $(i-1)$th subsystem. By (3), the actual control design can be accomplished after $m$ major recursion steps and, at the $i$th step, the state transformation is applied to the $i$th system after control design has been done for the $(i-1)$th and preceding subsystems. As will be given shortly, all fictitious controls are given by analytical expressions of same type.[3] Such a design fully exploit the cascaded structure of overall system dynamics and makes on-line implementation be a combination of simple additions, multiplications and successive substitutions. For notational convenience, we shall use $u_{m,j}$ or $x_{m+1,j}$ to denote the actual control variable $u_j$. That is, the fictitious control for the last subsystem is the actual control.

In what follows, design steps are given while details of algebraic derivations for a specific system are omitted as one can mimic those in Section 3. Again, we shall proceed with our design recursively. By induction, let us suppose that fictitious controls $u_{k,j}$, $k = 1, \dots, i-1$, have been chosen (as will be given by Eq. (8)) for subsystems 1 up to $(i-1)$, and we are to design a control for subsystem $i$. By the definition of new state variable $x_{i,j}$, we can rewrite equation of the $i$-th subsystem in terms of $x_{i,j}$ as

$$\begin{aligned}
\dot{x}_{i,j} &= A_{i,j}(z_{1,j}, \dots, z_{i,j}, t) \\
&\quad + B_{i,j}(z_{1,j}, \dots, z_{i-1,j}, t) z_{i+1,j} - \dot{u}_{i-1,j} \\
&= B_{i,j}(z_{1,j}, \dots, z_{i-1,j}, t) \\
&\quad \times [B_{i,j}^{-1}(z_{1,j}, \dots, z_{i-1,j}, t) A_{i,j}(z_{1,j}, \dots, z_{i-1,j}, t) \\
&\quad - B_{i,j}^{-1}(z_{1,j}, \dots, z_{i-1,j}, t) \dot{u}_{i-1,j} + z_{i+1,j}].
\end{aligned} \tag{4}$$

Since $u_{i-1,j}$ has been found, its time derivative can be rewritten as a sum of products of its partial derivatives and $\dot{z}_{k,j}$ $(k = 1, \dots, i-1)$ which have been found in the previous recursive steps. Therefore, the right-hand side of (4) is a function of $z_{k,j}$ and $x_{k,j}$, $k \le i+1$. Now, change the variables from $z_{k,j}$ to $x_{k,j}$ and group the dynamics of the $i$th subsystem according to the following expression:

$$\begin{aligned}
\dot{x}_{i,j} &= h_{i,j}^{-1}(x_{1,j}, \dots, x_{i-1,j}, t) f_{i,j}(x_{1,j}, \dots, x_{i,j}, t) \zeta_i(t) \\
&\quad + g_{i,j}(x_{1,j}, \dots, x_{i,j}, t) \\
&\quad + h_{i,j}^{-1}(x_{1,j}, \dots, x_{i-1,j}, t) u_{i,j} \\
&\quad + h_{i,j}^{-1}(x_{1,j}, \dots, x_{i-1,j}, t) x_{i+1,j},
\end{aligned} \tag{5}$$

where $\zeta_i(t) \in \Re^{p_i}$ is a vector of unknown but periodic time functions (to be learned by a learning contribution $\Delta_{i,j} \in \Re^{p_i}$ contained in learning control $u_{i,j}$), $f_i(\cdot) \in \Re^{n \times p_i}$ is a known and possibly nonlinear matrix function matrix, and vector $g_{i,j}(\cdot) \in \Re^n$ may contain nonlinear uncertainties and has the property that, for some known bounding functions $\rho_{g_{ik}}(\cdot)$, inequality

$$\|g_{i,j}(x_{1,j}, \dots, x_{i,j}, t)\| \le \sum_{k=1}^{i} \rho_{g_{ik}}(x_{1,j}, \dots, x_{i,j}, t) \cdot \|x_{k,j}\| \tag{6}$$

[1] This condition is sufficient as some of the unknown time functions (for example, those in $A_{m,j}(\cdot)$ and those in $A_{i,j}(\cdot)$ being covered under Assumption 3) may not have to be periodic.

[2] All bounding functions are assumed to be well-defined in the sense that they are locally uniformly bounded with respect to the state variables and uniformly bounded with respect to time. As shown in Qu (1993), bounding functions can be assumed, without loss of any generality, to be differentiable in order to proceed recursively with the control design.

[3] Specifically, fictitious control $u_{i,j}(t)$ contains a learning control part $(\Delta_{i,j})$ and a feedforward control part, and it is a closed-form function of $\Delta_{i,j-1}(t)$ plus $x_{1,j}(t)$ up to $x_{i,j}(t)$, that is, of $\Delta_{k,j}(t)$ plus $z_{k,j}(t)$, $k = 1, \dots, i$.

holds for all $x_{l,j}$ (with $l = 1, \dots, i$). Comparing (4) and (5), we know that

$$B_{i,j}(z_{1,j}, \dots, z_{i-1,j}, t) = h_{i,j}^{-1}(x_{1,j}, \dots, x_{i-1,j}, t).$$

Consequently, it follows from Assumptions 2 and 3 that $B_{i,j}(z_{1,j}, \dots, z_{i-1,j}, 0) = B_{i,j}(z_{1,j}, \dots, z_{i-1,j}, \delta T)$ and that inequalities

$$\underline{h}_i I_n \le h_{i,j}(x_{1,j}, \dots, x_{i-1,j}, t) \le \overline{h}_i I_n,$$
$$\left\| \frac{\mathrm{d}}{\mathrm{d}t} h_{i,j}(x_{1,j}, \dots, x_{i-1,j}, t) \right\| \le \rho_{h_i}(x_{1,j}, \dots, x_{i,j}, t), \tag{7}$$

hold for some known constants $\underline{h}_i$ and $\overline{h}_i$ and for a known function $\rho_{h_i}$. It follows from Assumptions 1, 2 and 3 that the re-grouping given by (5), (6) and (7) can always be obtained.

Learning control $u_{i,j}$, to be synthesized using Lyapunov direct method and to be given by (8), will be in terms of the transformed dynamics in (5) and of the bounding functions in properties (6) and (7), and its functional expression is the same for all $j$ and for all $i$. So, from a user point of view, the proposed design essentially involves symbolic recursive calculation of Eq. (5) and its properties.

Several observations are worth making at this point about the grouping. First, choices of functions $f_{i,j}(\cdot)$, $\zeta_i(t)$ and $g_{i,j}(\cdot)$ are not unique, a term in dynamics of $\dot{z}_{i,j}$ may be classified into either $f_{i,j}(\cdot)$ and $g_{i,j}(\cdot)$. The difference here is whether the designer wants to explicitly learn the unknown time function or to compensate for it through bounding it properly. Consequently, the user may construct several equivalent learning controls for any given system and make a selection. Second, upper and lower bounds $\underline{h}_i$ and $\overline{h}_i$ can be generalized to be positive-valued functions and, if matrix $h_{i,j}(\cdot)$ is known, it would be sufficient that $h_{i,j}(\cdot)$ is invertible rather than being positive definite.

Now, we are in a position to synthesize a learning control using Lyapunov direct method. For all $i = 1, \dots, m$ and for all $j$, the proposed learning control is

$$\begin{aligned} u_{i,j} = &-\Bigg[(m-i+1)x_{i,j} + \overline{h}_i \rho_{g_{ii}}(x_{1,j}, \dots, x_{i,j}, t) \cdot x_{i,j} \\ &+ \frac{1}{2}\rho_{h_i}(x_{1,j}, \dots, x_{i,j}, t) \cdot x_{i,j} + x_{i-1,j} \\ &+ \frac{1}{4}\overline{h}_i^2 x_{i,j} \sum_{k=1}^{i-1} \rho_{g_{ik}}^2(x_{1,j}, \dots, x_{i,j}, t)\Bigg] - f_{i,j} \cdot \Delta'_{i,j} \\ &\triangleq F_{i,j}(x_{1,j}, \dots, x_{i,j}, t) + L_{i,j}(x_{1,j}, \dots, x_{i,j}, t), \end{aligned} \tag{8}$$

where $x_{0,j} = 0$ is used for notational convenience, bounding functions $\rho_{h_i}(\cdot)$ and $\rho_{g_{ik}}(\cdot)$ are defined by (7) and (6), $F_{i,j}(\cdot)$ denotes the feedforward control part, and $L_{i,j}(\cdot)$ is the learning control part.

The lumped feedforward control $F_{i,j}(\cdot)$ is a robust control part that stabilizes asymptotically the system through compensating for uncertainties $g_{i,j}(\cdot)$ and $h_{i,j}(\cdot)$. Learning control part $L_{i,j}(\cdot)$ is designed to learn time function $\zeta_i(t)$ and to compensate for known nonlinear dynamics $f_{i,j}(\cdot)$. As mentioned before, the actual control is defined to be $u_j = u_{m,j}$.

The iterative learning contribution $\Delta'_{i,j}$ consists of two parts:

$$\Delta'_{i,j} = \Delta_{i,j} + \beta f_{i,j}^{\mathrm{T}}(x_{1,j}, \dots, x_{i,j}, t)\, x_{i,j}, \tag{9}$$

where $\beta \ge 0$ is a control gain that can be chosen freely by the designer. The term that learns unknown time function, $\Delta_{i,j}$, is updated from trial to trial by the learning law

$$\begin{aligned} \gamma \dot{\Delta}_{i,j} + \Delta_{i,j} = &(1-\gamma)\Delta_{i,j-1} \\ &+ \alpha f_{i,j}^{\mathrm{T}}(x_{1,j}, \dots, x_{i,j}, t)\, x_{i,j}, \end{aligned} \tag{10}$$

where $i = 1, \dots, m$, $j = 0, 1, \dots$, $\Delta_{i,-1} = 0$, $\alpha > 0$ is the learning control gain that can be chosen freely by the designer, and $0 \le \gamma < 1$ is a design parameter that yields either a difference or difference-differential learning law. Whenever $\gamma > 0$ is selected, $\Delta_{i,j}$ defined by (10) should be solved under initial condition $\Delta_{i,j}(0) = \Delta_{i,j-1}(\delta T)$, where $\delta T$ denotes the duration of all learning trials.

The introduction of $\Delta_{i,j}$ makes it easier to analyze stability and performance of the learning control. Specifically, it will be shown (in the proof) that $\Delta_{i,j}$ learns time function $\zeta_i(t)$ and ensures stability, and that the difference between $\Delta'_{i,j}$ and $\Delta_{i,j}$ improves convergence of the learning control.

The above learning control is derived based on Lyapunov's direct method using Lyapunov function: for $i = 1, \dots, m$ and for all $j$,

$$\begin{aligned} V_{i,j} = &\frac{1}{2}(1-\gamma)\int_0^{\delta T} \|\zeta_i(\tau) - \Delta_{i,j}(\tau)\|^2\, \mathrm{d}\tau \\ &+ \frac{1}{2}\gamma \|\zeta_i(\delta T) - \Delta_{i,j}(\delta T)\|^2, \end{aligned} \tag{11}$$

which consists of Euclidean norm and $L_2$ norm (Khalil, 1992) of learning error $[\zeta_i(t) - \Delta_{i,j}]$. To show design and effectiveness of the proposed learning control, we first present the following lemma which illustrates the property of Lyapunov function (11) for the $i$-th subsystem.

**Lemma.** *Consider system* (5) *under learning control* (8) *with learning law* (10). *Then, the incremental change of Lyapunov function,* $\delta V_{i,j} \triangleq V_{i,j} - V_{i,j-1}$, *satisfies the inequality that*

$$\begin{aligned} \delta V_{i,j} \le \alpha \Bigg[ &-\frac{1}{2} x_{i,j}^{\mathrm{T}} h_{i,j}\, x_{i,j}\Big|_0^{\delta T} - \beta \int_0^{\delta T} x_{i,j}^{\mathrm{T}} f_{i,j} \cdot f_{i,j}^{\mathrm{T}}\, x_{i,j}\, \mathrm{d}\tau \\ &- (m-i+1)\int_0^{\delta T} \|x_{i,j}\|^2\, \mathrm{d}\tau \end{aligned}$$

$$+\int_0^{\delta T} x_{i,j}^{\mathrm{T}}[(z_{i+1,j}-u_{i,j})-x_{i-1,j}]\,\mathrm{d}\tau$$
$$+\sum_{k=1}^{i-1}\int_0^{\delta T}\|x_{k,j}\|^2\,\mathrm{d}\tau\Bigg]+\frac{1}{4}\gamma\delta T\max_{s\in[0,\delta T]}\|\dot{\zeta}_i(s)+\zeta_i(s)\|^2.$$

**Proof.** It follows from the choice of initial condition of learning law (10) and from periodicity of $\zeta_i(\cdot)$ that the difference of Lyapunov function between two successive trials, $\delta V_{i,j}=V_{i,j}-V_{i,j-1}$, can be rewritten as

$$\delta V_{i,j}=\frac{1}{2}(1-\gamma)\int_0^{\delta T}[\|\zeta_i-\Delta_{i,j}\|^2-\|\zeta_i-\Delta_{i,j-1}\|^2]\,\mathrm{d}\tau$$
$$+\int_0^{\delta T}[\zeta_i-\Delta_{i,j}]^{\mathrm{T}}[\gamma\dot{\zeta}_i-\gamma\dot{\Delta}_{i,j}]^{\mathrm{T}}\,\mathrm{d}\tau.$$

It follows from (10) that

$$\begin{aligned}\delta V_{i,j}&=\frac{1}{2}(1-\gamma)\int_0^{\delta T}[\|\zeta_i-\Delta_{i,j}\|^2-\|\zeta_i-\Delta_{i,j-1}\|^2]\,\mathrm{d}\tau\\
&\quad+\int_0^{\delta T}[\zeta_i-\Delta_{i,j}]^{\mathrm{T}}\{\gamma[\dot{\zeta}_i+\zeta_i]-\alpha f_{i,j}^{\mathrm{T}}\cdot x_{i,j}\\
&\quad-[\zeta_i-\Delta_{i,j}]+(1-\gamma)[\zeta_i-\Delta_{i,j-1}]\}^{\mathrm{T}}\,\mathrm{d}\tau\\
&=-\frac{1}{2}(1+\gamma)\int_0^{\delta T}\|\zeta_i-\Delta_{i,j}\|^2\,\mathrm{d}\tau\\
&\quad-\frac{1}{2}(1-\gamma)\int_0^{\delta T}\|\zeta_i-\Delta_{i,j-1}\|^2\,\mathrm{d}\tau\\
&\quad+(1-\gamma)\int_0^{\delta T}[\zeta_i-\Delta_{i,j}]^{\mathrm{T}}[\zeta_i-\Delta_{i,j-1}]\,\mathrm{d}\tau\\
&\quad+\gamma\int_0^{\delta T}[\zeta_i-\Delta_{i,j}]^{\mathrm{T}}[\dot{\zeta}_i+\zeta_i]\,\mathrm{d}\tau\\
&\quad-\alpha\int_0^{\delta T}[\zeta_i-\Delta_{i,j}]^{\mathrm{T}}f_{i,j}^{\mathrm{T}}\cdot x_{i,j}\,\mathrm{d}\tau\\
&=-\gamma\int_0^{\delta T}\left\|\frac{1}{2}(\zeta_i-\dot{\zeta}_i)-\Delta_{i,j}\right\|^2\mathrm{d}\tau\\
&\quad-\frac{1}{2}(1-\gamma)\int_0^{\delta T}\|\Delta_{i,j}-\Delta_{i,j-1}\|^2\,\mathrm{d}\tau\\
&\quad+\frac{1}{4}\gamma\int_0^{\delta T}\|\dot{\zeta}_i+\zeta_i\|^2\,\mathrm{d}\tau-\alpha\int_0^{\delta T}[\zeta_i-\Delta_{i,j}]^{\mathrm{T}}f_{i,j}^{\mathrm{T}}\cdot x_{i,j}\,\mathrm{d}\tau\\
&\le\frac{1}{4}\gamma\delta T\max_{s\in[0,\delta T]}\|\dot{\zeta}_i(s)+\zeta_i(s)\|^2\\
&\quad-\alpha\int_0^{\delta T}[f_{i,j}\cdot(\zeta_i-\Delta_{i,j})]^{\mathrm{T}}\cdot x_{i,j}\,\mathrm{d}\tau.\end{aligned}\tag{12}$$

Applying control (8) together with learning term (9) to system (5) yields

$$\begin{aligned}\dot{x}_{i,j}&=h_{i,j}^{-1}f_{i,j}\zeta_i(t)+g_{i,j}-\beta h_i^{-1}\cdot f_{i,j}\cdot f_{i,j}^{\mathrm{T}}x_{i,j}\\
&\quad-(m-i+1)h_{i,j}^{-1}x_{i,j}\\
&\quad-\bar{h}_ih_{i,j}^{-1}x_{i,j}\rho_{g_{ii}}-\frac{1}{2}h_{i,j}^{-1}x_{i,j}\cdot\rho_{h_i}-\frac{1}{4}\bar{h}_i^2h_{i,j}^{-1}x_{i,j}\sum_{k=1}^{i-1}\rho_{g_{ik}}^2\\
&\quad-h_{i,j}^{-1}x_{i-1,j}-h_{i,j}^{-1}\cdot f_{i,j}\cdot\Delta_{i,j}+h_{i,j}^{-1}[z_{i+1,j}-u_{i,j}].\end{aligned}$$

Then, solving for $f_{i,j}\cdot[\zeta_i(t)-\Delta_{i,j}]$ from the above equation yields

$$\begin{aligned}&-\int_0^{\delta T}[f_{i,j}^{\mathrm{T}}\cdot(\zeta_i-\Delta_{i,j})]^{\mathrm{T}}\cdot x_{i,j}\,\mathrm{d}\tau\\
&=\int_0^{\delta T}-x_{i,j}^{\mathrm{T}}[h_{i,j}\dot{x}_{i,j}+\beta f_{i,j}\cdot f_{i,j}^{\mathrm{T}}x_{i,j}\\
&+(m-i+1)x_{i,j}+\bar{h}_ix_{i,j}\rho_{g_{ii}}\\
&+\frac{1}{2}x_{i,j}\rho_{h_i}+\frac{1}{4}\underline{h}_i^2x_{i,j}\sum_{k=1}^{i-1}\rho_{g_{ik}}^2+x_{i-1,j}\\
&-h_{i,j}\cdot g_{i,j}-(z_{i+1,j}-u_{i,j})]\,\mathrm{d}\tau\\
&\le-\frac{1}{2}x_{i,j}^{\mathrm{T}}h_{i,j}x_{i,j}\bigg|_0^{\delta T}+\frac{1}{2}\int_0^{\delta T}[\|x_{i,j}\|^2\|\dot{h}_i\|-\|x_{i,j}\|^2\rho_{h_i}]\,\mathrm{d}\tau\\
&-\beta\int_0^{\delta T}x_{i,j}^{\mathrm{T}}f_{i,j}\cdot f_{i,j}^{\mathrm{T}}x_{i,j}\,\mathrm{d}\tau-(m-i+1)\int_0^{\delta T}\|x_{i,j}\|^2\,\mathrm{d}\tau\\
&+\int_0^{\delta T}x_{i,j}^{\mathrm{T}}(z_{i+1,j}-u_{i,j})\,\mathrm{d}\tau-\int_0^{\delta T}x_{i,j}^{\mathrm{T}}x_{i-1,j}\,\mathrm{d}\tau\\
&-\int_0^{\delta T}\Bigg[\bar{h}_i\|x_{i,j}\|^2\rho_{g_{ii}}+\frac{1}{4}\bar{h}_i^2\|x_{i,j}\|^2\sum_{k=1}^{i-1}\rho_{g_{ik}}^2\\
&-\|x_{i,j}\|\cdot\|h_{i,j}\|\cdot\|g_{i,j}\|\Bigg]\,\mathrm{d}\tau\\
&\le-\frac{1}{2}x_{i,j}^{\mathrm{T}}h_{i,j}x_{i,j}\bigg|_0^{\delta T}-\beta\int_0^{\delta T}x_{i,j}^{\mathrm{T}}f_{i,j}\cdot f_{i,j}^{\mathrm{T}}x_{i,j}\,\mathrm{d}\tau\\
&-(m-i+1)\int_0^{\delta T}\|x_{i,j}\|^2\,\mathrm{d}\tau\\
&-\int_0^{\delta T}x_{i,j}^{\mathrm{T}}x_{i-1,j}\,\mathrm{d}tau+\int_0^{\delta T}x_{i,j}^{\mathrm{T}}(z_{i+1,j}-u_{i,j})\,\mathrm{d}\tau\\
&+\sum_{k=1}^{i-1}\int_0^{\delta T}\|x_{k,j}\|^2\,\mathrm{d}\tau\end{aligned}$$

in which the last inequality is obtained by applying definitions of the bounding functions in (7) and (6) and by using inequality $a^2+b^2\ge 2ab$ repeatedly. The proof can be completed by substituting the above result into (12). □

With the lemma in hand, we can now state the main result of this paper. In the following theorem, asymptotic stability of the overall system is concluded under the proposed learning control.

**Theorem.** *Suppose that subsystems in the form of* (5) *are formulated sequentially by fictitious control design and by state transformation* (3) *such that bounding functions in the form of* (7) *and* (6) *can be found. Then, under learning* *control* (8) *and* (10), *the transformed* $(n \times m)$*th order system* (*given by Eq.* (5)) *is globally asymptotically stable if* $\gamma = 0$ *and uniformly ultimately bounded if* $0 < \gamma < 1$. *That is, while all state variables of the original system* (*given by* (1) *and* (2)) *are globally and uniformly bounded, the system output tracks its given desired trajectory either asymptotically* (*for* $\gamma = 0$) *or with arbitrary accuracy in terms of* $L_2$ *norm* (*for small enough* $\gamma > 0$ *or large enough* $\alpha > 0$).

**Proof.** Let Lyapunov function of the overall system at the $j$th trial be $V_j = \sum_{k=1}^{m} V_{k,j}$. It follows from the lemma that

$$\delta V_j \triangleq \sum_{k=1}^{m} \delta V_{k,j} \le -\frac{\alpha}{2} \sum_{k=1}^{m} x_{k,j}^{\mathrm{T}} h_{k,j} x_{k,j}\Big|_0^{\delta T} - \alpha\beta \sum_{k=1}^{m} \int_0^{\delta T} x_{k,j}^{\mathrm{T}} f_{k,j} \cdot f_{k,j}^{\mathrm{T}} x_{k,j}\, \mathrm{d}\tau - \alpha \sum_{k=1}^{m} \int_0^{\delta T} \|x_{k,j}\|^2\, \mathrm{d}\tau + \frac{1}{4}\gamma m \delta T \max_{s \in [0,\, \delta T]} \|\dot{\zeta}_i(s) + \zeta_i(s)\|^2.$$

In learning control implementation, initial conditions at each trial are either manually set to zero or kept to the final conditions of the previous trail (that is, no resetting). In the case of resetting of initial conditions, the term $-\alpha/2 \sum_{k=1}^{m} x_{k,j}^{\mathrm{T}} h_{k,j} x_{k,j}|_0^{\delta T}$ is non-positive. In the second case that no resetting of initial condition is made, the sum of the initial- and final-condition term from the first trial to the $p$th trial is

$$-\frac{\alpha}{2} \sum_{j=1}^{p} \sum_{k=1}^{m} x_{k,j}^{\mathrm{T}} h_{k,j} x_{k,j}\Big|_0^{\delta T} = -\frac{\alpha}{2} \sum_{k=1}^{m} x_{k,p}^{\mathrm{T}}(\delta T)\, h_{k,p} x_{k,p}(\delta T) + \frac{\alpha}{2} \sum_{k=1}^{m} x_{k,0}^{\mathrm{T}}(0)\, h_{k,0} x_{k,0}(0),$$

in which the last sum on the right-hand side is only one possible positive term and it remains constant as $p$ increases. Thus, we have that, in both cases of initial conditions,

$$V_p - V_0 = \sum_{j=1}^{p} \delta V_j \le C_{\mathrm{init}} + p\gamma C_\zeta - \alpha\beta \sum_{j=1}^{p} \sum_{k=1}^{m} \int_0^{\delta T} x_{i,j}^{\mathrm{T}} f_{i,j} \cdot f_{i,j}^{\mathrm{T}} x_{i,j}\, \mathrm{d}\tau - \alpha \sum_{j=1}^{p} \sum_{k=1}^{m} \int_0^{\delta T} \|x_{k,j}\|^2\, \mathrm{d}\tau,$$

where

$$C_{\mathrm{init}} = \frac{\alpha}{2} \sum_{k=1}^{m} x_{k,0}^{\mathrm{T}}(0)\, h_{k,0} x_{k,0}(0)$$

and

$$C_\zeta = \frac{1}{4} m \delta T \max_{s \in [0, \delta T]} \|\dot{\zeta}_i(s) + \zeta_i(s)\|^2.$$

Equivalently, we have that, for all integer $p \ge 0$,

$$\alpha\beta \sum_{j=1}^{p} \sum_{k=1}^{m} \int_0^{\delta T} x_{i,j}^{\mathrm{T}} f_{i,j} \cdot f_{i,j}^{\mathrm{T}} x_{i,j}\, \mathrm{d}\tau + \alpha \sum_{j=1}^{p} \sum_{k=1}^{m} \int_0^{\delta T} \|x_{k,j}\|^2\, \mathrm{d}\tau \le C_{\mathrm{init}} + V_0 + p\gamma C_\zeta. \tag{13}$$

If $\gamma = 0$, taking the limit of $p \to \infty$ on the left-hand side of (13) yields

$$\lim_{p \to \infty} \sum_{k=1}^{m} \int_0^{\delta T} x_{i,p}^{\mathrm{T}} f_{i,p} \cdot f_{i,p}^{\mathrm{T}} x_{i,p}\, \mathrm{d}\tau = \lim_{p \to \infty} \sum_{k=1}^{m} \int_0^{\delta T} \|x_{k,p}\|^2\, \mathrm{d}\tau = 0$$

from which global asymptotic stability of the state $x_{i,j}$ can be concluded.

If $0 < \gamma < 1$, dividing $p$ on both sides of (13) and then taking the limit of $p \to \infty$ yields

$$\alpha \lim_{p \to \infty} \frac{1}{p} \sum_{j=1}^{p} \sum_{k=1}^{m} \int_0^{\delta T} \|x_{k,p}\|^2\, \mathrm{d}\tau \le \gamma C_\zeta$$

which implies that

$$\alpha \lim_{p \to \infty} \sup_{j \ge p} \sum_{k=1}^{m} \int_0^{\delta T} \|x_{k,j}\|^2\, \mathrm{d}\tau \le \gamma C_\zeta.$$

The above inequality shows global uniform ultimate boundedness of the state $\|x_{k,p}\|$ and arbitrary output tracking accuracy through the choice of either $\gamma$ or $\alpha$.

Based on stability of $\|x_{k,p}\|$, stability of the state $\|z_{k,p}\|$ of the original system can be concluded using Eqs. (3) and (8). □

Many systems in engineering applications have the cascaded structure. The above theorem shows how to design learning control for these systems. Specifically, the design is proceeded in a recursive manner. That is, formulate dynamic equation of the first subsystem into (5) and (6) by which a stabilizing fictitious learning control can be selected to be (8) and (10); find dynamic equation governing transformed state $x_{2,j} = z_{2,j} - u_{1,j}$ and choose fictitious control $u_{2,j}$ as if it were the first subsystem; and repeat the same steps for the rest of subsystems. This process will be illustrated by an example in the next section.

It is worth pointing out that, although this paper deals with learning control and details of control design and analysis is quite different, the proposed design procedure is conceptually the same as the so-called backstepping design in adaptive control (Kanellakppoulos, Kokotovic, & Morse, 1991) or the recursive design in robust control (Qu, 1993). That is, fictitious controls constitute a recursive mapping whose final outcome is the actual control to be designed.

**Remark 1.** The roles of control gains $\alpha$ and $\beta$ can be easily analyzed from inequality (13). Consider first the case that $\gamma = 0$. If $\alpha$ is fixed, the larger the gain $\beta$, the faster the convergence of $f_{i,j}^{\mathrm{T}} x_{i,j}$, and therefore by (10) the faster is the convergence of the learning law. If $\beta$ is fixed, the larger the gain $\alpha$, the faster is the convergence of the state variables to zero. In the case of $0 < \gamma < 1$, larger values of $\alpha$ and $\beta$ imply smaller uniform and ultimate bounds. Although $\gamma$ being zero offers better stability result, it will be shown in the next section that $\gamma$ being positive may be needed for better noise rejection in implementation. Note that, for linear systems, filters with unity dc gain were introduced into learning control in Hara, Yamamoto, Omato, and Nakano (1988), Jeon and Tomizuka (1993) and their associated stability was proven using the small gain theorem. Here, a new first-order filter with non-unity dc gain is used in nonlinear learning control and its performance has been shown by Lyapunov's direct method for general, high-order nonlinear systems. Since decreasing the value of $\gamma$ makes stability of boundedness approach asymptotic stability, $\gamma$ should be made to be small if it is chosen to be positive.

**Remark 2.** In general, matrix $B_{i,j}(\cdot)$ may be a function of state variables from $z_{1,j}$ up to $z_{i,j}$ (rather than up to $z_{i-1,j}$). In this case, $h_{i,j}(\cdot) = h_{i,j}(x_{1,j}, \ldots, x_{i,j}, t)$. Consequently, the bounding function defined in (7) should be modified to be

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t} h_{i,j}(x_{1,j}, \ldots, x_{i,j}, t) \right\| \leq \rho_{h_i}(\cdot) + \rho'_{h_i}(\cdot) \|x_{i+1,j}\|.$$

The presence of term $\rho'_{h_i}(\cdot)\|x_{i+1,j}\|$ in stability argument for the $i$th subsystem can only be compensated by the choice of fictitious control $u_{i+1,j}$ in order to have well-defined state transformations. Thus, two modifications must be introduced in this case. First, system (5) must be augmented to include the $(i+1)$th subsystem of state $x_{m+1,j} = u_j$ as $\dot{x}_{m+1,j} = u_{m+1,j}$. It follows that $\zeta_{m+1} = 0$, $f_{m+1,j} = 0$, $g_{m+1,j} = 0$, and $h_{i+1,j} = I$. Stability analysis and learning control design can then be proceeded for the augmented system in the same way as before. Second, fictitious controls $u_{i,j}$ are modified to be

$$\begin{aligned} u_{i,j} = & -\Bigg[ (m-i+1)x_{i,j} + \bar{h}_i \, \rho_{g_{ii}}(x_{1,j}, \ldots, x_{i,j}, t) \cdot x_{i,j} \\ & + \tfrac{1}{2}\rho_{h_i}(x_{1,j}, \ldots, x_{i,j}, t) \cdot x_{i,j} \\ & + \frac{1}{4}\bar{h}_i^2 x_{i,j} \sum_{k=1}^{i-1} \rho_{g_{ik}}^2(x_{1,j}, \ldots, x_{i,j}, t) \\ & + x_{i-1,j} + \frac{1}{4}\|x_{i-1,j}\|^2 \rho'_{h_i} \cdot x_{i,j} \Bigg] - f_{i,j} \cdot \Delta'_{i,j}, \end{aligned}$$

where $i = 1, \ldots, m+1$. The iterative learning part $\Delta'_{i,j}$ is updated by learning law (10). It is easy to see that $\Delta'_{m+1,j} = 0$ for all $j$. By definition of the augmented state, the actual control is in this case $u_j = \int_{t_0}^{t} u_{m+1,j} \, \mathrm{d}\tau$.

**Remark 3.** The proposed control is also robust in the sense that exact knowledge of nonlinear dynamics $g_{i,j}(\cdot)$ in (5) is not required except for bounding functions on the magnitude. In the event that Assumption 2 fails, system (5) can still be obtained except that it may have a constantly bounded but non-periodic time function as bias. In this case, stability analysis can be done in the same fashion as that of $0 < \gamma < 1$ to conclude uniform boundedness.

## 3. Illustrative example

Consider a second-order system

$$\dot{z}_{1,j} = a_1(t) z_{1,j}^2 + z_{2,j}, \tag{14}$$

$$\dot{z}_{2,j} = a_2(t)(1 + z_{1,j}^2 z_{2,j}) + a_3 u_j(t), \tag{15}$$

where subscript $j$ is the index of learning trials, $z_{1,j}$ and $z_{2,j}$ are state variables, and $u_j(t)$ is the control input, $a_1(t)$ and $a_2(t)$ are periodic time functions whose magnitudes are bounded by 1, and $a_3$ is an unknown constant bounded as $1 \leq a_3 \leq 2$.

Based on the formulation in the previous section, the state of the first subsystem should be defined to be the output tracking error as $x_{1,j} = z_{1,j} - z^{\mathrm{d}}$, where $z^{\mathrm{d}}$ is a given desired trajectory. Thus, dynamic equation of $x_{1,j}$ can be derived from (14) as follows:

$$\begin{aligned} \dot{x}_{1,j} &= \dot{z}_{1,j} - \dot{z}^{\mathrm{d}} = a_1(t)(x_{1,j} + z^{\mathrm{d}})^2 - \dot{z}^{\mathrm{d}} + u_{1,j} + x_{2,j} \\ &\triangleq h_{1,j}^{-1} f_{1,j} \zeta_1(t) + g_{1,j} + h_{1,j}^{-1} u_{1,j} + h_{1,j}^{-1} x_{2,j}, \end{aligned} \tag{16}$$

where $u_{1,j}$ is the fictitious control, and $x_{2,j} = z_{2,j} - u_{1,j}$ is the second, transformed state variable. The last equation in the above derivation requires the designer to cast the dynamics of the first subsystem into the standard form of (5). Once functions $h_{1,j}$, $f_{1,j}$, $\zeta_1(t)$ and $g_{1,j}$ are determined, fictitious control $u_{1,j}$ can be found, and then dynamic equation of $x_{2,j}$ can be in turn derived. That is,

$$\begin{aligned} \dot{x}_{2,j} &= \dot{z}_{2,j} - \dot{u}_{1,j} = a_2(t)[1 + z_{1,j}^2 z_{2,j}] + a_3(t) u_j - \dot{u}_{1,j} \\ &\triangleq h_{2,j}^{-1} f_{2,j} \zeta_2(t) + g_{2,j} + h_{2,j}^{-1} u_j. \end{aligned} \tag{17}$$

Finally, actual control $u_j$ can be found by determining functions $h_{2,j}$, $f_{2,j}$, $\zeta_2(t)$ and $g_{2,j}$.

There are many possible choices of functions $f_{i,j}$, $\zeta_i(t)$ and $g_{i,j}$, and different choices of these functions yield different but equivalent learning controls. To show versatility of our learning control scheme, *two typical choices* of learning controls, linear learning part and linear feedforward part, will be made through properly deriving (16) and (17), respectively. In the view that the overall learning control is always nonlinear due to the nonlinear nature of the system, the two typical choices are simply the extreme cases.

For Eq. (16), we shall design a fictitious learning control with a linear learning part. To this end, we know from (9) and (10) that $f_{i,j}$ must be independent of $x_{k,j}$ $(k \le j)$, for instance, $f_{i,j} = 1$ if possible. Under the choice of $f_{1,j} = 1$, comparing Eq. (16) with its preceding one yields

$$h_{1,j} = f_{1,j} = 1, \qquad \zeta_1 = a_1(t)(z^{\mathrm{d}})^2 - \dot{z}^{\mathrm{d}},$$
$$g_{1,j} = 2a_1(t)z^{\mathrm{d}}x_{1,j} + a_1(t)x_{1,j}^2.$$

Then, it follows that $\rho_{h_1} = 0$ and that

$$|g_{1,j}| \le 2|z^{\mathrm{d}}||x_{1,j}| + |x_{1,j}|^2$$
$$\le [1.5 + (z^{\mathrm{d}})^2 + 0.5x_{1,j}^2] \cdot |x_{1,j}| \triangleq \rho_{g_{11}} |x_{1,j}|.$$

Thus, fictitious control $u_{1,j}$ is defined by (8) with the above functions, specifically,

$$u_{1,j} = -(2 + \rho_{g_{11}})x_{1,j} - \Delta'_{1,j}$$
$$= -(2 + \beta + \rho_{g_{11}})x_{1,j} - \Delta_{1,j},$$
$$\gamma\dot{\Delta}_{1,j} + \Delta_{1,j} = (1-\gamma)\Delta_{1,j-1} + \alpha x_{1,j}. \tag{18}$$

Upon having fictitious control (18), the term $-\dot{u}_{1,j}$ in differential equation of $x_{2,j}$ can be derived as follows. First, it follows that

$$-\dot{u}_{1,j} = (2 + \beta + \rho_{g_{11}} + x_{1,j}^2)\dot{x}_{1,j} + 2z^{\mathrm{d}}\dot{z}^{\mathrm{d}}x_{1,j} + \dot{\Delta}_{1,j}.$$

Second, $\dot{x}_{1,j}$ and $\dot{\Delta}_{1,j}$ (with $0 < \gamma < 1$) can be rewritten by (16) and (18) in terms of $x_{i,j}$, $\Delta_{1,j}$, $\Delta_{1,j-1}$, and $\zeta_1(t)$. In the case that $\gamma = 0$, it follows from (18) that $\dot{\Delta}_{1,j} = \dot{\Delta}_{1,j-1} + \alpha\dot{x}_{1,j}$. Note that, at the $j$th trial, $\Delta_{1,j-1}$ is a known time function and hence $\dot{\Delta}_{1,j-1}$ can be calculated. To avoid high noise sensitivity of derivative operation, learning law with $\gamma > 0$ can be used by trading off asymptotic stability. With these facts in mind, we shall proceed with design and, for comparison and illustration, shall make actual learning control $u_j$ be one with linear feedforward part.

It follows from (8) that linear feedforward part can be obtained by setting $g_{2,j} = 0$. By comparing Eq. (17) with its preceding one, we have that, under the choice of $g_{2,j} = 0$,

$$h_{2,j}^{-1} = a_3(t), \qquad \zeta_2(t) = \begin{bmatrix} a_2(t)/a_3(t) \\ a_1(t)/a_3(t) \\ 1/a_3(t) \end{bmatrix}, \qquad \rho_{h_2} = 0.$$

Function $f_{2,j}$ has two expressions: if $0 < \gamma < 1$,

$$f_{2,j} = \begin{bmatrix} 1 + z_{1,j}^2 z_{2,j} \\ (2 + \beta + \rho_{g_{11}} + x_{1,j}^2)z_{1,j}^2 \\ (F_3 + 2z^{\mathrm{d}}\dot{z}^{\mathrm{d}}x_{1,j} + [-\Delta_{1,j} + (1-\gamma)\Delta_{1,j-1} + \alpha x_{1,j}]/\gamma \end{bmatrix}^{\mathrm{T}},$$

where $F_3 = (2 + \beta + \rho_{g_{11}} + x_{1,j}^2)(z_{2,j} - \dot{z}^{\mathrm{d}})$, and if $\gamma = 0$,

$$f_{2,j} = \begin{bmatrix} 1 + z_{1,j}^2 z_{2,j} \\ (2 + \alpha + \beta + \rho_{g_{11}} + x_{1,j}^2)z_{1,j}^2 \\ (2 + \alpha + \beta + \rho_{g_{11}} + x_{1,j}^2)(z_{2,j} - \dot{z}^{\mathrm{d}}) + 2z^{\mathrm{d}}\dot{z}^{\mathrm{d}}x_{1,j} + \dot{\Delta}_{1,j-1} \end{bmatrix}^{\mathrm{T}}.$$

Since this is the last subsystem and hence no more state transformation is needed, learning law (10) with $\gamma = 0$

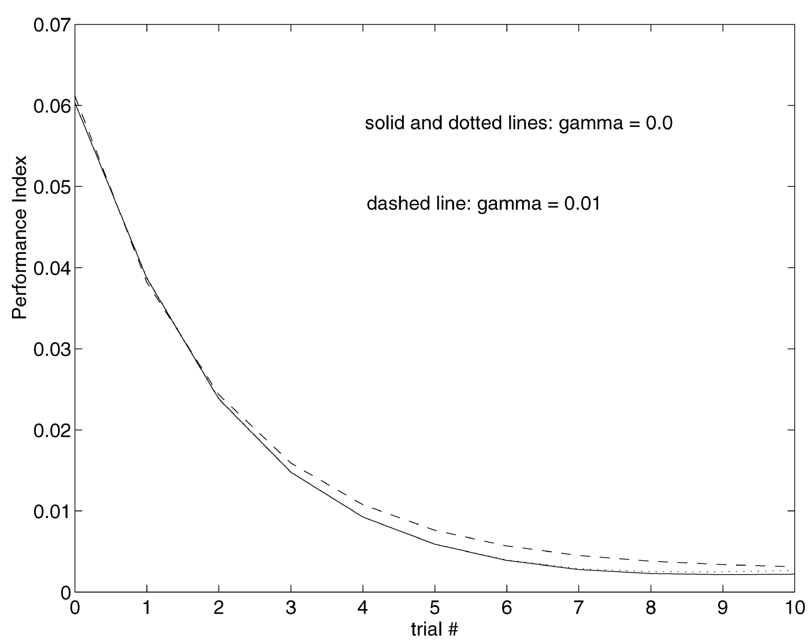


Fig. 1. Tracking performance.

can always be selected. Thus, actual control $u_j$ is defined by (8) with the above functions, i.e.,

$$u_j = -[x_{1,j} + x_{2,j}] - f_{2,j}\Delta'_{2,j}$$
$$= -x_{1,j} - (1 + \beta f_{2,j} f_{2,j}^{\mathrm{T}}) x_{2,j} - f_{2,j}\Delta_{2,j}.$$

Simulations were carried out using SIMNON© to demonstrate effectiveness of the proposed learning scheme. In the simulations, the following choices were made for system (14) and (15): Initial conditions are $z_{1,0}(0) = z_{2,0}(0) = 0$; desired trajectory is given by $z^{\mathrm{d}} = (1 - \cos t)$; learning trials are of length $\delta T = 2$ s; unknown periodic functions are chosen to be $a_1(t) = \cos^3(0.5t)$, $a_2(t) = \cos(3.0t)$, and $a_3(t) = 1$; learning controls $u_{1,j}$ and $u_j$ are implemented with gains $\alpha = 5$ and $\beta = 5$. Both types of learning laws were simulated: difference updating law (i.e., $\gamma = 0$) and differential-difference updating law with $\gamma = 0.01$.

The performance index used to evaluate tracking performance is defined to be

$$PI(j) = \int_0^{\delta T} (z_{1,j} - z^{\mathrm{d}})^2 \, \mathrm{d}\tau.$$

Simulation results are given in Fig. 1. Performance under learning control with $\gamma = 0$, shown by the solid line, is the best as expected. For simple calculation, $\dot{\Delta}_{1,j-1}$ can be removed from matrix $f_{2,j}$, whose performance is shown by the dotted line. It is clear that $\dot{\Delta}_{1,j-1}$ is not critical for stability but its existence will enhance convergence. The dashed line represents performance under learning control with $\gamma = 0.01$. As shown by theoretical analysis, filtering of $0 < \gamma < 1$ would degrade somewhat the learning performance.

## 4. Conclusion

Since most physical systems are nonlinear and perfect knowledge of their dynamics are usually unavailable, control design should be both nonlinearly based and robust. Most existing learning control scheme are based on Lipschitz condition and, in addition, they require derivative measurement of the state and resetting of initial conditions. The proposed nonlinear learning control design not only extends the backward recursive (backstepping) design but also overcomes the shortcomings of the existing learning designs. Two classes of learning controls have been derived, and the designer has much flexibility in choosing various combinations of feedforward and learning control parts. The trade-off is that the resulting control laws require more computation time, which is acceptable using the current computer technology.

Future study is needed to develop a learning control design method for nonlinear systems not satisfying the cascaded structure. Such an effort will call for nonlinear analysis and design methods, such as the recursive-interlacing design (Qu, 1995) that are more sophisticated and advanced than the one-directional (backward or forward) design.

## References


Arimoto, S. (1996). *Control theory of non-linear mechanical systems: A passivity-based and circuit-theoretic approach*. Oxford, UK: Oxford University Press and Clarendon Press.

Arimoto, S., Kawamura, S., & Miyazaki, F. (1984a). Bettering operation of dynamic systems by learning: A new control theory for servomechanism or mechatronics systems. *Proceedings of the IEEE conference on decision and control*, Las Vegas, Nevada (pp. 1064–1069).

Arimoto, S., Kawamura, S., & Miyazaki, F. (1984b). Iterative learning control for robotic systems. *Proceedings of IECON*, Tokyo, Japan (pp. 22–26).

Arimoto, S., Kawamura, S., Miyazaki, F., & Tamaki, S. (1984). Learning control theory for dynamical systems. *Proceedings of IECON*, Tokyo, Japan (pp. 22–26).

Bondi, P., Casalino, G., & Gambardella, L. (1988). On the iterative learning control theory for robotic manipulators. *IEEE Journal of Robotics and Automation*, *4*, 14–22.

Ham, C., & Qu, Z. (1997). A new nonlinear learning control for robotic manipulators. *Advanced Robotics*, *11*(1), 1–15.

Ham, C., Qu, Z., & Johnson, R. (2000). A nonlinear iterative learning control for robot manipulators in the presence of actuator dynamics. *International Journal of Robotics and Automation*, in press.

Ham, C., Qu, Z., & Park, M. (1994). A new learning control of robot manipulators. *Korean automatic control conference*, Seoul, Korea (pp. 697–702).

Hara, S., Yamamoto, Y., Omata, T., & Nakano, M. (1988). Repetitive control system: a new type servo system for periodic exogenous signals. *IEEE Transactions on Automatic Control*, *33*(7), 659–667.

Hauser, J. (1987). Learning control for a class of nonlinear systems. *Proceedings of the IEEE conference on decision and control*, Los Angeles, CA (pp. 859–860).

Heinzinger, G., Fenwick, D., Paden, B., & Miyazaki, F. (1992). Stability of learning control with disturbances and uncertain initial conditions. *IEEE Transactions on Automatic Control*, *37*(1), 110–114.

Horowitz, R., Messner, W., & Moore, J. B. (1991). Exponential convergence of a learning controller for robot manipulators. *IEEE Transactions on Automatic Control*, *36*(7), 890–894.

Jeon, D., & Tomizuka, M. (1993). Learning hybrid force and position control of robotic manipulators. *IEEE Transactions on Robotics and Automation*, *9*(4), 423–431.

Kanellakppoulos, I., Kokotovic, P. V., & Morse, A. S. (1991). Systematic design of adaptive controllers for feedback linearizable systems. *IEEE Transactions on Automatic Control*, *36*(11), 1241–1253.

Kawamura, S., Miyazaki, F., & Arimoto, S. (1985). Application of learning methods for dynamic controls of robotic manipulators. *Proceedings of IEEE Conference on Decision and Control*, Ft. Lauderale, FL (pp. 1381–1386).

Khalil, H. (1992). *Nonlinear systems*. New York, NY: Macmillan.

Kuc, T., Lee, J. S., & Nam, K. (1992). An iterative learning control theory for a class of nonlinear dynamic systems. *Automatica*, *28*(6), 1215–1221.

Messner, W., Horowitz, R., Kao, W., & Boals, M. (1991). A new adaptive learning rule. *IEEE Transactions on Automatic Control*, *36*(2), 188–197.

Miller III, T., Glanz, F., & Kraft III, G. (1987). Application of a general learning algorithm to the control of robotic manipulators motion. *International Journal of Robotics*, *6*, 84–98.

Moore, M. (1993). *Iterative learning control for deterministic systems*. London: Springer.

Qu, Z. (1993). Robust control of nonlinear uncertain systems under generalized matching conditions. *Automatica*, *29*(4), 985–998.

Qu, Z. (1995). Robust control design for nonlinear uncertain systems without generalized matching conditions. *IEEE Transactions on Automatic Control*, *40*(8), 1453–1460.

Qu, Z., & Dawson, D. M. (1996). *Robust tracking control of robot manipulators*. New York, NY: IEEE Press.

Qu, Z., Dorsey, J., Dawson, D., & Johnson, R. (1993). Linear learning control of robot motion. *Journal of Robotic Systems*, *10*(1), 123–140.

Qu, Z., & Zhuang, H. (1993). Nonlinear learning control of robot manipulators without requiring acceleration measurement. *International Journal of Adaptive Control and Signal Processing*, *7*(2), 77–90.

Rouche, N., Habets, P., & Laloy, M. (1977). *Stability theory by Lyapunov's direct method*. New York: Springer.

Slotine, J. J., & Li, W. (1991). *Applied nonlinear control*. Englewood Cliffs, NJ: Prentice-Hall.

Sugie, T., & Ono, T. (1991). An iterative learning control law for dynamical systems. *Automatica*, *27*, 729–732.

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